On the lattices and $K3$ surfaces admitting symplectic automorphism[1]

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Abstract

In this talk, we will discuss a study on symplectic automorphisms on $K3$ surfaces. The main source of this talk is the article in preparation entitled "Primitive closure of the lattices associated to symplectic automorphisms on $K3$ surfaces (temporary)" by the presenter.

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Contents

1 Introduction

A study of $K3$ surfaces spreads in a wide range of areas in mathematics. We are interested from the viewpoints of algebraic geometry and singularity theory.

As an example, a $K3$ surface is obtained as the minimal model of a double covering of the projective plane branching at a sextic curve with at most ADE singularities. By identifying "Gorenstein model" and its minimal model up to birational equivalence, such a surface is regarded as a general anticanonical member of the weighted projective space with weights $1, 1, 1, 3$. This weight system also gives a compactified simple $K3$ singularity in $\mathbb{C}^3$. Thus we may consider the Milnor lattice associated to the hypersurface singularity. A $K3$ surface admits the Picard lattice, which is the group $H^1(X, \mathcal{O}_X^*)$ with a natural pairing inherited by $H^2(X, \mathbb{Z})$. One of our motivation is to find out some intrinsic relation between the Milnor lattice of a simple $K3$ singularity and the Picard lattice of the associated $K3$ surface.

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Related to algebraic curves in $K3$ surfaces, it is quite important to study whether or not a given semigroup is admitted by a pointed algebraic curve. For a double covering type, we have technique to investigate this question, while in genral, for an n-covering, we have no technique. We think we have to study algebro-topological aspects for the coverings to study the im-/possibility of this admittance, which is our second motivation.

Finally, what is in common in the above topics is the existence of automorphism on a $K3$ surface. Finite automorphism groups acting symplectically on $K3$ surfaces are well-studied and all classified by Nikulin [3], Mukai [2] and Xiao [7]. If a $K3$ surface X admits a symplectic automorphism group G, then, the minimal model $Y := \widetilde{X/G}$ of the quotient X/G is also birationally isomorphic to a $K3$ surface. It is interesting to compare the Picard lattice of X and that of Y. The classes of (-2)-curves in the exceptional divisor of a minimal resolution of the singular locus of X/G live in the Picard lattice of Y, forming a sublattice, say L_G.

By Torelli-type theorem, in order to understand the geometry of Y, it is important to study the Picard lattice of Y, and in particular, the structure of L_G in the Picard lattice. In fact, it is not necessarily true that L_G itself is a primitive sublattice of the $K3$ lattice Λ_{K3}, while the Picard lattice of Y is. Our problem is to determine whether or not it is possible to construct explicitly a primitive sublattice $\tilde{L}_G$ such that $L_G \subset \tilde{L}_G \subset \Lambda_{K3}$ holds, and if it is true, to find an explicit generator of the primitive model. Among such groups G, Nikulin [3] and Whitcher [6] study the problem for all Abelian cases and non-Abelian with $G = [G, G]$, respectively. Our aim is to consider the problem for the remaining cases. Here is our main theorem of this talk:

Main Theorem. Suppose that a finite group G acts symplectically on a $K3$ surface and neither the commutator subgroup $[G, G]$ nor the abelianization $Q := G/[G, G]$ of G is trivial. Then, there exists a generator for the quotient $\tilde{L}_G/L_G$ satisfying the condition $(*)$. Moreover, if Q is a cyclic group of order 2 or 3, then the existance of the generator is unique up to isomorphism.

2 Preliminary

2.1 Basic Facts

We start with recalling basic facts on $K3$ surfaces and symplectic automorphisms on them.

Definition 2.1. A $K3$ *surface* is a compact complex 2-dimensional smooth algebraic variety with trivial canonical divisor and irregularity zero. ■

A *lattice* is a non-degenerate finitely-generated $\mathbb{Z}$-module. Denote by U the hyperbolic lattice of rank 2 and E_8 the negative-definite even unimodular lattice of rank 8. For a $K3$ surface X, the Hodge decomposition gives

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

where $H^{2,0}(X) = \overline{H^{0,2}}(X)$ and $H^{1,1}(X) = \overline{H^{1,1}}(X)$.

Facts 2.2. Let X be a $K3$ surface.

- The surface X admits a nowhere-vanishing holomorphic 2-form ω_X that is unique up to constant, and $H^{2,0}(X) = \mathbb{C}\omega_X$.

- The cohomology group $H^2(X, \mathbb{Z})$ is a negative-definite even unimodular lattice with signature $(3, 19)$: $H^2(X, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2}$. We call the even unimodular lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ the $K3$ *lattice*, which is denoted by Λ_{K3}.

- The *Picard lattice* of X, denoted by $\mathrm{Pic}\,(X) := H^1(X, \mathcal{O}_X^*)$, is a torsion-free primitive sublattice of $H^2(X, \mathbb{Z})$ of signature $(1, \rho - 1)$, where ρ is called the *Picard number.*

Let $g \in \mathrm{Aut}(X)$ faithfully act on X. The action of g naturally induces a transformation on ω_X by

$$g^*\omega_X = \alpha\omega_X \quad (\alpha \in \mathbb{C}^*).$$

Definition 2.3. (1) The action of g on X is called *symplectic* if $\alpha = 1$, and lest *non-symplectic.*

(2) A finite subgroup G of the automorphism group $\mathrm{Aut}(X)$ of a $K3$ surface X acts *symplectically* on X if all $g \in G$ acts symplectically on X. ■

Facts 2.4. If a finite subgroup $G \subseteq \mathrm{Aut}(X)$ acts symplectically on X, then the quotient space X/G has at most ADE singularities. Thus, the minimal model $Y := \widetilde{X/G}$ is again a $K3$ surface.

Here, we fix the notations as in the list below:
X : $K3$ surface,
$G \subseteq \mathrm{Aut}(X)$: finite group, symplectically acting on X,
$\mathrm{Sing}(X/G)$: the singular locus of X/G,
$\pi : Y := \widetilde{X/G} \to X$: minimal resolution of $\mathrm{Sing}(X/G)$,
L_G : lattice spanned by all classes of (-2)-curves in the exceptional divisor of π.
In general, for an even lattice L,
$L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$: the dual lattice of L,
$A_L := L^*/L$ the discriminant group of L,
$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$: the discriminant quadratic form on A_L,
$b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$: the discriminant bilinear form on A_L.

2.2 History

We first present a brief history of the classifications of symplectic automorphism groups G on a $K3$ surface, and their fundamental properties.

Denote by C_n the cyclic group of order n.

- Nikulin [3, Theorem 4.5] classifies abelian cases. There are fourteen of them in all:

$$C_2^k\,(k = 1, \ldots, 4), \quad C_3^l\,(l = 1, 2), \quad C_4^m\,(m = 1, 2),$$
$$C_n\,(n = 5, 6, 7, 8), \quad C_2 \times C_h\,(h = 4, 6).$$

- Mukai [2] shows that each G (not necessarily abelian) is a subgroup of the Mathieu group M_{23} of order 23.

- Xiao [7] completes the classification of G to conclude that there are 81 classes up to isomorphism, and the configuration of $\mathrm{Sing}(X/G)$ is determined.

Remark 2.5. *The lattice L_G is not necessarily a primitive sublattice of the $K3$ lattice as in Example 2.6.*

Example 2.6 ([3], $G = \mathbb{Z}_2$)**.** Suppose $G = C_2$. Then, we have $L_G = A_1^{\oplus 8}$, which is not a primitive sublattice of Λ_{K3}. Indeed,

$$x = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$$

is a torsion element in Λ_{K3}/L_G since $x^2 = -2$, and $2x \in L_G$. ■

Our next natural question is the existence and properties of a primitive sublattice $\widetilde{L}_G$ of L_G in the Picard lattice of Y. We summarize a history of the studies concerning $\widetilde{L}_G$.

- Nikulin [3, Theorem 7.2] : for all abelian G, $\widetilde{L}_G$ is determined, and uniqueness of its generator is proved.

- Xiao [7] : L_G is determined for each G. Moreover, he proves

  Lemma 2.7 ([7])**.** *The quotient $\widetilde{L}_G/L_G$ is isomorphic to the dual of the abelization group $Q := G/[G, G]$.* □

- Whitcher [6] : for all non-abelian G with $G/[G, G] = \{1\}$, determines the non-/uniqueness of the generators of $\widetilde{L}_G$.

As an example, we produce a part of Nikulin's result in [3]. In this context, we assume that
$G \subseteq \mathrm{Aut}(X)$: abelian group of order $m := |G|$,
$\{id\} \neq G_i \subset G$: cyclic subgroup of G of order $m_i := |G_i|$ ($i = 1, \ldots, N$),
k_i : the number of points in X that are stational by G_i.
Then, by an analysis of the Euler characteristic, there is a relation :

$$24(m-1) = \sum_{i=1}^{N} k_i(m_i^2 - 1). \qquad (\star)$$

By ($\star$), one can determine G.

Theorem 2.8 (Theorem 7.2 [3])**.** *There exists a unique generator for $\widetilde{L}_G/L_G$ for abelian G.* □

Note that, in his paper, our $\widetilde{L}_G$ is denoted by $M_{(G)}$. The generator in each case is explicitly given as in the following table:

#	G	Additional Element(s)	rk$M_{(G)}$	det $M_{(G)}$	$A_{M_{(G)}}$
$1a$	$\mathbb{Z}_2$	$\sum_{l=1}^{8} f_{1l}^{(2)}$	8	2^6	$\mathbb{Z}_2^6$
$1a$	$\mathbb{Z}_3$	$\sum_{l=1}^{6} f_{1l}^{(3)}$	12	3^4	$\mathbb{Z}_3^4$
$1a$	$\mathbb{Z}_5$	$f_{11}^{(5)} + f_{12}^{(5)} + 2f_{13}^{(5)} + 2f_{14}^{(5)}$	16	5^2	$\mathbb{Z}_5^2$
$1a$	$\mathbb{Z}_7$	$f_{11}^{(7)} + 2f_{12}^{(7)} + 3f_{13}^{(7)}$	18	7	$\mathbb{Z}_7$
$1b$	$\mathbb{Z}_4$	$f_{11}^{(2)} + f_{12}^{(2)} + f_{21}^{(4)} + f_{22}^{(4)} + f_{23}^{(4)} + f_{24}^{(4)}$	14	2^6	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
$1c$	$\mathbb{Z}_6$	$f_{11}^{(2)} + f_{12}^{(2)} + f_{21}^{(3)} + f_{22}^{(3)} + f_{31}^{(6)} + f_{32}^{(64)}$	16	$2^2 \cdot 3^2$	$\mathbb{Z}_6^2$
$1d$	$\mathbb{Z}_8$	$f_{11}^{(2)} + f_{21}^{(4)} + f_{31}^{(8)} + 3f_{32}^{(8)}$	18	2^3	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$2a$	$\mathbb{Z}_2^2$	$\frac{1}{2}\sum_{(\varepsilon_1,\varepsilon_2),\varepsilon_q=1}\sum_{l=1}^{4} e_{(\varepsilon_1,\,\varepsilon_2)l}$ $(q = 1, 2)$	12	2^8	$\mathbb{Z}_2^8$
$2a$	$\mathbb{Z}_2^3$	$\frac{1}{2}\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3),\varepsilon_q=1}\sum_{l=1}^{2} e_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)l}$ $(q = 1, 2, 3)$	14	2^8	$\mathbb{Z}_2^8$
$2a$	$\mathbb{Z}_2^4$	$\frac{1}{2}\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4),\varepsilon_q=1} e_{(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)1}$ $(q = 1, 2, 3, 4)$	15	-2^7	$\mathbb{Z}_2^7$
$2b$	$\mathbb{Z}_3^2$	$f_{11}^{(3)} + f_{12}^{(3)} + f_{21}^{(3)} + f_{22}^{(3)} + f_{31}^{(3)} + f_{32}^{(3)}$, $f_{21}^{(3)} + f_{22}^{(3)} - f_{31}^{(3)} - f_{32}^{(3)} + f_{41}^{(3)} + f_{42}^{(3)}$	16	3^4	$\mathbb{Z}_3^4$
$2c$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$f_{11}^{(2)} + f_{12}^{(2)} + f_{21}^{(2)} + f_{22}^{(2)} + f_{41}^{(4)} + f_{42}^{(4)}$, $f_{11}^{(2)} + f_{12}^{(2)} + f_{31}^{(4)} + f_{32}^{(4)} + f_{41}^{(4)} + f_{42}^{(4)}$	16	2^6	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
$2d$	$\mathbb{Z}_4^2$	$f_{11}^{(4)} + f_{21}^{(4)} + f_{31}^{(4)} + f_{41}^{(4)} + f_{61}^{(4)}$, $2f_{21}^{(4)} + f_{31}^{(4)} - f_{41}^{(4)} + f_{51}^{(4)} + f_{61}^{(4)}$	18	2^4	$\mathbb{Z}_4^2$
$2e$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$f_{21}^{(2)} + f_{31}^{(2)} + 3f_{51}^{(6)} + 3f_{61}^{(6)}$, $f_{11}^{(2)} + f_{21}^{(2)} + f_{41}^{(6)} + f_{51}^{(6)} + 2f_{61}^{(6)}$	18	$2^2 \cdot 3$	$\mathbb{Z}_2 \times \mathbb{Z}_6$

In the above list, we mean:

$$f_{il}^{(m_i)} := \sum_{r=1}^{m_i-1} \frac{r}{m_i} e_{ilr},$$

and e_{ilr}'s (resp. $e_{(\varepsilon_1,\ldots,\varepsilon_k)l}$'s) are canonical generators of the lattice $\widetilde{L}_G$ (forming appropriate trees in accordance with $\mathrm{Sing}(X/G)$).

Example 2.9 ([3], $G = \mathbb{Z}_2$, $L_G = A_1^{\oplus 8}$)**.** Suppose $G = \mathbb{Z}_2$. Then, one obtains the primitive sublattice

$$\widetilde{L}_G = L_G + \mathbb{Z}g$$

of Λ_{K3} with the UNIQUE additional element

$$g := \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8).$$

Here, e_i is the generator of the i-th copy of A_1 in L_G.

Note that $q_{L_G}(e_i) = -2$, and $q_{L_G}(g) = -4$. ■

In general, since $L_G \subseteq \widetilde{L}_G$, we have that $\widetilde{L}_G^* \subseteq L_G^*$ (the dual-lattice process is contravariant). It is trivial by definition that $L_G \subseteq L_G^*$, and $\widetilde{L}_G \subseteq \widetilde{L}_G^*$. Combining them, and we get

$$L_G \subseteq \widetilde{L}_G \subseteq \widetilde{L}_G^* \subseteq L_G^*.$$

Thus, $\widetilde{L}_G/L_G \subseteq L_G^*/L_G = A_{L_G}$. Therefore, We may search a generator for $\widetilde{L}_G/L_G$ in the discriminant group of L_G.

Motivated by [3], we set the condition $(*)$ as follows:

$$(*) \begin{cases} \bullet \ \ q_{L_G}(e) \equiv 0 \mod 2, \text{ and } e^2 \neq -2, \\ \bullet \ \ \forall d \in L_G, b_{L_G}(d,\, e) \in \mathbb{Z}\, (i.e., \widetilde{L}_G \text{ is a } \mathbb{Z}\text{-lattice}), \text{ and} \\ \bullet \ \ \text{if } L_G^*/L_G \simeq \langle e_1 \rangle \simeq \langle e_2 \rangle \text{ with } e_1 \neq e_2, \text{ then}, b_{L_G}(e_1,\, e_2) \in \mathbb{Z} \\ \quad\ \text{(``compatibility'')}. \end{cases}$$

Problem 2.10. *Describe the smallest primitive sublattice $\widetilde{L}_G$ s.t.*

$$L_G \subseteq \widetilde{L}_G \subseteq \Lambda_{K3}.$$

Equivalently, describe a generator $e \in L_G^*/L_G$ with

$$\widetilde{L}_G = L_G + \mathbb{Z}e$$

satisfying the condition $(*)$.

According to the background results, we may proceed to give an answer to Problem 2.10 for non-abelian G's with neither $[G,\, G]$ nor Q is trivial.

3 Main Theorem and a sketch of the proof

We re-produce our main theorem.

Main Theorem 1. Suppose that a finite group G acts symplectically on a $K3$ surface and neither the commutator subgroup $[G,\, G]$ nor the abelianization $Q := G/[G,\, G]$ of G is trivial. Then, there exists a generator for the quotient $\widetilde{L}_G/L_G$ satisfying the condition $(*)$. Moreover, if Q is a cyclic group of order 2 or 3, the existance of the generator is unique up to isomorphism.

In the following three subsections, we give a sketch of the proof of our main theorem.

3.1 Existence

Suppose that the abelianization of G contains a factor C_n as

$$Q := G/[G, G] = \cdots \times C_n \times \cdots .$$

As we have discussed before, we may search a generator in the discriminant group A_{K_G}.

Since we know explicitly a formula for the discriminant quadratic form q_{L_G} on A_{L_G}, we can compute the self-intersection number (norm)

$$q_{L_G}(g) \quad \text{for} \quad \mathrm{ord}(g) = n$$

to determine which $g \in A_{L_G}$ satisfies the conditions

$$q_{L_G}(g) \in 2\mathbb{Z} \quad \text{and} \quad q_{L_G}(g) \leq -4.$$

In case that there exist two candidates $g_1, g_2 \in A_{L_G}$ for the generator, determine whether or not the intersection number satisfies the condition

$$b_{L_G}(g_1, g_2) \in \mathbb{Z}.$$

Since there is a relation

$$2b_{L_G}(g_1, g_2) \equiv q_{L_G}(g_1 + g_2) - q_{L_G}(g_1) - q_{L_G}(g_2) \mod 2,$$

we may well see if

$$q_{L_G}(g_1 + g_2) \in 2\mathbb{Z}$$

holds true.

Next, we show the uniqueness of the generator in the cases $Q = C_2$, and C_3.

Let M be a lattice that is the direct sum of lattices of ADE-type. Occasionally we use the following well-known facts for such a lattice M.

(i) There exists an induced homomorphism $O(M) \to O(A_M)$ between the automorphism group of the lattice M and that of discriminant group A_M [4, §1-4°].

(ii) If the Dynkin diagram $D(M)$ of M admits a $\mathbb{Z}_2$-symmetry due to a reflection, then, so does the discriminant group A_M.

For notations of groups, we refer [7].

3.2 $Q = C_2$ case.

We construct a generator explicitly by a case-by-case analysis for

$$\begin{aligned} G \;=\; & \mathfrak{S}_4(\#34), T_{48}(\#54), \mathfrak{A}_{4,3}(\#61), 2^4D_6(\#65), \\ & 4^2D_6(\#67), \mathfrak{S}_5(\#70), \mathfrak{A}_{4,4}(\#78), F_{384}(\#80). \end{aligned}$$

In other cases, we use the following two Lemmas.

Lemma 3.1 ($G = D_6(\#6), D_{10}(\#16), \mathfrak{A}_{3,3}(\#30)$)**.** *If $q_{L_G}(g)$ of an element $g \in A_{L_G}$ of order 2 is given by*

$$q_{L_G}(g) = \sum_{i=1}^{8} [-\frac{[a_i]_2^2}{2}]_{-2},$$

then, g contains non-trivial entries as in the table.

Norm	Element
-2	$(\cdots [1]_2,\ [1]_2,\ [1]_2,\ [1]_2,\ [0]_2,\ [0]_2,\ [0]_2,\ [0]_2 \cdots)$
-4	$(\cdots [1]_2,\ [1]_2,\ [1]_2,\ [1]_2,\ [1]_2,\ [1]_2,\ [1]_2,\ [1]_2 \cdots)$

Therefore, there exists a unique generator of $\widetilde{L_G}/L_G$ with the condition $(*)$ up to $O(A_M)$. $\square$

Lemma 3.2 ($G = 2^4 D_{10}(\#73), T_{192}(\#77)$)**.** *If $q_{L_G}(g)$ of an element $g \in A_{L_G}$ of order 2 is given by*

$$q_{L_G}(g) = \sum_{j=1}^{3} [-\frac{3[b_j]_4^2}{4}]_{-2} + \sum_{k=1}^{2} [-\frac{[c_k]_2^2}{2}]_{-2},$$

then, g contains non-trivial entries as in the table. Denote by $m := \#\{j \in \{1,2,3\} \mid [b_j]_4 = [2]_4\}$ and $n := \#\{k \in \{1,2\} \mid [c_k]_4 = [1]_4\}$.

Norm	(m, n)
-2	$(1,2), (2,0)$
Norm	**Element**
-4	$(\cdots [0]_5,\ [0]_5,\ [2]_4,\ [2]_4,\ [2]_4,\ [1]_2,\ [1]_2 \cdots)$

Therefore, there exists a unique generator of $\widetilde{L_G}/L_G$ with the condition $(*)$ up to $O(A_M)$. $\square$

3.3 $Q = C_3$ case.

Similarly we construct explicitly the generator. In particular, we use the following Lemmas.

Lemma 3.3. *Consider a lattice M admitting $\mathbb{Z}_2$-summetry. If the self-intersection number (norm) of an element g of order 3 in A_M is given by*

$$g^2 = \sum_{i=1}^{6} [-\frac{2[a_i]_3^2}{3}]_{-2},$$

then, g contains non-trivial entries up to permutation as in the table below.

Norm	Conditions
-2	$[a_i]_3 = [0]_3$ for $i = 4,5,6$, and $\#\{i \in \{1,2,3\} \mid [a_i]_3 = [2]_3\}$ is odd
-4	$[a_i]_3 \neq [0]_3 \forall i$, and $\#\{i \in \{1,\dots,6\} \mid [a_i]_3 = [2]_3\}$ is even

Therefore, there exists a unique generator

$$(\cdots,\ [1]_3,\ [1]_3,\ [1]_3,\ [1]_3,\ [1]_3,\ [1]_3,\ \cdots)$$

(of norm -4) of A_M up to $O(A_M)$ symmetry with the condition $(*)$. $\square$

Lemma 3.4. *Consider a lattice M admitting $\mathbb{Z}_2$-summetry. If the self-intersection number (norm) of an element g of order 3 in A_M is given by*

$$g^2 = \sum_{i=1}^{2}[-\frac{4[a_i]_3^2}{3}]_{-2} + \sum_{j=1}^{2}[-\frac{2[c_j]_3^2}{3}]_{-2},$$

then, g contains non-trivial entries up to permutation as in the table below.

Norm	Conditions
-2	$[a_1]_3 = [c_1]_3 \neq [0]_3$ and $[a_2]_3 = [c_2]_3 = [0]_3$
-4	$[a_i]_3$ and $[c_j]_3 (\forall i, \forall j)$ are non-zero

Therefore, there exists a unique generator

$$(\cdots, [1]_3, [1]_3, [1]_3, [1]_3, \cdots)$$

(of norm -4) of A_M up to $O(A_M)$ symmetry with the condition $(*)$. $\square$

4 Summary and Prospect

4.1 Summary

In this talk, by giving an explicit generator, we have described the smallest primitive closure $\widetilde{L}_G$ of the lattice L_G in the $K3$ lattice Λ_{K3} in the cases where $G \subseteq \mathrm{Aut}(X)$ acts symplectically on X, neither $[G, G]$ nor Q is trivial.

4.2 Prospects I: Other cases

An idea: general theory of du Val singularities. [2]

In some cases, we expect to be able to use techniques of double covering of rational double points (RDP's for short), globally a ramified point. An RDP is a germ of isolated singularity $(\mathcal{X}, 0)$ which is known to be isomorphic to the quotient singularity

$$(\mathbb{C}^2/\mathcal{G}, 0),$$

where $\mathcal{G}$ is a finite subgroup of $SL_2(\mathbb{C})$. It is known that such a group $\mathcal{G}$ is up to isomorphic classified into the following five cases, and the corresponding RDP's are given in the far right column :

Cyclic	$C_{2m} = \left\langle \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{pmatrix} \right\rangle$	A_{2m-1}
Binary Dihedral	$BD_n = \left\langle \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$	D_n
Binary Tetrahedral	$BT_{24} = \left\langle \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right\rangle$	E_6
Binary Octagonal	$BO_{48} = \left\langle \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right\rangle$	E_7
Binary Icosahedral	$BI_{120} = \left\langle \begin{pmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5-\zeta_5^4 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & -\zeta_5+\zeta_5^4 \end{pmatrix} \right\rangle$	E_8

[2] Here, we follow the notations in [1] and refer [5].

Since the group C_{2n} is a normal subgroup of BD_{4n}, the group BD_{4n}/C_{2n} gives a covering transformation of

$$\sigma : \mathbb{C}^2/C_{2n} \to \mathbb{C}^2/BD_{4n}$$

of order 2. Thus the mapping σ is a ramifying double covering of du Val singularities from an A_{2n-1}-singularity to a D_{4n}-singularity. Similarly, there is a double covering $D_{8n} \to D_{4n}$ due to the fact that the group BD_{8n} is a normal subgroup of BD_{4n}.

4.3 Prospects II

In future, we are intended

- to compute the invariants of the lattice $\widetilde{L}_G$:
  the rank, the discriminant group, the discriminant form.
- to describe a polarization of the $K3$ surface Y.
- to reveal an elliptic structure (if any) of Y.
- to study the relations between the Picard lattice of X, that of Y and the lattice $\widetilde{L}_G$.

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