Characteristic classes of manifold bundles and graph homology

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1. Background

In recent joint work with Ib Madsen [3], we found a surprising connection between differential topology of high dimensional manifolds and automorphisms of free groups. Namely, we found that the stable rational cohomology,

$$\lim_{g\to\infty} H^*(B\widetilde{\mathrm{Diff}}_D(W_g);\mathbb{Q}),$$

of the block diffeomorphism group of the $2n$-manifold ($2n \geq 6$)

$$W_g = \# S^n \times S^n,$$

relative to an embedded disk $D = D^{2n} \subset W_g$, could be computed in terms of the homology of automorphism groups of free groups. The connection is relayed by a certain graph complex, a version of which is described below.

In principle, this means that classes in the homology of automorphism groups of free groups, or equivalently graph homology classes, give rise to characteristic classes of block bundles with fiber W_g, relative to a disk. However, the computation in [3] does not yield a workable definition of such characteristic classes and it does not show whether similar classes can be defined for manifolds other than W_g.

The overall goal of the work presented here is to solve these problems: We introduce new families of rational characteristic classes of M-bundles associated to graph homology classes, defined for arbitrary simply connected closed oriented manifolds M. When specialized to W_g, the new classes lead to a better understanding of the computation of [3]. In particular, we are able to solve the main conjecture of [3] about the relation to the generalized Miller-Morita-Mumford classes.

1.1. The graph complex

The graph complex we study will be denoted F^m and is defined as a certain twisted Feynman transform of the Lie operad, in the sense of Getzler-Kapranov [5]. More concretely, F^m is a direct sum of finite dimensional chain complexes $\mathsf{F}^m((g,n))$ that may be described as follows. Let $\mathsf{F}^m((g,n))^i$ be the vector space over $\mathbb{Q}$ spanned by connected graphs G with

- $\dim H_1(G) = g$,
- n univalent external vertices, or 'hairs', labeled $1,\ldots,n$,
- i internal vertices of valence ≥ 3, each equipped with a cyclic order of its adjacent half-edges,

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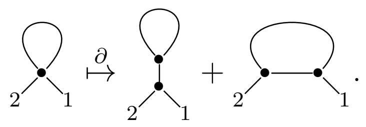
- an orientation of the vertices and an orientation of each edge. If m is odd, then add an orientation of $H_1(G)$.

We furthermore impose certain 'shuffle relations' locally at each internal vertex, but we omit the details here. There is a differential $\partial\colon \mathsf{F}^m((g,n))^i \to \mathsf{F}^m((g,n))^{i+1}$ given by 'edge expansions', e.g.,



This yields a finite cochain complex,

$$\mathsf{F}^m((g,n))^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathsf{F}^m((g,n))^{2g+n-2}.$$

We regrade it in order to view it as a chain complex:

$$\mathsf{F}^m((g,n))_{2g+n-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathsf{F}^m((g,n))_0.$$

The homology of this graph complex turns out to be expressible in terms of the homology of automorphism groups of free groups. To state this more precisely, let $A_{g,n}$ denote the discrete group of homotopy classes of homotopy self-equivalences of a bouquet of g circles, relative to n marked points. Then

$$A_{g,0} \cong \mathrm{Out}(F_g), \quad A_{g,1} \cong \mathrm{Aut}(F_g), \quad A_{g,n} \cong \mathrm{Aut}(F_g) \ltimes F_g^{n-1},$$

where F_g is the free group on g generators. The following can be deduced from results of Kontsevich [8, 9] and Conant-Kassabov-Vogtmann [4].

Theorem 1.1 (Kontsevich, Conant-Kassabov-Vogtmann)**.** *Suppose m is even. For all $g+n \geq 2$ and all k,*

$$H_k(\mathsf{F}^m((g,n))) \cong H_k(A_{g,n};\mathbb{Q}).$$

The homology groups $H_k(A_{g,n};\mathbb{Q})$ have been studied extensively, but they remain largely unknown.

1.2. TM-fibrations

Let M be a closed oriented manifold with tangent bundle TM. The new characteristic classes will be defined not only for smooth M-bundles but for a generalization that we call TM-fibrations [2].

Definition 1.2. A TM-fibration over a space B consists of

- a fibration $M \to E \xrightarrow{\pi} B$, and
- an oriented vector bundle $T_\pi E$ over the total space E such that $T_\pi E|_{\pi^{-1}(b)} \sim TM$ for every $b \in B$.

Every smooth oriented M-bundle (i.e., fiber bundle $\pi\colon E \to B$ with fiber M and structure group $\mathrm{Diff}^+(M)$) gives rise to a TM-fibration by letting $T_\pi E$ be the fiberwise tangent bundle. One can also show that block bundles give rise to T^sM-fibrations, where T^sM is the stable tangent bundle. We remark that not every TM-fibration comes from a smooth bundle.

1.3. Characteristic classes

A characteristic class λ of TM-fibrations is the association of a cohomology class

$$\lambda(\pi) \in H^*(B)$$

to each TM-fibration $M \to E \xrightarrow{\pi} B$, such that

$$f^*(\lambda(\pi)) = \lambda(f^*(\pi)) \in H^*(B')$$

for every map $f \colon B' \to B$. By specialization, every characteristic class of TM-fibrations gives rise to a characteristic class of smooth M-bundles.

The well-known Miller-Morita-Mumford classes, originally defined and studied for surface bundles, may be generalized to give characteristic classes of TM-fibrations, one class κ_c for each $c \in H^*(BSO(m))$, defined by

$$\kappa_c(\pi) := \int_M c(T_\pi E) \in H^{*-m}(B).$$

2. Results

The following gives more precise statements of our main results. These will appear in forthcoming work [1].

2.1. Characteristic classes associated to graph homology classes

Theorem 2.1. *Let M be a simply connected closed oriented m-manifold. For every graph homology class $\alpha \in H_k(\mathsf{F}^m((g,n)))$, there is a family κ^α of rational characteristic classes of TM-fibrations $M \to E \xrightarrow{\pi} B$ of simply connected spaces, one class*

$$\kappa^\alpha_{c_1,\ldots,c_n}(\pi) \in H^*(B;\mathbb{Q})$$

for every n-tuple $c_1,\ldots,c_n \in H^*(BSO(m);\mathbb{Q})$.

The cohomological degree is given by

$$|\kappa^\alpha_{c_1,\ldots,c_n}(\pi)| = |c_1| + \ldots + |c_n| - k + m(g-1).$$

Remark 2.2. The characteristic classes κ^α associated to 'vacuum' graph classes, $\alpha \in H_k(\mathsf{F}((g,0)))$, are defined for arbitrary fibrations with fiber M (no bundle over the total space is required). In this case, we recover classes introduced by Matsuyuki [10].

2.2. Detection theorem

A natural questions is whether the new classes are non-trivial. We have the following detection result.

Theorem 2.3. *Let $m \geq 4$ be even. If the graph homology class $\alpha \in H_k(\mathsf{F}^m((g,n)))$ is non-zero, then the family κ^α is non-trivial, i.e., there exists a simply connected m-manifold M, a TM-fibration of simply connected spaces, $\pi \colon E \to B$, and cohomology classes $c_1,\ldots,c_n \in H^*(BSO(m))$, such that*

$$\kappa^\alpha_{c_1,\ldots,c_n}(\pi) \neq 0 \in H^*(B;\mathbb{Q}).$$

The case m odd is work in progress. It is an open problem to decide whether graph homology classes α can be detected by evaluating the characteristic classes $\kappa^\alpha_{c_1,\ldots,c_n}$ on smooth manifold bundles, rather than on TM-fibrations.

2.3. Zero dimensional graph homology and Miller-Morita-Mumford classes

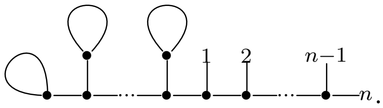
It will be clear from the construction that if $1 \in \mathsf{F}^m((0,1)) \cong \mathbb{Q}$ denotes a generator, then we recover the generalized Miller-Morita-Mumford class,

$$\kappa_c^1 = \kappa_c.$$

More generally, we have the following. For m even,

$$H_0(\mathsf{F}^m((g,n))) \cong H_0(A_{g,n};\mathbb{Q}) \cong \mathbb{Q},$$

and a generator $\epsilon_{g,n}$ may be taken to be the class of the graph



Theorem 2.4. *We have that*

$$\kappa^{\epsilon_{g,n}}_{c_1,\ldots,c_n} = \kappa_{e^g c_1 \ldots c_n},$$

where e is the fiberwise Euler class of [6].

By applying the above result to the $T^s W_g$-fibration associated to the universal block bundle over a suitable cover of $B\widetilde{\mathrm{Diff}}_D(W_g)$, and by tracing the new classes through the computation in [3], we are able to solve Conjecture 1.11 of [3].

Our approach is algebraic. To define the new classes and prove our results, we use rational homotopy theory. In particular, we develop a theory of relative C_∞-algebra models for fibrations. We comment that some of our techniques are similar to techniques used by Kajiura, Matsuyuki and Terashima [7].

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