Quillen rational homotopy theory revisited

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Abstract

This paper surveys the main properties of the model and realization functors,

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}$$

which are based in the cosimplicial complete differential graded Lie algebra $\mathfrak{L}_{\Delta^\bullet}$. This let us extend the Quillen approach to rational homotopy theory to non simply connected spaces and to any complete differential graded Lie algebra.

Introduction

In his celebrated and seminal paper [21], D. Quillen developed the "Lie" approach to rational homotopy theory. It is based in the construction of a couple of functors,

$$\mathbf{sset_1} \underset{\langle - \rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl_1}$$

between the categories of reduced or simply connected simplicial sets, those with only one simplex in dimensions 0 and 1, and that of reduced dgl's, that is, differential graded Lie algebras positively graded. These functors are defined as the composition of several pairs of adjoint functors (the upper arrow denotes left adjoint), in fact Quillen pairs, with respect to the corresponding model category structures,

$$\lambda\colon\ \mathbf{sset_1} \underset{W}{\overset{G}{\rightleftarrows}} \mathbf{sgp_0} \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sch_0} \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{sla_1} \underset{N}{\overset{N^*}{\leftrightarrows}} \mathbf{dgl_1} \colon \langle - \rangle_Q.$$

Here, $\mathbf{sgp_0}$, $\mathbf{sch_0}$ and $\mathbf{sla_1}$ denote respectively the categories of connected simplicial groups, connected complete Hopf algebras, and reduced simplicial Lie algebras. Each of these pairs induces Quillen equivalences on the corresponding homotopy categories when localizing on the family of rational homotopy equivalences in $\mathbf{sset_1}$, $\mathbf{sgp_0}$, and on the family of quasi-isomorphisms in $\mathbf{sch_0}$, $\mathbf{sla_1}$, $\mathbf{dgl_1}$ [21, Thm. I].

The complexity of the functors λ and Quillen realization $\langle - \rangle_Q$ strongly contrasts with the conceptual simplicity of the pair of adjoint functors in which the Sullivan "commutative" approach to rational homotopy theory is based [3, 22]. These are defined by the PL-forms $\mathscr{A}(-)$ on simplicial sets and the Sullivan realization functor $\langle - \rangle_S$ on commutative differential graded algebras (cdga's henceforth):

$$\mathbf{sset} \underset{\langle - \rangle_S}{\overset{\mathscr{A}(-)}{\rightleftarrows}} \mathbf{cdga}\,.$$

Explicitely, given a cdga A, its realization is

$$\langle A \rangle_S = \operatorname{Hom}_{\mathbf{cdga}}(A, \mathscr{A}_\bullet)$$

where $\mathscr{A}_\bullet = \mathscr{A}(\Delta^\bullet)$ is the simplicial set of PL-forms on the standard simplices. In other words, $\langle A \rangle_S$ is "corepresentable" by $\mathscr{A}_\bullet$.

In fact, the lack of an Eckmann-Hilton dual of the simplicial $\mathscr{A}_\bullet$ has puzzled rational homotopy theorists since the birth of the theory. On the other hand, there are many situations in a wide range of mathematics, from algebraic geometry to mathematical physics, where a suitable extension of the Quillen functor to non necessarily reduced dgl's would be most welcome.

These problems are attacked in the work reviewed by this survey, whose departure point is the following observation and subsequent general question raised by R. Lawrence and D. Sullivan in [17]:

The rational singular chains on a cellular complex are naturally endowed with a structure of cocommutative, coassociative infinity coalgebra and hence, taking the commutators of a "generalized bar construction" it should give rise to a complete dgl (in fact, all our dgl's would be of this kind, see next section for a precise definition). What is the topological and geometrical meaning of this dgl? Allowing 1-cells, what is the relation of this dgl with the fundamental group of the given complex?

In the same reference they carefully construct such a dgl for the interval. It consists of a free dgl,

$$\mathfrak{L}_{\Delta^1} = (\widehat{\mathbb{L}}(a, b, x), \partial),$$

in which a and b are Maurer-Cartan elements representing the endpoints of the interval, x is a degree 0 element representing the 1-cell, and

$$\partial x = \operatorname{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_x^n (b - a)$$

where the B_n's are the Bernoulli numbers.

We begin by extending this to any simplex and construct, for each $n \geq 1$, a free dgl $\mathfrak{L}_{\Delta^n} = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), \partial)$ in which $s^{-1}\Delta^n$ together with the linear part of the differential ∂ is the (desuspension) of the rational simplicial chain complex of the standard n-simplex Δ^n, and the vertices correspond to Maurer-Cartan elements. We then show that the family

$$\mathfrak{L}_{\Delta^\bullet} = \{\mathfrak{L}_{\Delta^n}\}_{n \geq 0}$$

is a cosimplicial dgl and therefore, we may geometrically realize any dgl L as the simplicial set

$$\langle L \rangle = \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^\bullet}, L).$$

On the other hand, The $\mathfrak{L}$ construction can be extended to any simplicial set X by defining its dgl model as

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X} \mathfrak{L}_{\Delta^{|\sigma|}}.$$

It turns out that the model and realization functors

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}$$

are adjoint and they extend the original Quillen functors in different directions, all of them carefully covered by section §3. Here, we mention two:

On the one hand, $\langle L\rangle \simeq \langle L\rangle_Q$ for any reduced finite type dgl L. This shows that the Quillen realization functor is representable by the cosimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$ which becomes the Eckmann-Hilton dual of $\mathscr{A}_\bullet$. Moreover, under no restriction, our realization coincide, up to homotoy type, with any other known realization functor for dgl's including the Deligne-Getzler-Hinich simplicial functor [1, 13, 14].

On the other hand, unlike the Quillen λ functor, our model functor reflects geometrical properties of non nilpotent spaces. Indeed, the non trivial component $\langle \mathfrak{L}^a_X\rangle$ of the realization of the model of a connected finite simplicial set X has the homotopy type of the Bousfield-Kan $\mathbb{Q}$-completion of X [2]. In particular, $H_0(\mathfrak{L}^a_X)$, with the group structure given by the Baker-Campbell-Hausdorff product, recovers the Malcev completion of the fundamental group $\pi_1(X)$.

After that, we embed the model and realization functors in a suitable homotopy theoretical framework. Indeed, we endow the category of dgl's with a model category structure for which a dgl morphism $f\colon A\to B$ is a fibration if it is surjective in non negative degrees; f is a weak equivalence if $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \stackrel{\cong}{\to} \widetilde{\mathrm{MC}}(B)$ is a bijection and $f^a\colon A^a \stackrel{\simeq}{\to} B^{f(a)}$ is a quasi-isomorphism for every $a\in \widetilde{\mathrm{MC}}(A)$; finally f is a cofibration if it has the left lifting property with respect to trivial fibrations. As an immediate consequence we deduce that the model and realization functor form a Quillen pair. In particular, they induce adjoint functors in the homotopy categories,

$$\mathrm{Ho}\,\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{dgl},$$

and both preserve weak equivalences and homotopies.

This survey is extracted from [19] and it contains the main results of a project which begun some years ago in collaboration with U. Buijs, Y. Félix and D. Tanré to all of whom I am deeply grateful. All of these results can be found in [4, 5, 6, 7, 8].

1 Differential graded Lie algebras

Throughout this paper we assume that $\mathbb{Q}$ is the base field. Direct and inverse limits are denoted by $\varinjlim$ and $\varinjlim$ respectively.

A *graded Lie algebra* consists of a $\mathbb{Z}$-graded vector space $L = \oplus_{p\in\mathbb{Z}} L_p$ together with a bilinear product

$$[\,,\,]\colon L_p \otimes L_q \longrightarrow L_{p+q}, \quad p, q \in \mathbb{Z},$$

called the *Lie bracket*, satisfying *antisimmetry*,

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

and *Jacobi identity*,

$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0.$$

Here, $|x|$ denotes the degree of x. The commutator operator $[a, b] = a\otimes b - (-1)^{|a||b|} b\otimes a$ is a Lie bracket on $T(V)$, the tensor algebra on the graded vector space V. The *free Lie algebra* $\mathbb{L}(V)$ generated by V is the sub Lie algebra of $T(V)$ generated by V.

A *differential graded Lie algebra* is a graded Lie algebra L endowed with a *differential*, that is, a linear derivation ∂ of degree -1 such that $\partial^2 = 0$. By abusing of notation we say that a differential graded Lie algebra is *free* if it is so as graded Lie algebra.

Given a differential graded Lie algebra L, a *Maurer-Cartan element* is an element $a \in L_{-1}$ satisfying the *Maurer-Cartan equation*

$$\partial a + \frac{1}{2}[a, a] = 0.$$

We denote by $\mathrm{MC}(L)$ the set of Maurer-Cartan elements which is clearly preserved by morphisms. Given $a \in \mathrm{MC}(L)$ the derivation $\partial_a = \partial + \mathrm{ad}_a$ is again a differential on L. Here ad_a denotes the usual adjoint operator, $\mathrm{ad}_a b = [a, b]$. The *component* of L at $a \in \mathrm{MC}(L)$ is the truncation of the perturbed (L, ∂_a) at non negative degrees,

$$L^a = (L, \partial_a)/(L_{<0} \oplus J) \cong L_{>0} \oplus (L_0 \cap \ker \partial_a)\,,$$

in which J is a complement of $\ker \partial_a$ in L_0.

The *completion* $\widehat{L}$ of a differential graded Lie algebra L is

$$\widehat{L} = \varinjlim_n L/L^n$$

where $L^1 = L$, $L^n = [L, L^{n-1}]$ for $n \geq 2$, and the limit is taken on the topology arising from this filtration. An element $\overline{a}$ of $\widehat{L}$ is then a sequence $\overline{a} = (a_1, a_2, \cdots)$ with $a_i \in L/L^i$ and $a_i = a_{i-1}$ in L/L^{i-1}.We write $\widehat{\mathbb{L}}(V) = \widehat{\mathbb{L}(V)}$. Each element of $\widehat{\mathbb{L}}(V)$ can be seen as a series $\sum_n x_n$ with $x_n \in \mathbb{L}^n(V)$ for all n.

A differential graded Lie algebra L is *complete* if the natural morphism $L \overset{\cong}{\to} \widehat{L}$ is an isomorphism. Observe that, reduced differential graded Lie algebras, which are concentrated in positive degrees, are always complete.

We denote by **dgl** the category of *complete differential graded Lie algebras*, dgl's henceforth.

Given $L = \widehat{\mathbb{L}}(V)$ a free dgl and $v \in V$, we will often write $\partial v = \sum_{n\geq 1} \partial_n v$ where $\partial_n v \in \mathbb{L}^n(V)$. Observe that, if θ is a derivation of L satisfying $\theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)$ and $[\theta, \partial] = 0$, then $e^\theta = \sum_{n\geq 0} \frac{\theta^n}{n!}$ is an automorphism of L and in particular, it induces a bijection on the Maurer-Cartan set.

Recall that given a dgl L, the Baker-Campbell-Hausdorff product $*$ equips the vector space L_0 with a group structure. Since $a * (-a) = 0$ we often use the notation $-a = a^{-1}$.

The *gauge action*, see for instance [18, §4], of $(L_0, *)$ on $\mathrm{MC}(L)$ is defined by

$$x \mathcal{G} a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(\partial x) = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(a)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(\partial x)}{(i+1)!}.$$

Here and from now on, 1 inside an operator will denote the identity. We denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$ the orbit set, that is, the set of equivalence classes of Maurer-Cartan elements modulo the gauge action.

Geometrically [15, 17], interpreting Maurer-Cartan elements as points in a space, one thinks of x as a flow taking $x \,\mathcal{G}\, a$ to a in unit time. For the more topological oriented reader [10], the points a and $x \,\mathcal{G}\, a$ are in the same path component.

The *Deligne groupoid* of L has $\mathrm{MC}(L)$ as objects, and elements $x \in L_0$ as arrows from $x \,\mathcal{G}\, z$ to z.

A fundamental object, which illustrates all of the above concepts and facts, turns out to be the starting point of our work:

Definition 1.1. [17] *The Lawrence-Sullivan model for the interval*, LS-interval henceforth, is the dgl

$$\mathfrak{L}_{\Delta^1} = (\widehat{\mathbb{L}}(a, b, x), \partial),$$

in which a and b are Maurer-Cartan elements, x is of degree 0 and

$$\partial x = \mathrm{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \mathrm{ad}_x^n (b - a) = \mathrm{ad}_x b + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}(b - a),$$

where the B_n's are the Bernoulli numbers.

Let $(\widehat{\mathbb{L}}(a_0, a_1, a_2, x_1, x_2), \partial)$ be two glued LS-models of the interval. That is, a_0, a_1 and a_2 are Maurer-Cartan elements, $\partial x_1 = \mathrm{ad}_{x_1}(a_1) + \frac{\mathrm{ad}_{x_1}}{e^{\mathrm{ad}_{x_1}} - 1}(a_1 - a_0)$ and $\partial x_2 = \mathrm{ad}_{x_2}(a_2) + \frac{\mathrm{ad}_{x_2}}{e^{\mathrm{ad}_{x_2}} - 1}(a_2 - a_1)$. Then, the “subdivision of the interval” is given by:

In [5, Thm. 2.3] the reader may find a complete description of the Deligne groupoid of the LS-interval as two disjoint rational lines.

2 The cosimplicial dgl $\mathfrak{L}_{\Delta^\bullet}$

Given $n \geq 0$, let Δ^n be the standard n-simplex ,

$$\Delta^n_p = \{(i_0, \ldots, i_p) \mid 0 \leq i_0 < \cdots < i_p \leq n\}, \quad \text{if} \quad p \leq n,$$

and denote by $s^{-1}\Delta^n$ the graded vector space of desuspended rational simplicial chains on Δ^n with the usual boundary operator,

$$da_{i_0\dots i_p}=\sum_{j=0}^{p}(-1)^j a_{i_0\dots\widehat{i_j}\dots i_p}.$$

Here, $a_{i_0\dots i_p}$ denotes the generator of degree $p-1$ represented by the p-simplex $(i_0,\dots,i_p)\in\Delta^n_p$. Consider $(\widehat{\mathbb{L}}(s^{-1}\Delta^n),d)$ the free dgl generated by $s^{-1}\Delta^n$ with the differential induced by d.

For each $0\le i\le n$ consider the i-th *coface* map $\delta_j\colon\{0,\dots,n-1\}\to\{0,\dots,n\}$ and the i-th *codegeneracy* map $\sigma_i\colon\{0,\dots,n+1\}\to\{0,\dots,n\}$ defined by:

$$\delta_i(j)=\begin{cases} j, & \text{if } j<i,\\ j+1, & \text{if } j\ge i,\end{cases}\qquad \sigma_i(j)=\begin{cases} j & \text{if } j\le i,\\ j-1 & \text{if } j>i,\end{cases}$$

and use the same notation for the induced dgl morphisms,

$$\delta_i\colon(\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}),d)\longrightarrow(\widehat{\mathbb{L}}(s^{-1}\Delta^{n}),d),\quad \sigma_i\colon(\widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}),d)\longrightarrow(\widehat{\mathbb{L}}(s^{-1}\Delta^{n}),d),$$

defined by

$$\delta_i(a_{j_0\dots j_p})=a_{\ell_0\dots\ell_p}\quad\text{with}\quad \ell_k=\begin{cases} j_k, & \text{if } j_k<i,\\ j_k+1, & \text{if } j_k\ge i.\end{cases}$$

$$\sigma_i(a_{\ell_0\dots\ell_q})=\begin{cases} a_{\sigma_i(\ell_0)\dots\sigma_i(\ell_q)} & \text{if } \sigma_i(\ell_0)<\dots<\sigma_i(\ell_q),\\ 0 & \text{otherwise,}\end{cases}$$

The following is the core result on which our dgl realization is based.

Theorem 2.1. [4, thms. 2.3 and 2.8] *There is a cosimplicial dgl, unique up to dgl isomorphism,*

$$\mathfrak{L}_{\Delta^\bullet}=\{\mathfrak{L}_{\Delta^n}\}_{n\ge0}=\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n),\partial)\}_{n\ge0},$$

such that,

(1) *For each $n\ge0$ and each $i=0,\dots,n$, the generator $a_i\in s^{-1}\Delta^n_0$ is a Maurer-Cartan element, $\partial a_i=-\frac{1}{2}[a_i,a_i]$.*

(2) *The linear part ∂_1 of ∂ is precisely the desuspension $s^{-1}d$ of d.*

(3) *The cofaces and codegeneracies are the morphisms δ_i's and σ_i's defined above.*

3 The model and realization functors

Given a simplicial set X, identify as usual any simplex $\sigma\in X_n$ with a simplicial map $\sigma\colon\underline{\Delta}^n\to X$. Here, $\underline{\Delta}^n$ denote the simplicial set whose p-simplices are integer sequences $0\le i_0\le\dots\le i_p\le n$. Then, X can be recovered from its simplices as the colimit

$$X=\varinjlim_{\sigma\in X}\underline{\Delta}^{|\sigma|}.$$

Definition 3.1. The *model* of any simplicial set X is defined as the dgl

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X} \mathfrak{L}_{\Delta^{|\sigma|}}.$$

In fact, Theorem 2.1 is a special case of the following: It can be proven that the model of X is the free complete Lie algebra

$$\mathfrak{L}_X = (\widehat{\mathbb{L}}(s^{-1}X), \partial)$$

where, abusing of notation, $s^{-1}X$ denotes the desuspension of the normalized chains on X. Recall that these are the simplicial chains on X modulo degeneracies. In other words, $s^{-1}X$ is generated by the non degenerate simplices of X. The differential ∂ is completely determined by the following:

(1) The non degenerate 0-simplices are Maurer-Cartan elements.

(2) The linear part ∂_1 of ∂ is precisely the desuspension of the differential in the normalized chains on X.

(3) If $j\colon Y \subset X$ is a subsimplicial set, then $\mathfrak{L}(j) = \widehat{\mathbb{L}}(s^{-1}j)$.

On the other hand the cosimplicial structure on $\mathfrak{L}_{\Delta^\bullet}$ gives rise to the following.

Definition 3.2. The *realization* of a dgl L is defined as the simplicial set

$$\langle L\rangle = \operatorname{Hom}_{\mathbf{dgl}}(\mathfrak{L}_{\Delta^\bullet}, L).$$

Theorem 3.3. *the model and realization functors are adjoint,*

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{dgl}\,.$$

The first results describing the homotopy type of the realization of a given dgl are the following.

Theorem 3.4. [4, Thm. 4.6] *For any dgl,* $\langle L\rangle \simeq \dot\cup_{z\in\widetilde{\mathrm{MC}}(L)}\langle L^z\rangle$. *In particular,* $\pi_0\langle L\rangle \cong \widetilde{\mathrm{MC}}(L)$.

Theorem 3.5. [4, Prop. 4.5] *Let L be a non negatively graded dgl and $z \in \mathrm{MC}(L)$. Then, $\langle L^z\rangle$ is a connected simplicial set and there are natural group isomorphisms*

$$\pi_n\langle L^z\rangle \cong H_{n-1}(L^z), \qquad n \geq 1,$$

in which $H_0(L,d)$ is considered with the group structure given by the Baker-Campbell-Hausdorff product.

Finally, concerning the realization functor, we state that, under the usual bounding and finite type assumptions, it extends the original Quillen realization functor $\langle - \rangle_Q$ [21], and the realization $\langle \mathscr{C}^*(-)\rangle_S$ of the cdga given by the Chevalley-Eilenberg cochain functor $\mathscr{C}^*$ on L [3]. This is the composite of the functors,

$$\mathscr{C}^* = (-)^\sharp \circ \mathscr{C}\colon\ \mathbf{dgl}_f \to \mathbf{cdga} \quad \text{and} \quad \langle - \rangle_S\colon\ \mathbf{cdga} \to \mathbf{sset},$$

where $\mathbf{dgl}_f$ is the full subcategory of $\mathbf{dgl}$ of finite type dgl's. The second one is the Sullivan realization functor defined by $\langle A\rangle_S = \operatorname{Hom}_{\mathbf{cdga}}(A, \mathscr{A}_\bullet)$.

Theorem 3.6. [4, Thm. 8.1] *Let L be a finite type dgl with $H_q(L) = 0$ for $q < 0$. Then,*

$$\langle L \rangle \simeq \langle \mathscr{C}^*(L) \rangle_S. \text{ If in addition } L \text{ is reduced, } \langle L \rangle \simeq \langle L \rangle_Q.$$

This exhibits the Quillen realization as a functor representable by $\mathfrak{L}_{\Delta^\bullet}$

We also show in [7, Thm.4.8] that, with full generality, our realization is homotopy equivalent to the Deligne-Getzler-Hinich simplicial functor on L [13, 14].

We now analyze the main properties of the model functor:

Theorem 3.7. [6] *For any finite simplicial set X,*

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) \cong \pi_0(X^+).$$

Here, X^+ denotes the disjoint union of X with a point. This, together with Theorem 3.4, gives,

$$\pi_0 \langle \mathfrak{L}_X \rangle = \pi_0(X^+).$$

Moreover, we are able to determine the homotopy type of each of these components.

Theorem 3.8. [8, Thm. 2.7] *Given X a finite simplicial set, $\langle \mathfrak{L}_X^0 \rangle$ is contractible.*

For the non trivial components we have:

Theorem 3.9. [8, Thm. 2.9] *Let X be a connected finite simplicial set and let $z \in \mathfrak{L}_X$ be a non trivial Maurer-Cartan element. Then, $\langle \mathfrak{L}_X^z \rangle \simeq \mathbb{Q}_\infty X$, the $\mathbb{Q}$-completion of X [2].*

In particular, by [12, Cor. 7.4], and taking into account Theorem 3.5 for $n = 1$, we deduce:

Corollary 3.10. *$H_0(\mathfrak{L}_X^z)$ is the Malcev Lie completion of the fundamental group $\pi_1(X)$.*

4 A model category structure on dgl

Henceforth, by *model category* we mean the original closed model category definition of Quillen [20]. In the category **sset** of simplicial sets we consider the classical model category structure, see for instance [2, Chap. VII], in which fibrations are Kan fibrations, cofibrations are injective simplicial mpas, and weak equivalences are homotopy weak equivalences. Then, we have:

Theorem 4.1. [8, Thm. 3.1] *There is a model category structure on* **dgl** *for which:*

- *A morphism $f\colon A \to B$ is a fibration if it is surjective in non negative degrees.*
- *A morphism $f\colon A \to B$ is a weak equivalence if $\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(B)$ is a bijection and $f^a\colon A^a \xrightarrow{\simeq} B^{f(a)}$ is a quasi-isomorphism for every $a \in \widetilde{\mathrm{MC}}(A)$.*

- *A morphism is a cofibration if it has the left lifting property with respect to trivial fibrations.*

Corollary 4.2. *the realization and model functors, form a Quillen pair. In particular, they preserve weak equivalences and induce adjoint functors in the homotopy categories,*

$$\mathrm{Ho}\,\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{dgl}\,.$$

We end this section with the following important observation which compare our model structure in **dgl** with other known model structures:

Remark 4.3. (1) One may consider in the category **dgl** the classical model structure given on categories of unbounded chain complexes enriched with some algebraic structure, see for instance [14, §2]. Fibrations are surjective morphisms, weak equivalences are quasi-isomorphisms and cofibrations are morphisms satisfying the left lifting property with respect to trivial fibrations.

Then, the zero map $0 \to \mathbb{L}(a)$, in which a is a Maurer-Cartan element, is not surjective but it is a fibration in our model structure. The same example is a quasi-isomorphism but it is not a weak equivalence in our structure. Contrarily, consider the abelian dgl L generated by a single cycle of negative degree. Then, the zero map $0 \to L$ is a weak equivalence in our structure but it is not a quasi-isomorphism.

(2) On the other hand, in [16, Thm. 9.16], A. Lazarev and M. Markl define a model category structure on the full subcategory of **dgl** formed by the *profinite complete dgl's* where:

f is a *fibration* if it is a surjection.

f is a *weak equivalence* if $\mathscr{C}^*(f)$ is a quasi-isomorphism.

f is a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

Here $\mathscr{C}^*$ is a generalization of the usual cochain functor [16, §7]. In [8, Thm. 6.12] we show that if f is a *weak equivalence* in this structure, it is so in our model structure. However, this inclusion is strict: let L be the abelian Lie algebra generated by a single cycle of degree -1. As observed in (1) the zero map $f\colon 0 \to L$ is a weak equivalence in our model structure but $\mathscr{C}^*(f)$ is not a quasi-isomorphism. Also, it is obvious that the class of fibrations in the above structure is also properly contained in our class of fibrations.

A final word

Needless to say what would be the natural continuation of the work presented in this survey: the literature is plenty of deep results describing the non torsion behaviour of the homotopy type of simply connected complexes, all of them using the Quillen approach to rational homotopy theory. Is it possible to extend these results to general complexes by means of the new framework reviewed in this paper?

On the other hand, there are deep results concerning rational invariants of "highly non simply connected" spaces. Illustrative examples include the Mumford conjecture on the rational cohomology ring of the moduli space of Riemann surfaces, and the rational homological stability problem in general, and that of configuration spaces in particular. Would it be possible to use our new machinery to attack related problems?

We finish with another general question which may attract experts in various mathematical subjects to this new approach to rational homotopy theory:

Let R be a local commutative algebra with maximal ideal $\mathfrak{M}$ and let $\Bbbk = R/\mathfrak{M}$. Let A be an $\Bbbk$-vector space endowed with some additional structure. An R-deformation of A is another such structure in $A \otimes_{\Bbbk} R$ such that, modulo $\mathfrak{M}$, it reduces to the original one in A. The *Deligne principle* asserts that, whenever $\Bbbk$ is of characteristic zero, every deformation functor is governed by a dgl. That is, denoting by $\operatorname{Def}(A;R)$ = the set of equivalence classes of R-deformations of A, there exists a dgl L such that

$$\operatorname{Def}(A;R) \cong \widetilde{\mathrm{MC}}(L).$$

In words of Kontsevich, finding the appropriate L for a given deformation functor is an art. Nevertheless, we may consider its realization $\langle L\rangle$ and think of it as the "homotopy moduli space" of $\operatorname{Def}(A;R)$. Is it then possible to translate homotopy invariants of $\langle L\rangle$ into properties related with deformation phenomena?

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