PRESENTATIONS OF (IMMERSED) SURFACE-KNOTS BY MARKED GRAPH DIAGRAMS

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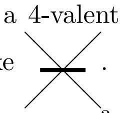
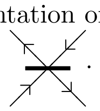
1. Introduction

An *immersed surface-link* is a generically immersed closed oriented surface in the 4-space $\mathbb{R}^4$. When the surface has only one component, it is also called an *immersed surface-knot*. When the surface consists of 2-spheres, it is also called an *immersed sphere-link* or simply an *immersed 2-link*. When the immersion is an embedding, it is also called a *surface-link*. Two (immersed) surface-links $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if there is an orientation-preserving auto-homeomorphism h of $\mathbb{R}^4$ sending $\mathcal{L}$ to $\mathcal{L}'$ orientation-preservingly. An immersed 2-link is studied in [11] in relation to a cross-sectional link. A normal form of an immersed surface-link introduced by S. Kamada and K. Kawamura in [5] is used to define a marked graph diagram of an immersed surface-link. In [6], we extend the method of presenting surface-links by marked graph diagrams to presenting immersed surface-links. We also give some local moves on marked graph diagrams that preserve the ambient isotopy classes of their presenting immersed surface-links, which are extension of moves given by Yoshikawa [19] for presentation of embedded surface-links. In [13], with an example described by a marked graph diagram of an immersed 2-knot, it is shown as the main theorem (Theorem 3.6) that *for any positive integer n, there are infinitely many immersed 2-knots with only n essential double point singularities, that is, infinitely many immersed 2-knots with n double point singularities which are not equivalent to the connected sum of any immersed 2-knot and any unknotted immersed sphere.*

2. Marked graph representation of immersed surface-links

In this section, we review (oriented) marked graph diagrams representing immersed surface-links described in [6]. A *marked graph* is a 4-valent graph in $\mathbb{R}^3$ each of whose vertices is a vertex with a marker looks like . Two marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods of markers. As usual, a marked graph in $\mathbb{R}^3$ can be described by a link diagram on $\mathbb{R}^2$ with some 4-valent vertices equipped with markers, called a *marked graph diagram.* An *orientation* of a marked graph G in $\mathbb{R}^3$ is a choice of an orientation for each edge of G. An orientation of a marked graph G is said to be *consistent* if every vertex in G looks like . A marked graph G in $\mathbb{R}^3$ is said to be *orientable* if G admits a consistent orientation. Otherwise, it is said to be *non-orientable*. By an *oriented marked graph* we mean an orientable marked graph in $\mathbb{R}^3$ with a fixed consistent orientation. Two oriented marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhood, marker and consistent orientation. For $t \in \mathbb{R}$, we denote by $\mathbb{R}^3_t$ the hyperplane of $\mathbb{R}^4$ whose fourth coordinate is equal to $t \in \mathbb{R}$, i.e., $\mathbb{R}^3_t = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$. An immersed surface-link $\mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be described in terms of its *cross-sections* $\mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3_t,\ t \in \mathbb{R}$ (cf. [3]). It is shown [5] that any immersed surface-link $\mathcal{L}$, there is an immersed surface-link $\mathcal{L}' \subset \mathbb{R}^3[-2, 2]$ satisfying the following conditions:

(1) The intersections $\mathcal{L}'_1$ and $\mathcal{L}'_{-1}$ are H-trivial links;
(2) All saddle points of $\mathcal{L}'$ are in $\mathbb{R}^3[0]$;
(3) All maximal points of $\mathcal{L}'$ are in $\mathbb{R}^3[2]$;
(4) All minimal points of $\mathcal{L}'$ are in $\mathbb{R}^3[-2]$;
(5) The intersections $\mathcal{L}' \cap (\mathbb{R}^3[1, 2])$ and $\mathcal{L}' \cap (\mathbb{R}^3[-2, -1])$ are disjoint unions of a disjoint system of trivial knot cones and a disjoint system of Hopf link cones.

We call $\mathcal{L}'$ a *normal form* of $\mathcal{L}$. Let $\mathcal{L}$ be an immersed surface-link in $\mathbb{R}^4$, and $\mathcal{L}'$ a normal form of $\mathcal{L}$. Then $\mathcal{L}'_0$ is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Fig. 1. We choose an orientation for each edge of $\mathcal{L}'_0$ that coincides with the induced orientation on the boundary of $\mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ from the orientation of $\mathcal{L}'$. The resulting oriented marked graph G is called an *oriented marked graph* of $\mathcal{L}$. As usual, G is described by a link diagram D with rigid marked vertices. Such a diagram D is called an *oriented marked graph diagram* or an *oriented ch-diagram* (cf. [17]) of $\mathcal{L}$.

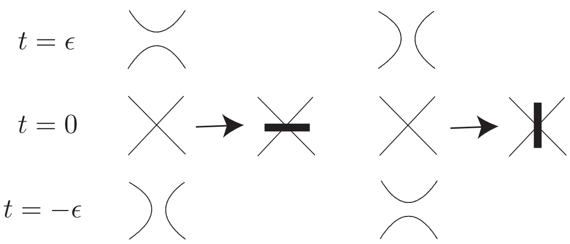


FIGURE 1. Marking of a vertex

Let D be an oriented marked graph diagram. We obtain two links $L_-(D)$ and $L_+(D)$ from D by replacing each marked vertex in D as shown in Fig. 2. Links $L_-(D)$ and $L_+(D)$ are also called the *negative resolution* and the *positive resolution* of D, respectively. By replacing a neighborhood of each marked vertex v_i $(1 \leq i \leq n)$ with an oriented band B_i as illustrated in Fig. 2. Denote the disjoint union $B_1 \sqcup \cdots \sqcup B_n$ of bands by $\mathcal{B}(D)$. A link L is H-*trivial* if L is a split union of trivial knots and Hopf links. A marked graph diagram D is said to be H-*admissible* if both resolutions $L_-(D)$ and $L_+(D)$ are H-trivial classical link diagrams as shown in Fig. 3.

From now on, we recall how to construct an immersed surface-link $\mathcal{L}$ in $\mathbb{R}^4$ from a given H-admissible oriented marked graph diagram (cf. [5, 6]). Let D be an H-admissible oriented marked graph diagram. We define a surface-link $\mathcal{F}(D) \subset \mathbb{R}^3 \times [-1, 1]$, called the *proper surface associated with D*, by

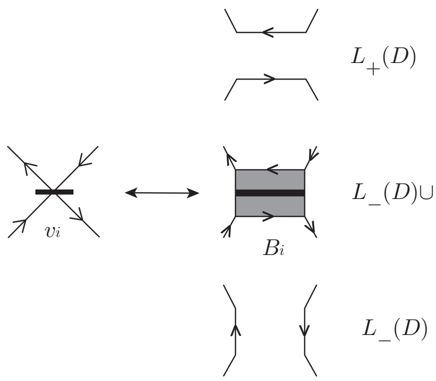


FIGURE 2. Marked vertex resolutions

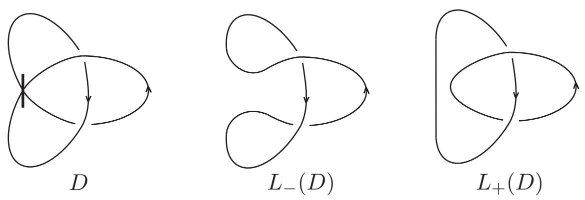


FIGURE 3. An H-admissible marked graph diagram

$$(\mathbb{R}^3_t, \mathcal{F}(D) \cap \mathbb{R}^3_t) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ (\mathbb{R}^3, L_-(D) \cup \mathcal{B}(D)) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0. \end{cases}$$

It is known that a marked graph diagram D is orientable if and only if the proper surface $\mathcal{F}(D)$ associated with D is an orientable surface. Since D has a consistent orientation, the resolutions $L_+(D)$ and $L_-(D)$ have the orientations induced from the orientation of D. We choose an orientation for the proper surface $\mathcal{F}(D)$ so that the induced orientation of the cross-section $L_+(D) = \mathcal{F}(D)_1 = \mathcal{F}(D) \cap \mathbb{R}^3_1$ at $t = 1$ matches the orientation of $L_+(D)$. Let $[a, b]$ be a closed interval with $a < b$. For a link L, let $\hat{L} * [a, b]$ (or $\check{L} * [a, b]$) be a cone with $L[a]$ (or $L[b]$) as the base and a point in $\mathbb{R}^3[b]$ (or $\mathbb{R}^3[a]$), respectively. Let $H = (O_1 \cup \cdots \cup O_m) \cup (P_1 \cup \cdots \cup P_n)$ be an H-trivial link in $\mathbb{R}^3$, where O_i is a trivial knot and P_j is a Hopf link for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

- Let $H_\wedge[a, b]$ be a disjoint union of a disjoint system of trivial knot cones $\hat{O}_i * [a, b] (i = 1, \ldots, m)$ and a disjoint system of Hopf link cones $\hat{P}_j * [a, b] (j = 1, \ldots, n)$ in $\mathbb{R}^3[a, b]$.
- Let $H_\vee[a, b]$ be a disjoint union of a disjoint system of trivial knot cones $\check{O}_i * [a, b] (i = 1, \ldots, m)$ and a disjoint system of Hopf link cones $\check{P}_j * [a, b] (j = 1, \ldots, n)$ in $\mathbb{R}^3[a, b]$.

By capping off $\mathcal{F}(D)$ with $L_+(D)_\wedge[1, 2]$ and $L_-(D)_\vee[-2, -1]$, we obtain an oriented immersed surface-link $\mathcal{S}(D)$ in $\mathbb{R}^4$. We call the oriented immersed surface-link $\mathcal{S}(D)$ the *oriented immersed surface-link associated with* D. It is straightforward from the construction of $\mathcal{S}(D)$ that D is an oriented marked graph diagram of the oriented immersed surface-link $\mathcal{S}(D)$.

Definition 2.1. An immersed surface-link $\mathcal{L}$ is *presented* by an H-admissible marked graph diagram D if $\mathcal{L}$ is ambient isotopic to $\mathcal{S}(D)$ constructed from D.

Theorem 2.2. Let $\mathcal{L}$ be an immersed surface-link. Then there is an H-admissible marked graph diagram D such that $\mathcal{L}$ is presented by D.

We discuss moves on marked graph diagrams which preserve the ambient isotopy classes of the immersed surface-links presented by the diagrams.

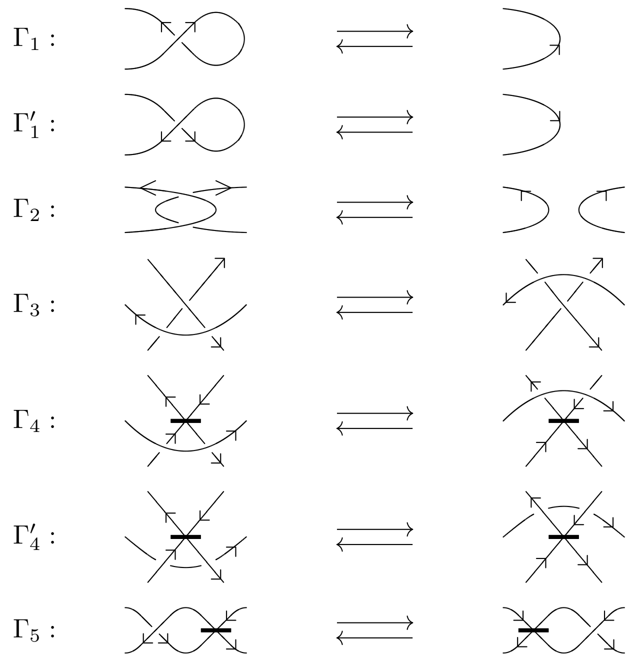


FIGURE 4. Moves of Type I

The moves depicted in Fig. 4 on marked graph diagrams are called moves of type I, which do not change the equivalence classes of marked graphs in $\mathbb{R}^3$.

The moves depicted in Fig. 5 on marked graph diagrams are called moves of type II, which change the equivalence classes of marked graphs in $\mathbb{R}^3$. When a marked graph diagram D is H-admissible (or admissible) then the result obtained from D by any move of type II is also H-admissible (or admissible) and the immersed surface-link (or surface-link) presented by the diagrams are ambient isotopic, respectively.

It is known that two admissible marked graph diagrams present ambinet isitopic surface-links if and only if they are related by the moves of type I and II (cf. [14, 18, 19]). These moves are called *Yoshikawa moves.*

Let D be a link diagram of an H-trivial link L. A crossing point p of D is an *unlinking crossing point* if it is a crossing between two components of the same Hopf link of L and if the crossing change at p makes the Hopf link into a trivial link.

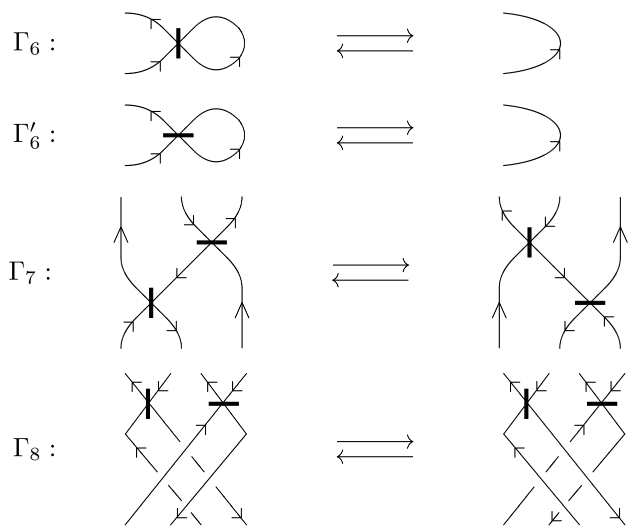


FIGURE 5. Moves of Type II

Definition 2.3. Let D be an H-admissible marked graph diagram and let D_- and D_+ be the diagrams of the lower resolution $L_-(D)$ and the upper resolution $L_+(D)$, respectively. A crossing point p of D is an *lower singular point* (or an *upper singular point*, respectively) if p is an unlinking crossing point of D_- (or D_+).

We introduce new moves for H-admissible marked graph diagrams. They are the moves Γ_9, Γ_9' and Γ_9 in Fig. 6, which we call moves of type III. For the moves (a) of Γ_9 and Γ_9' in Fig. 6 we require a condition that the components l^+ (in the resolution $L_+(D)$) and l^- (in the resolution $L_-(D)$) are trivial, respectively. For the moves (b) of Γ_9 and Γ_9', we require a condition that p is an upper singular point and a lower singular point, respectively.

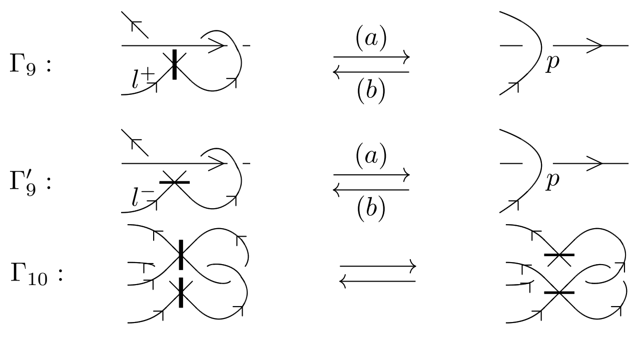


FIGURE 6. Moves of Type III

The *generalized Yoshikawa moves* for marked graph diagrams are the moves $\Gamma_1, \dots, \Gamma_5$ (Type I), $\Gamma_6, \dots, \Gamma_8$ (Type II), and $\Gamma_9, \Gamma'_9, \Gamma_{10}$ (Type III) as shown in Fig. 4, Fig. 5, and Fig. 6, respectively.

Definition 2.4. Let D and D' be marked graph diagrams. Marked graph diagrams D and D' are *stably equivalent* if they are related by a finite sequence of generalized Yoshikawa moves.

Theorem 2.5. ([6]) Let $\mathcal{L}$ and $\mathcal{L}'$ be immersed surface-links, and D and D' their marked graph diagrams, respectively. If D and D' are stably equivalent, then $\mathcal{L}$ and $\mathcal{L}'$ are ambient isotopic.

Definition 2.6 (cf. [5])**.** A *positive* (or *negative*) *standard singular 2-knot*, denoted by $S(+)$ (or $S(-)$) is the immersed 2-knot of the marked graph diagram D (or D') in Fig. 7, respectively. An *unknotted immersed sphere* is defined to be the connected sum $mS(+)\#nS(-)$ for any non-negative integers m, n with $m+n>0$.

A double point singularity p of an immersed 2-knot S is *inessential* if S is the connected sum of an immersed 2-knot and an unknotted immersed sphere such that p belongs to the unknotted immersed sphere. Otherwise, p is *essential*.

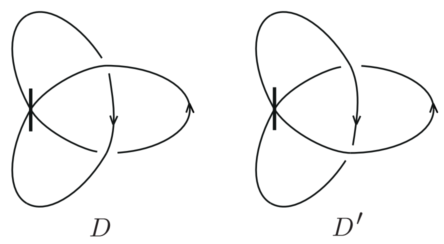


FIGURE 7. Standard singular 2-knot

3. Confirming immersed 2-knots with essential singularity

In this section, the main theorem will be shown with an example of infinitely many immersed 2-knots with essential singularity. For an immersed 2-knot K, let $E(K) = \mathrm{Cl}(S^4 \setminus \mathrm{N}(K))$. Let $\tilde{E}(K)$ be the infinite cyclic covering of $E(K)$. Then the homology $H(K) = H_1(\tilde{E}(K))$ is a finitely generated Λ-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. This module is called the *first Alexander module* of K (cf. [15]). Let

$$DH(K) = \{x \in H(K) |\ \exists \{\lambda_i\}_{1 \le i \le m} : \text{coprime } (m \ge 2) \text{ with } \lambda_i x = 0,\ \forall i\},$$

called the *annihilator* Λ*-submodule*, which is known to be equal to the integral torsion part of the Alexander module $H(K)$ (cf. [9, Section 3]). Let $\epsilon(K)$ be the first elementary ideal of $DH(K)$. A Λ-ideal is *symmetric* if the ideal is unchanged by replacing t by t^{-1}. Let $DH(K)^* = \mathrm{hom}(DH(K), \mathbb{Q}/\mathbb{Z})$ have the induced Λ-module structure, called the *dual* Λ-module of $DH(K)$. The following lemma is used in our argument.

Lemma 3.1. If K is a 2-knot such that the dual Λ-module $DH(K)^*$ is Λ-isomorphic to $DH(K)$, then the first elementary ideal $\epsilon(K)$ is symmetric.

For any marked graph diagram D of K, the fundamental group $\pi(K)$ of K is generated by the connected components of D, namely, the connected components obtained from D by cutting the under-crossing points and the relations $s_3 = s_2^{-1}s_1s_2$ for all crossings as in (a) or (b) in Fig. 8.

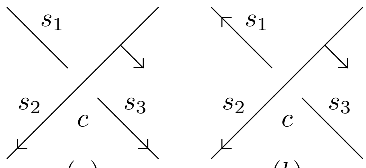


FIGURE 8. Labels at a crossing or a vertex

A computation of the Alexander module $H(K)$ and the ideal $\epsilon(K)$ is shown in a concrete example as follows:

Example 3.2. Let K be the immersed 2-knot of D in Fig. 9. The immersed 2-knot K has only one double point. The fundamental group $\pi(K)$ is computed as follows: $\pi(K) =< x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15} | \, x_1 = x_2^{-1}x_3x_2,\ x_2 = x_3^{-1}x_5x_3,\ x_1 = x_3^{-1}x_4x_3,\ x_2 = x_1^{-1}x_3x_1, x_6 = x_2^{-1}x_1x_2,\ x_6 = x_1^{-1}x_7x_1,\ x_1 = x_7^{-1}x_8x_7,\ x_2 = x_7^{-1}x_9x_7, x_{10} = x_2^{-1}x_7x_2,\ x_{10} = x_1^{-1}x_{11}x_1,\ x_1 = x_{11}^{-1}x_{12}x_{11},\ x_2 = x_{11}^{-1}x_{13}x_{11}, x_{14} = x_2^{-1}x_{11}x_2,\ x_{14} = x_1^{-1}x_2x_1, x_1 = x_2^{-1}x_{15}x_2 >.$

This group $\pi(K)$ is isomorphic to the group $< x_1, x_2 | \, r_1, r_2 >$, where

$$r_1 : x_2x_1x_2^{-1} = x_1x_2x_1^{-1}, \quad r_2 : (x_1x_2^{-1})^3x_1(x_1x_2^{-1})^{-3} = x_2.$$

Then the following Λ-semi-exact sequence

$$\Lambda[r_1^*, r_2^*] \xrightarrow{d_2} \Lambda[x_1^*, x_2^*] \xrightarrow{d_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of the group presentation of $\pi(K)$ is obtained by using the fundamental formula of the Fox differential calculus in [1], where $\Lambda[r_1^*, r_2^*]$ and $\Lambda[x_1^*, x_2^*]$ are free Λ-modules with bases r_i^* $(i = 1, 2)$ and x_j^* $(j = 1, 2)$, respectively, and the Λ-homomorphisms ε, d_1 and d_2 are given as follows:

$$\varepsilon(t) = 1,\ d_1(x_j^*) = t - 1\,(j = 1, 2),\ d_2(r_i^*) = \sum_{j=1}^{u} \frac{\partial r_i}{\partial x_j} x_j^*\,(i = 1, 2)$$

for the Fox differential calculus $\frac{\partial r_i}{\partial x_j}$ regarded as an element of Λ by letting x_j to t. The Alexander module $H(K)$ is identified with the quotient Λ-module $\mathrm{Ker}(d_1)/\mathrm{Im}(d_2)$ (see [10, Theorem 7.1.5]). The Alexander matrix $M_K = (m_{ij})$ defined by $m_{ij} = \frac{\partial r_i}{\partial x_j}$ is a presentation matrix of the Λ-homomorphism d_2 and calculated as follows:

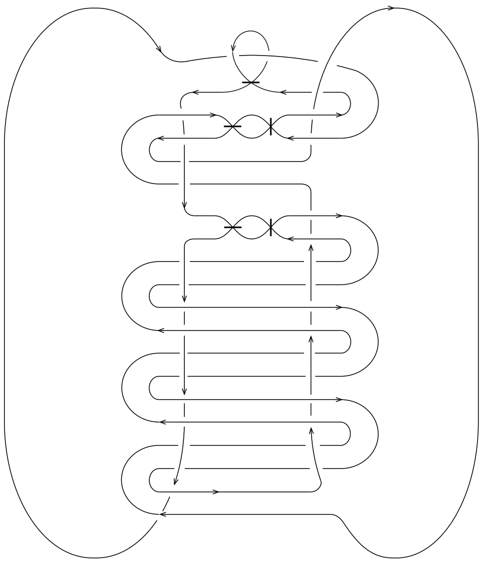
$$M_K = \begin{bmatrix} 2t-1 & 1-2t \\ 4-3t & 3t-4 \end{bmatrix}.$$

Hence we have

$$H(K) \cong \Lambda/(2t - 1, 4 - 3t),$$

which is equal to $DH(K)$. Thus, the first elementary ideal $\epsilon(K)$ of K is

$$\begin{aligned}\epsilon(K) &=< 2t-1, 4-3t > \\ &=< 2t-1, 4-3t, 3(2t-1)+2(4-3t) > \\ &=< 2t-1, 5 > .\end{aligned}$$



D

FIGURE 9. An H-admissible marked graph diagram D

The following lemma is useful in a computation for a symmetric ideal.

Lemma 3.3. ([13]) The following statements are equivalent:

(1) The ideal $< 2t-1, m >$ is symmetric.
(2) An integer m is $\pm 2^r$ or $\pm 2^r 3$ for any integer $r \geq 0$.

Lemma 3.4. ([13]) There are infinitely many immersed 2-knots with one essential double point singularity.

Let J be one of the immersed 2-knots $K_n, K'_n (n = 1, 2, 3, \dots)$ such that the first elementary ideal $\epsilon(J)$ is asymmetric. Then the following corollary is obtained.

Corollary 3.5. The connected sum $J\#U$ of J and any immersed 2-knot U such that the group orders $|DH(J)|$ and $|DH(U)|$ are coprime is an immersed 2-knot with at least one essential double point singularity.

Finally, the ideal $(2t-1, 5)$ is known to be the first elementary ideal of a ribbon torus-knot in [4].

By using an immersed 2-knot in Lemma 3.4, the following main theorem is proved.

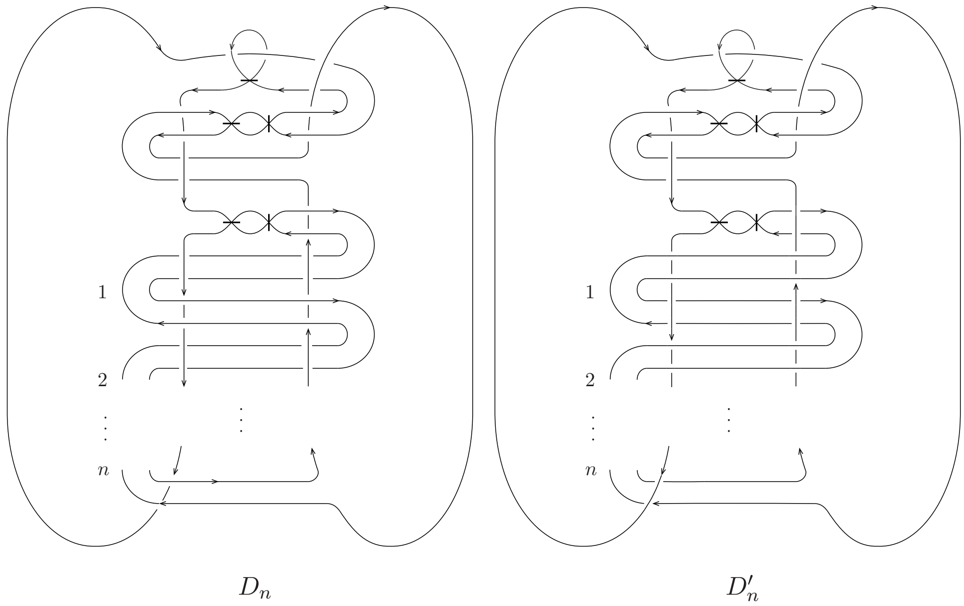


FIGURE 10. H-admissible marked graph diagrams D_n and D'_n

Theorem 3.6. ([13]) Let $K = nK^*_m$ be the connected sum of n copies of an immersed 2-knot K^*_m with one essential double point singularity whose first elementary ideal is $< 2t-1, m >$ for any integer $m \geq 5$ without factors 2 and 3. Then K gives infinitely many immersed 2-knots with n double point singularities every of which is essential.

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