Some spectral properties of perturbed Maass operator

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1 Eigenvalue asymptotics for perturbed Maass operator

In the first part, I talk about the eigenvalue asymptotics for the Maass operator perturbed by a decaying scalar potential. (For omitted proofs in this part, see [Shi].)

Let $\mathbb{H}$ be the hyperbolic plane equipped with the metric $y^{-2}(dx \otimes dx + dy \otimes dy)$. The Riemannian volume is given by $y^{-2} dx \wedge dy$. The Riemannian measure on $\mathbb{H}$ is given by $dxdy/y^2$ and the hyperbolic distance $d_{\mathbb{H}}(z, z_0)$ on $\mathbb{H}$ is given by $\cosh(d(z, z_0)) = (|x - x_0|^2 + y^2 + y_0^2)/(2yy_0)$ for any $z = (x, y), z_0 = (x_0, y_0) \in \mathbb{H}$.

Let B be a real constant. We introduce the (unperturbed) Maass operator as

$$H(B) = y^2 \left((D_x - B/y)^2 + D_y^2\right)$$

on $L^2(\mathbb{H})$. Here, we write D_x for $-\sqrt{-1}\partial/\partial x$, etc.

The operator $H(B)$ has a physical interpretation as the Hamiltonian which governs a non-relativistic, charged particle moving on $\mathbb{H}$ under the influence of the magnetic field of constant strength B perpendicular to $\mathbb{H}$.

The spectral properties of the Maass operator has been investigated by many authors (see [Roe], [Els], [Fay], [Gro], [C-H], [Com], [A-P], [I-M], [K-L] and references therein). We recall some basic results. The Maass operator $H(B)$ is essentially self-adjoint on $C_0^\infty(\mathbb{H})$, the set of all complex-valued, smooth functions with compact support on $\mathbb{H}$ ([Roe], Satz 3.2). (In what follows we use the same notation for an operator and its operator closure if there is no fear of confusion.) The spectrum of $H(B)$ consists of the absolutely continuous part $[B^2 + 1/4, \infty)$ and the discrete Landau levels $\{E_n\}_{n=0}^{N(|B|-1/2)}$, where $E_n = (2n+1)|B| - n(n+1)$ and $N(x)$ denotes the largest integer less than x. In case $|B| \le 1/2$, the set of discrete Landau levels is empty. If $|B| > 1/2$, each of E_n's is an eigenvalue of infinite multiplicity. In the rest of this part, we may restrict ourselves to the case $B > 1/2$, provided we are concerned with the discrete Landau levels, since the Maass operator $H(B)$ with B is unitarily equivalent to the one with $-B$ via the transform $(x, y) \mapsto (-x, y)$.

For a measurable function V on $\mathbb{H}$, we say V *decays at infinity* if for any $\varepsilon > 0$ there exists a compact subset K of $\mathbb{H}$ such that $|V(x, y)| < \varepsilon$ outside K. Any bounded, measurable function V decaying at infinity is relatively compact with respect to $H(B)$ (see [I-S], Lemma 3.10), so the perturbed Maass operator $H(B, V) = H(B) + V$ is a well-defined self-adjoint operator when V is real-valued, and the essential spectrum of $H(B, V)$ coincides with that of $H(B)$. (Note that, examining the proof, one can easily find that Lemma 3.10 in [I-S] is still valid if we drop the continuity condition of V.) Then the perturbed

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operator $H(B,V)$ may have the discrete spectrum (i.e., discrete eigenvalues of finite multiplicity) in the spectral gaps.

To formulate our results, we make the following condition $(V)_\varepsilon$ on the perturbation V.

$(V)_\varepsilon$ The perturbation V is a real-valued, bounded, measurable and non-negative function on $\mathbb{H}$. Moreover, there exist $z_0 \in \mathbb{H}$ and positive constants ε and C_V such that the asymptotic relation

$$\lim_{d(z,z_0)\to\infty} \exp\left(\varepsilon d_{\mathbb{H}}(z,z_0)\right)V(z) = C_V \tag{1.1}$$

holds.

Let n be any non-negative integer n satisfying $0 \le n \le N(B-1/2)$ and let $\varepsilon > 0$. We introduce the notations $\beta_n = 2B - 2n - 1$ (> 0) and

$$\begin{aligned}\Theta_n(\varepsilon) &= \frac{\Gamma(\beta_n+\varepsilon)\Gamma(\beta_n+n+1)}{\Gamma(\beta_n)\Gamma(n+1)\Gamma(\beta_n+1)} F_2(\beta_n+\varepsilon; -n, -n; \beta_n+1, \beta_n+1; 1, 1) \\ &= \frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n)}\, {}_3F_2\left(\begin{matrix} -n, 1-\varepsilon, \varepsilon \\ \beta_n+1, 1 \end{matrix}; 1\right).\end{aligned}$$

Here, $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ is the gamma function, ${}_3F_2$ is the Gauss hypergeometric function and

$$F_2(a; b, b'; c, c'; x, y) = \sum_{l,m=0}^{\infty} \frac{(a)_{l+m}(b)_l(b')_m}{(c)_l(c')_m} \frac{x^l y^m}{l!m!}$$

is the Appell hypergeometric series and $(x)_0 = 1$ and $(x)_m = x(x+1)\cdots(x+m-1)$ if $m \ge 1$. We note that, because of the parameter $-n$, the hypergeometric series in the expressions of $\Theta_n(\varepsilon)$ terminate and it turns out that $\Theta_n(\varepsilon)$ is positive.

For any real numbers a, b and for any self-adjoint operator T in a Hilbert space, we set $N(a < T < b) = \dim \operatorname{ran}(P_T((a,b)))$, where $P_T(I)$ stands for the spectral projection for T on an open interval I.

The main results of this part are the following two theorems.

Theorem 1.1 *Assume that* $|B| > 1/2$. *Let* $\{E_n\}_{n=0}^{N(|B|-1/2)}$ *be as above. Let* E' *be any point between* E_n *and* E_{n+1}, *where we set* $E_{n+1} = B^2 + 1/4$ *for* $n = N(|B| - 1/2)$. *Then the condition* $(V)_\varepsilon$ *implies that*

$$N(E_n + E < H(B,V) < E') = \frac{1}{4\pi}(\Theta_n(\varepsilon))^{1/\varepsilon}\mathrm{Vol}_{\mathbb{H}}\{z \in \mathbb{H} | V(z) > E\}(1 + o(1)) \tag{1.2}$$

as $E \searrow 0$, *where* $\mathrm{Vol}_{\mathbb{H}}$ *is the Riemannian volume on* $\mathbb{H}$.

For any $z_0 \in \mathbb{H}$, we denote by F_{T,t,z_0} the characteristic function on the set $\{z \in \mathbb{H} | t \le d(z_0, z) \le T\}$.

Theorem 1.2 *Assume that* $|B| > 1/2$ *and* V *is bounded, measurable, non-negative on* $\mathbb{H}$ *and decays at infinity. Let* E' *be any point between* E_n *and* E_{n+1}, *where we set* $E_{n+1} = B^2 + 1/4$ *for* $n = N(|B| - 1/2)$. *Let* $z_0 \in \mathbb{H}$ *and* $0 \le t < T$. *Then the following assertions hold:*

(i) If there exists a positive constant c *such that* $0 \le V(z) \le cF_{T,t,z_0}(z)$ *holds for all* $z \in \mathbb{H}$, *then we have*

$$|\log \tanh^2(T/2)| \limsup_{E\searrow 0} N(E_n + E < H(B,V) < E')/|\log E| \le 1.$$

(ii) If there exists a positive constant c such that $cF_{T,t,z_0}(z) \le V(z)$ holds for all $z \in \mathbb{H}$. then we have

$$|\log \tanh^2 (T/2)| \liminf_{E \searrow 0} N(E_n + E < H(B,V) < E')/|\log E| \ge 1.$$

(iii) In particular, if there exist positive constants c, c' such that $cF_{T,t,z_0}(z) \le V(z) \le c' F_{T,t,z_0}(z)$ holds for all $z \in \mathbb{H}$, then we have

$$|\log \tanh^2 (T/2)| \lim_{E \searrow 0} N(E_n + E < H(B,V) < E')/|\log E| = 1.$$

Let $\mathbb{D}$ be the Poincaré disk $\{w = \rho e^{\sqrt{-1}\theta} | 0 \le \rho < 1, \, 0 \le \theta < 2\pi\}$ equipped with the standard measure $4r(1-\rho^2)^{-2} d\rho d\theta$. The Cayley transform φ_1 is defined by $z \to (z - \sqrt{-1})/(z + \sqrt{-1})$ for each $z \in \mathbb{H}$, and it defines an isometric diffeomorphism between $\mathbb{H}$ and $\mathbb{D}$.

We note that, in the case of $z_0 = \sqrt{-1}$, the asymptotic relation (1.1) is equivalent to the condition that $\lim_{|w| \nearrow 1} V(\varphi_1^{-1}(w))(1-|w|^2)^{-\varepsilon} = 4^{-\varepsilon} C_V$ holds uniformly in θ for $w = |w| e^{\sqrt{-1}\theta} \in \mathbb{D}$, which follows from the relation $1 - |w|^2 = \cosh^{-2}(d_{\mathbb{D}}(w,0)/2)$.

Remark 1.3 *Let V satisfy $(V)_\varepsilon$ for some $\varepsilon > 0$ and let F_{T,t,z_0} be the function as in Theorem 1.2. Then a simple calculation shows that*

$$\begin{aligned}
\lim_{E \searrow 0} E^{1/\varepsilon} \mathrm{Vol}_{\mathbb{H}} \{z \in \mathbb{H} | \, V(z) > E\} &= \pi C_V^{1/\varepsilon}, \\
\lim_{E \searrow 0} \mathrm{Vol}_{\mathbb{H}} \{z \in \mathbb{H} | \, F_{T,t,z_0}(z) > E\} &= 4\pi (\cosh^2 T - \cosh^2 t).
\end{aligned}$$

In the Euclidean case, Raikov ([Rai], [Rai2]) has obtained the asymptotic distribution of the number of the discrete spectrum near the boundary of the essential spectrum of the Schrödinger operators with constant magnetic fields and power-like decreasing electric potentials. In the two dimensional case, the leading asymptotics are independent of the level-number n, and behaves quasi-classically, i.e., behaves like $(B/2\pi)\mathrm{Vol}_{\mathbf{R}^2} \{x \in \mathbf{R}^2 | V(x) > E\}$ as $E \searrow 0$ (see, e.g., [R-W], Remark 2.5). Here, B is the strength of the constant magnetic field and $B/(2\pi)$ is the density of states for the n-th Landau level of the Landau Hamiltonian.

Recently, several authors ([R-W], [M-R]) investigate the asymptotics for the case where the decay of the electric potentials V is Gaussian or faster. They show that the asymptotics are non-classical if the decay of V is faster than Gaussian (in an appropriate sense), or support of V is compact. The leading asymptotics are independent of n, and in the case of compact support, it does not depend on V.

On the other hand, our results shows that the asymptotic behaviour of $N(E_n + E < H(B,V) < E')$ has the form (1.2) as $E \searrow 0$. The density of states of the Maass operator can be found in [Com], Eq.(5.14)–(5.16), Eq.(B.19). In particular, the density of states for the n-th discrete Landau level is given by $\beta_n/(4\pi)$, which depends on n. The quantity $\beta_n/(4\pi)$ does not coincide with the leading coefficient $\Theta_n(\varepsilon)^{1/\varepsilon}/(4\pi)$ in (1.2) unless $\varepsilon = 1$. Obviously, $\Theta_n(\varepsilon)^{1/\varepsilon}$ depends on both n and ε. So, this is different from the flat case.

2 A formal flat space limit for the Maass operator

In the second part, I talk about a flat space limit for the Maass operator and show that the leading coefficient of the asymptotics (1.2) converges to the corresponding one for the Landau Hamiltonian on the Euclidean plane in the flat space limit. However, the statements in this part are more or less well-known (and trivial) facts (see [Com], Section IV).

Let $R > 0$. We introduce the space $\mathbb{D}_R = \{w = u + iv \in \mathbb{C} |\ |w| < R\}$ equipped with the Riemannian metric $g_{\mathbb{D}_R} = 4(1 - |w/R|^2)^{-2}(du \otimes du + dv \otimes dv)$. Here, $|w| = (|u|^2 + |v|^2)^{1/2}$. (In what follows we denote by $|\cdot|$ the standard Euclidean norm.) Note that the metric is expressed as $4(1 - (\rho/R)^2)^{-2}(d\rho \otimes d\rho + \rho^2 d\theta \otimes d\theta)$ in the polar coordinates $w = \rho e^{\sqrt{-1}\theta}$ $(0 \leq \rho < R, 0 \leq \theta < 2\pi)$, and is also expressed as $dr \otimes dr + R^2 \sinh^2(r/R) d\theta \otimes d\theta$ in the geodesic polar coordinates (r, θ) given by $w = R \tanh(r/(2R)) e^{\sqrt{-1}\theta}$. Moreover, the Ricci curvature is given by $-R^{-2} g_{\mathbb{D}_R}$. In the case of $R = 1$, we may write $\mathbb{D}$ and $\mathbb{H}$ for $\mathbb{D}_1$ and $\mathbb{H}_1$, respectively.

The space $\mathbb{D}_R$ has a different model $\mathbb{H}_R = \{z = x + \sqrt{-1}y \in \mathbb{C} | x \in \mathbb{R}, y > 0\}$ equipped with the metric $R^2 y^{-2}(dx \otimes dx + dy \otimes dy)$. Indeed, the spaces $\mathbb{D}_R$ and $\mathbb{H}_R$ are isomorphic as a Riemannian manifold, via the transform $\varphi_R(z) = R(z - i)/(z + i)$. This follows from the facts that $g_{\mathbb{H}_R} = {}^t(D\varphi_R) g_{\mathbb{D}_R} (D\varphi_R)$, where $(D\varphi_R)$ is the Jacobian of φ_R, and that the usual Cayley transform gives an isometric differomorphism from $\mathbb{H}$ to $\mathbb{D}$.

Remark 2.1 *As is easily seen, the space $\mathbb{D}_R$ converges to the Euclidean plane $\mathbb{R}^2$ with the metric $4(du \otimes du + dv \otimes dv)$, at least, in a formal level. Indeed, one can find that $\mathbb{D}_R$ converges in the (pointed) measured Gromov-Hausdorff topology.*

For any $B \geq 0$ and $R > 0$, we define the (scaled) Maass operator

$$H_R(B) = R^{-2} y^2 \left((D_x - R^2 B/y)^2 + D_y^2\right)$$

on $L^2(\mathbb{H}_R)$ with the Riemannian volume $d\mathrm{Vol}_{\mathbb{H}_R} = R^2 y^{-2} dx dy$. In the case of $R = 1$ we may write $H(B)$ for $H_1(B)$.

This operator is the Bochner Laplacian on $\mathbb{H}_R$ with the connection one form $\mathbf{a} = B y^{-1} dx$. Then $\mathbf{a}$ gives the curvature two form $\omega_B = BR^2 y^{-2} dx \wedge dy = B d\mathrm{Vol}_{\mathbb{H}_R}$, so we call $H_R(B)$ the Schrödinger operator with constant magnetic field B (or the Landau Hamiltonian) as in the Euclidean case.

The operator $H_R(B)$ is essentially self-adjoint on $C_0^\infty(\mathbb{H}_R)$.

Lemma 2.2 *The spectrum of $H_R(B)$ is explicitly given by*

$$\mathrm{Spec}(H_R(B)) = \bigcup_{l=0}^{N(BR^2 - 1/2)} \{(2l+1)B - l(l+1)R^{-2}\} \bigcup [B^2 R^2 + 1/(2R)^2, \infty),$$

where $N(x)$ stands for the largest integer strictly less than x. Moreover, the first part of the spectrum consists of the eigenvalues of infinite multiplicity, which is regarded as empty in the case of $BR^2 < 1/2$, and the second part consists of the absolutely continuous spectrum.

Proof. Define the unitary transform U_R from $L^2(\mathbb{H}_R)$ to $L^2(\mathbb{H})$ by $U_R f(z) = R f(z)$. Then we can find that $U_R H_R(B) U_R^{-1} = R^{-2} H(R^2 B)$ holds on $L^2(\mathbb{H})$, from which we have $\mathrm{Spec}(H_R(B)) = R^{-2}\mathrm{Spec}(H(R^2 B))$. The results follows from the well-known result $\mathrm{Spec}(H(B)) = \cup_{l=0}^{N(B-1/2)} \{(2l+1)B - l(l+1)\} \cup [B^2 + 1/4, \infty)$ (see, e.g., Elstrodt [Els]). ∎

From Lemma 2.2, we observe that the set $\mathrm{Spec}(H_R(B))$ 'converges' to the set $\cup_{l=0}^{\infty}\{(2l+1)B\}$ in the flat space limit $R \to \infty$ (at least, in the Hausdorff topology on any compact set in $\mathbb{R}$).

Lemma 2.3 *The operator $H_R(B)$ is unitarily equivalent to the operator*

$$\tilde{H}_R(B) = \frac{1}{4}(1-|w/R|^2)^2\left((D_u - b_1)^2 + D_v^2\right)$$

on $L^2(\mathbb{D}_R)$, where $w = u + iv$ as before and

$$\begin{aligned} b_1(u,v) &= -2BR\frac{1}{1-(u/R)^2}\frac{v/R}{1-|w/R|^2} \\ &\quad -2BR(1-(u/R)^2)^{-3/2}\tanh^{-1}\frac{v/R}{(1-(u/R)^2)^{1/2}}. \end{aligned}$$

Proof. Let φ_R be as in the previous section. By the transformation rule for connection forms, we find that

$$(\varphi_R)_* H_R(B)(\varphi_R)_*^{-1} = \frac{1}{4}(1-|w/R|^2)^2\left((D_u-\tilde{a}_1)^2 + (D_v - \tilde{a}_2)^2\right)$$

with $\tilde{a}_1 = R^2 a_1 \frac{\partial x}{\partial u}$ and $\tilde{a}_2 = R^2 a_1 \frac{\partial x}{\partial v}$.

Using the relations

$$\begin{aligned} \frac{\partial x}{\partial u} &= -\frac{4}{R}\frac{(v/R)(1-u/R)}{((1-u/R)^2+(v/R)^2)^2}, \\ \frac{\partial x}{\partial v} &= -\frac{2}{R}\frac{(1-u/R)^2-(v/R)^2}{((1-u/R)^2+(v/R)^2)^2}, \\ y &= \frac{1-(u/R)^2-(v/R)^2}{(1-u/R)^2+(v/R)^2}, \end{aligned}$$

we find that the gauge function

$$\varphi = 2BR^2\left(\tan^{-1}\frac{v/R}{1-u/R} - \frac{u/R}{\sqrt{1-(u/R)^2}}\tanh^{-1}\frac{v/R}{\sqrt{1-(u/R)^2}}\right)$$

satisfies $\partial_v\phi + \tilde{a}_2 = 0$ and $b_1 = \tilde{a}_1 + \partial_u\varphi$. Then a simple gauge transform procedure shows that the operator $\tilde{H}_R(B)$ is given by $e^{\sqrt{-1}\varphi}(\varphi_R)_* H_R(B)(\varphi_R)_*^{-1}e^{-\sqrt{-1}\varphi}$. ∎

Observation 2.4 *The operator $\tilde{H}_R(B)$ tends to the Landau Hamiltonian $\frac{1}{4}((D_u+4Bv)^2 + D_v^2)$ on the space $\mathbb{R}^2$ with the metric $4(du\otimes du + dv\otimes dv)$, which has the essential spectrum $\cup_{l=0}^{\infty}\{(2l+1)B\}$, as $R\to\infty$ in a formal level.*

Proof. For each u and v, we can observe from Lemma 2.3 that

$$\begin{aligned} b_1 &= -2Bv\frac{1}{1-(u/R)^2}\frac{1}{1-|w/R|^2} \\ &\quad -2Bv(1-(u/R)^2)^{-2}\frac{(1-(u/R)^2)^{1/2}}{v/R}\tanh^{-1}\frac{v/R}{(1-(u/R)^2)^{1/2}} \\ &\to -2Bv-2Bv = -4Bv \end{aligned}$$

as $R \to \infty$, where we used the fact that $\lim_{x\to 0} \tanh^{-1} x/x = 1$. In the same way as in the proof of Lemma 2.2, we can show that the spectrum of the above Landau Hamiltonian is given by $\cup_{l=0}^{\infty}\{(2l+1)B\}$, which coincides with that of $\frac{1}{4}((D_u + 4Bv)^2 + D_v^2)$ on $\mathbb{R}^2$ with standard Euclidean metric. ∎

Let $\varepsilon > 0$ and $C_V > 0$ and put $V_R(z) = C_V \exp(-\varepsilon d_{\mathbb{H}_R}(\sqrt{-1}, z))$ for any $z \in \mathbb{H}_R$. We consider the following unitary equivalences:

$$L^2(\mathbb{D}_R) \overset{(\varphi_R)^*}{\cong} L^2(\mathbb{H}_R) \overset{U_R}{\cong} L^2(\mathbb{H}) \overset{(\varphi_1)_*}{\cong} L^2(\mathbb{D}).$$

Then the multiplication operator V_R on $\mathbb{H}_R$ is unitarily equivalent to each of the multiplication operators $C_V \exp(-\varepsilon d_{\mathbb{D}_R}(0, w))$ on $L^2(\mathbb{D}_R)$, $C_V \exp(-R\varepsilon d_{\mathbb{H}}(\sqrt{-1}, z))$ on $L^2(\mathbb{H})$ and $C_V \exp(-R\varepsilon d_{\mathbb{D}}(0, w'))$ on $L^2(\mathbb{D})$, where $w = \varphi_R(z)$ and $w' = \varphi_1(z)$.

Hence it follows that, for each large $R > 0$,

$$\begin{aligned}
& N(E_n^{(R)} + E < H_R(B) + V_R < E') \\
= \ & N(R^2 E_n^{(R)} + R^2 E < H(R^2 B) + R^2 V_R < R^2 E') \\
= \ & N(E_n(BR^2) + R^2 E < H(R^2 B, R^2 V_R) < R^2 E') \\
= \ & \frac{1}{4\pi}(\Theta_n(R\varepsilon))^{\frac{1}{R\varepsilon}} \mathrm{Vol}_{\mathbb{H}}\{z \in \mathbb{H} | R^2 V_R(z) > R^2 E\}(1 + o(1))
\end{aligned} \tag{2.1}$$

holds as $E \searrow 0$, where we set $E_n^{(R)} = (2n+1)B - n(n+1)/R^2$ and $E_n(BR^2) = (2n+1)BR^2 - n(n+1)$, and we used the unitary equivalence $H_R(B) \cong R^{-2} H(BR^2)$ in the first equality and used Theorem 1.1 in the first part with $B \to BR^2$, $V = V_R$, $\varepsilon \to R\varepsilon$ in the last line.

The next obvious but natural fact is the main claim in this part.

Observation 2.5 *For each small $E > 0$ and each $\varepsilon > 0$, the leading coefficient on the right-hand-side of (2.1) converges to $\frac{4B}{2\pi}\mathrm{Vol}_{\mathbb{R}^2}\{w \in \mathbb{R}^2 | C_V \exp(-2\varepsilon|w|) > E\}$ as $R \to \infty$.*

Proof. In this proof we use the notation $\beta_n^{(R)} = 2BR^2 - 2n - 1$. Then we have

$$\begin{aligned}
\Theta_n(\varepsilon) &= \frac{\Gamma(\beta_n^{(R)} + R\varepsilon)}{\Gamma(\beta_n^{R})}\, {}_3F_2\left(\begin{matrix} -n, 1 - R\varepsilon, R\varepsilon \\ \beta_n^{(R)} + 1, 1 \end{matrix}; 1\right). \\
&= (2BR^2)^{R\varepsilon} \cdot (2BR^2)^{-R\varepsilon} \frac{\Gamma(\beta_n^{(R)} + R\varepsilon)}{\Gamma(\beta_n^{R})}\, {}_3F_2\left(\begin{matrix} -n, 1 - R\varepsilon, R\varepsilon \\ \beta_n^{(R)} + 1, 1 \end{matrix}; 1\right).
\end{aligned} \tag{2.2}$$

First, by the Stirling asymptotic formula, we find that

$$\lim_{R\to\infty} (2BR^2)^{-R\varepsilon} \frac{\Gamma(\beta_n^{(R)} + R\varepsilon)}{\Gamma(\beta_n^{R})} = \lim_{R\to\infty} (2BR^2)^{-R\varepsilon} \frac{\Gamma(2BR^2 - 2n - 1 + R\varepsilon)}{\Gamma(2BR^2 - 2n - 1)} = 1, \tag{2.3}$$

and

$$\lim_{R\to\infty} {}_3F_2\left(\begin{matrix} -n, 1 - R\varepsilon, R\varepsilon \\ \beta_n^{(R)} + 1, 1 \end{matrix}; 1\right) = {}_1F_1\left(\begin{matrix} -n \\ 1 \end{matrix}; -\frac{\varepsilon^2}{2B}\right) = L_n(-\frac{\varepsilon^2}{2B}), \tag{2.4}$$

where we used the formulas $(x)_j = \Gamma(x+j)/\Gamma(x)$ and $(-x)_j = (-1)^j\Gamma(x+1)/\Gamma(x-j+1)$. Here, $L_n(z)$ is the Laguerre polynomial, which does not vanish on the negative real line. Finally, we find that

$$\begin{aligned} R^2\mathrm{Vol}_{\mathbb{H}}\{z|V_R(z) > E\} &= \mathrm{Vol}_{\mathbb{H}_R}\{z|V_R(z) > E\} \\ &= \mathrm{Vol}_{\mathbb{D}_R}\{w|C_V \exp(-\varepsilon d_{\mathbb{D}_R}(0,w)) > E\} \\ &\to 4\mathrm{Vol}_{\mathbb{R}^2}\{w|C_V e^{-2\varepsilon|w|} > E\} \end{aligned} \tag{2.5}$$

as $R \to \infty$ for each $E > 0$. Then the claim follows from (2.1)–(2.5) since

$$\begin{aligned} & N((2n+1)B + E < \frac{1}{4}((D_u + 4Bv)^2 + D_v^2) + C_v e^{-2\varepsilon|w|} < E') \\ = & \frac{4B}{2\pi}\mathrm{Vol}_{\mathbb{R}^2}\{w|C_v e^{-2\varepsilon|w|} > E\}(1+o(1)) \end{aligned}$$

holds as $E \searrow 0$ (see [R-W]). ▮

3 Decay of the magnetic eigenfunctions on $\mathbb{H}$

In the third part, following [Shi2], I will talk about some decay properties of the magnetic eigenfunctions of perturbed Maass operator $H_V = H(B) + V$ and a related flat space limit if I have the time.

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