ON PATH INTEGRAL FOR THE RADIAL DIRAC EQUATION

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Abstract. For the radial Dirac equation, we construct a countably additive path space measure on the space of continuous paths living on the real half-line to give a path integral representation to its Green function.

Introduction and Results

Consider the radial Dirac equation, namely, the radial part of the Dirac equation in spherical coordinates, given, for $\mathbf{C}^2$-valued functions $\psi(r,t) = {}^t(\psi_1(r,t), \psi_2(r,t))$, by

$$\frac{\partial}{\partial t}\psi(r,t) = -i[\tau_k + V(r)]\psi(r,t), \tag{1}$$

with a real-valued spherically symmetric potential $V = V(r)$, i.e. a function in the real half-line $\mathbf{R}_+ := (0,\infty)$, where the variables (r,t) lie in radial space-time $\mathbf{R}_+ \times \mathbf{R}$. Here τ_k is the free radial Dirac operator with mass m defined for $k = \pm 1, \pm 2, \cdots$, by

$$\tau_k : f \mapsto \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial r} + \begin{pmatrix} m & -k/r \\ -k/r & -m \end{pmatrix} \right] \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} \tag{2a}$$

for suitably smooth $\mathbf{C}^2$-valued functions $f = {}^t(f_1, f_2)$ in $\mathbf{R}_+$, in short,

$$\tau_k = -i\sigma_2 \frac{\partial}{\partial r} - \sigma_1 \frac{k}{r} + m\sigma_3, \tag{2b}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The free radial Dirac operator τ_k arises from the spin-angular momentum decomposition of the free Dirac operator in three space dimensions. The nonzero integer k represents an eigenvalue of the "spin-orbit operator" (see [BD], [Th], [W]).

The operator τ_k is a symmetric operator with domain $D[\tau_k] = C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$ in $L^2(\mathbf{R}_+;\mathbf{C}^2)$. Here $L^2(\mathbf{R}_+;\mathbf{C}^2)$ is the Hilbert space of the $\mathbf{C}^2$-valued square-integrable functions in $\mathbf{R}_+$ with respect to the Lebesgue measure dr, and $C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$ the locally convex space of the $\mathbf{C}^2$-valued C^∞ functions in $\mathbf{R}_+$ with compact support. It has a singularity at $r=0$ as in (2ab). This is indeed harmless if we consider it as an operator in the L^2 space, but is a problem to construct a path space measure, for we need to consider it as an operator in the L^∞ space (see also [IT 1, IT 3]). In this context, in [IJ], we have made an explicit construction of the propagator, namely, the integral kernel $\mathcal{K}_t(r,\rho)$ of $e^{-it\tau_k}$ for $k=1$, and shown that, though it turns out to be a distribution *of order zero* in the variables $(r,\rho)\in \mathbf{R}_+\times\mathbf{R}_+ = {\mathbf{R}_+}^2$, there exists no countably additive path space measure to give a path integral representation to the solution of the Cauchy problem for the radial Dirac equation (1).

The aim of this note is to construct a countably additive path space measure to represent by path integral, though not the propagator, the Green function for the radial Dirac equation (1). The main idea is to combine our method of constructiong a path space measure developed for the one-dimensional Dirac equation in the papers [I 1, I 2, IT 1, IT 2, IT 3], in particular, in their a little more refined review [I 3], with the following simple but intriguing procedure of dealing with the singularity, which was invented by Duru and Kleinert [DK] and has since been employed by many physicists (see [InKG, p. 6], [K, Chap. 12 and 14], [GS, pp. 77–83]) to perform space-time transformations in path integrals.

The free radial Dirac operator τ_k has a singularity at $r=0$. However, if we multiply this operator τ_k by some (nonnegative) functions $a(r)$ and $b(r)$ from the left and right sides, respectively, then $a(r)\tau_k b(r)$ becomes no more singular. Let us take $a(r)=b(r)=r^{1/2}$, and put $T_k = r^{1/2}\tau_k r^{1/2}$. Then $-iT_k$ becomes

$$\begin{aligned} -iT_k &= -ir^{1/2}\tau_k r^{1/2} = -\sigma_2 r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2} + i(k\sigma_1 - mr\sigma_3) \\ &= -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} r^{1/2}\Big(\frac{\partial}{\partial r}\Big)r^{1/2} + i\begin{pmatrix} -mr & k \\ k & mr \end{pmatrix} \\ &=: -iT_{k0} - iT_{k1}. \end{aligned} \tag{3}$$

We also consider the operator

$$H(k,rV) := T_k + rV(r), \tag{4}$$

relevant to the radial Dirac operator in (2ab). Then T_k is a symmetric operator with domain $D[T_k] = C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$ in $L^2(\mathbf{R}_+;\mathbf{C}^2)$. For the potentials $V(x)$ we are concerned with, we can show $H(k,rV)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$, as well as T_k. We shall denote their closures or unique selfadjoint extensions again by the same $H(k,rV)$ and T_k.

Then consider, for $H(k,rV)$ instead of $\tau_k + V(r)$, the Cauchy problem

$$\frac{\partial}{\partial t}u(r,t) = -iH(k,rV)u(r,t), \quad u(r,0) = g(r) = {}^t(g_1(r), g_2(r)) \tag{5}$$

for $t \in \mathbf{R}$ or the solution $u(r,t) = \big(e^{-itH(k,rV)}g\big)(r)$. Since T_k has no more singularity at $r=0$, we expect to be able to construct a path space measure associated with the semigroup $e^{-itH(k,rV)}$. Since the resolvent is expressed by the Laplace transform of the semigroup, we shall find the following relation between the desired resolvent kernel of $\tau_k + V$ and the semigroup $e^{-itH(k,rV)}$, though a little formally expressed ,

$$\begin{aligned}[(\tau_k + V - \lambda)^{-1}](r,\rho) &= r^{1/2}[(H(k,rV) - \lambda r)^{-1}](r,\rho)\rho^{1/2} \\ &= i\int_0^\infty r^{1/2} e^{-it(H(k,rV)-\lambda r)}\rho^{1/2}\,dt,\end{aligned}$$

for suitable real or complex numbers λ. In this way, we might get, for the original radial Dirac operator $\tau_k + V(r)$, a path integral representation of the resolvent kernel and the Green function if λ might be taken to be zero, though we were unable to find such a representation for the propagator $e^{-it(\tau_k+V)}$ itself.

For the potential $V(x)$, we assume that it is a real-valued function in $\mathbf{R}_+$ such that $V(r) = V_1(r) + V_2(r)$, where $V_1(r) = e/r$ with a real constant e satisfying $|e| \le \sqrt{k^2 - \frac{1}{4}}$ and $V_2(r)$ is a locally square-integrable function in $\mathbf{R}_+$ which is bounded near $r = 0$.

Note that this class of potentials $V(r)$ contains the Coulomb potential.

In this case it can be shown that the radial Dirac operator $\tau_k + V$ in (1) is essentially selfadjoint on $C_0^\infty(\mathbf{R}_+;\mathbf{C}^2)$. We denote its unique selfadjoint extension in $L^2(\mathbf{R}_+;\mathbf{C}^2)$ also by the same $\tau_k + V$. Thus this operator has a real spectrum. Further, as to its spectrum, for instance, the following results are known ([W], [Tha, Theorems 4.18, 4.19]). If $V_2(r) := V_{21}(r) + V_{22}(r)$ satisfies for some $r_0 > 0$ that $V_{21} \in L^1((r_0,\infty))$ and V_{22} is of bounded variation in $[r_0,\infty)$ with $\lim_{r\to\infty} V_{22}(r) = 0$, then the operator $\tau_k + V$ has a purely absolutely continuous spectrum in the real line $\mathbf{R}$ outside the closed interval $[-m,m]$. If $V_1(r)$ is absent (i.e. $e = 0$) and if $\lim_{r\to\infty}|V_2(r)| = \infty$ and $\int_{r_0}^\infty \left|\frac{V_2'(r)}{V_2(r)^2}\right| dr < \infty$ for some $r_0 > 0$, then $\tau_k + V$ has the whole real line $\mathbf{R}$ as a purely absolutely continuous spectrum.

We will show the following path integral representation for the resolvent kernel $[(\tau_k + V \mp i\varepsilon)^{-1}](r,\rho)$ for $\varepsilon > 0$ and the Green function for the radial Dirac operator $\tau_k + V(r)$ in (1).

The set of all complex 2×2-matrices is denoted by $M_2(\mathbf{C}) = \mathbf{C}^2 \otimes (\mathbf{C}^2)'$. With $[0,\infty) = \overline{\mathbf{R}_+}$ being the closed real half-line, let $C_{00}^\infty([0,\infty)^2; M_2(\mathbf{C}))$ be the locally convex space of the $M_2(\mathbf{C})$-valued C^∞ functions $M(r,\rho)$ in $[0,\infty)\times[0,\infty)$ which have compact support and whose derivatives $\partial r^m \partial\rho^n M(r,\rho)$, for all nonnegative integers m and n, vanish at $(r,0)$ and $(0,\rho)$ for all $r \ge 0$ and $\rho \ge 0$. Let $C_{00}^\infty([0,\infty)^2; M_2(\mathbf{C}))'$ be its dual space. As $C_0^\infty({\mathbf{R}_+}^2; M_2(\mathbf{C}))$ is a subspace of $C_{00}^\infty([0,\infty)^2; M_2(\mathbf{C}))$, so $C_{00}^\infty([0,\infty)^2; M_2(\mathbf{C}))'$ is a subspace of the space $\mathcal{D}'({\mathbf{R}_+}^2; M_2(\mathbf{C}))$ of the $M_2(\mathbf{C})$-valued distributions in $\mathbf{R}_+ \times \mathbf{R}_+$. By $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ we denote respectively the sesquilinear and bilinear inner products of a dual pairing.

The main result of this note is the following theorems. The notation $|0,t|$ stands for the interval $0 \le s \le t$ or $0 \ge s \ge t$ according as $t > 0$ or $t < 0$.

Theorem 1. *Let $V(r)$ be a potential mentioned above.*

(i) *For every $\varepsilon > 0$, there exists a 2×2-matrix distribution-valued, precisely speaking, $C_{00}^{\infty}([0,\infty)^2; M_2(\mathbf{C}))'$-valued, countably additive path space measure $\mu_{t,0}$ on the Banach space $C([0,t] \to [0,\infty))$ of the continuous paths $R: [0,t] \to [0,\infty)$ such that the resolvent kernel function $[(\tau_k + V \mp i\varepsilon)^{-1}](r,\rho)$ for the radial Dirac operator $\tau_k + V$ in (1) admits a path integral representation: for every pair $(f,g) \in C_{00}^{\infty}([0,\infty);\mathbf{C}^2) \times C_{00}^{\infty}([0,\infty);\mathbf{C}^2)$,*

$$\begin{aligned}(f,(\tau_k+V\mp i\varepsilon)^{-1}g) &= \int_0^{\infty}\int_0^{\infty} {}^t\overline{f(r)}[(\tau_k+V\mp i\varepsilon)^{-1}](r,\rho)g(\rho)\,drd\rho \\ &= i\int_0^{\pm\infty} dt \int_{C([0,t]\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \\ &\qquad \times R(t)^{1/2}R(0)^{1/2}\exp\Big[-\int_0^t \big(iV(R(s))R(s) \pm \varepsilon R(s)\big)\,ds\Big]. \end{aligned} \tag{6}$$

In particular, the resolvent kernel function $[(\tau_k + V \mp i\varepsilon)^{-1}](r,\rho)$ is a distribution of order zero in $\mathbf{R}_+ \times \mathbf{R}_+$.

(ii) *The measure $\mu_{t,0}$ is concentrated on the set of the continuous paths $R: [0,t] \to [0,\infty)$ for which there exists a finite partition: $0 = t_0 \lesseqgtr t_1 \lesseqgtr \cdots \lesseqgtr t_{n+1} = t$ of the interval $[0,t]$ such that for $t_{j-1} \lesseqgtr s \lesseqgtr t_j$, $1 \le j \le n+1$,*

$$R(s) = R(0)e^{\pm[\sum_{p=1}^{j-1}(-1)^{p-1}(t_p - t_{p-1}) + (-1)^{j-1}(s - t_{j-1})]}. \tag{7}$$

Therefore each of these continuous paths $R(\cdot)$ is, for some finite n, an n-vertex piecewise smooth curve in radial space-time, starting from $R(0)$ at time 0 and reaching $R(t)$ at time t, and exponentially growing or decreasing in each partioned short time interval.

We denote by $G_{\pm}(r,\rho)$ the Green function for the radial Dirac operator $\tau_k + V$ to be given as the limit of the integral kernel of the resolvent $(\tau_k + V \mp i\varepsilon)^{-1}$ as $\varepsilon \to +0$, if this limit exists.

Theorem 2. *For the same potential $V(r)$ as in Theorem 1, suppose that 0 is not an eigenvalue of the radial Dirac operator $\tau_k + V$. Suppose that the Green function $G_{\pm}(r,\rho)$ for the radial Dirac operator $\tau_k + V$ in (1) exists. Then it is a distribution of order zero in $(r,\rho) \in \mathbf{R}_+ \times \mathbf{R}_+$, and admits a path integral representation: for every pair $(f,g) \in C_{00}^{\infty}([0,\infty);\mathbf{C}^2) \times C_{00}^{\infty}([0,\infty);\mathbf{C}^2)$,*

(i)

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty} {}^t\overline{f(r)}G_{\pm}(r,\rho)g(\rho)\,drd\rho \\ &\quad = i\lim_{\varepsilon\to+0}\int_0^{\pm\infty} dt \int_{C([0,t]\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \\ &\qquad \times R(t)^{1/2}R(0)^{1/2}\exp\Big[-\int_0^t \big(iV(R(s))R(s) \pm \varepsilon R(s)\big)\,ds\Big]; \end{aligned} \tag{8}$$

(ii)

$$\int_0^\infty \int_0^\infty {}^t\overline{f(r)}G_\pm(r,\rho)g(\rho)\,drd\rho$$

$$= i \lim_{\varepsilon\to+0} \int_0^{\pm\infty} dt \int_{C(|0,t|\to[0,\infty))} \langle {}^t\overline{f(R(t))}, d\mu_{t,0}(R)\, g(R(0))\rangle \quad (9)$$

$$\times R(t)^{1/2}R(0)^{1/2}\exp\Big[-i\int_0^t V(R(s))R(s)\,ds \mp \varepsilon t\Big].$$

If we formally take the delta functions at the two points $r > 0$ and $\rho > 0$ respectively for f and g, the formula (8)/(9) looks like

$$G_\pm(r,\rho) = i\int_0^{\pm\infty} dt \int_{C(|0,t|\to[0,\infty)),R(0)=\rho,R(t)=r} \times r^{1/2}\rho^{1/2}\exp\Big[-i\int_0^t V(R(s))R(s)\,ds\Big]d\mu_{t,0}(R). \quad (10)$$

For the Dirac equation in the one-dimensional space, i.e. the whole real line **R**, a path integral measure to represent the propagator was constructed first in [I 1], [I 2, IT 1] and then further studied in [IT 2], [IT 3] and [I 3]. The problem was treated in [BlCS] from a different point of view based on Poisson process, and there are further related works [G], [GJKS], [CS] and [Z] on the subject.

This note is only to give the first step to describe the idea, and application to some problem in quantum field theory will be discussed elsewhere.

To prove the theorems, we investigate first the Cauchy problem (5) in the L^∞ norm for the relevant operator to our radial Dirac operator, and also show essential selfadjointness of both the radial Dirac operator (2ab) and its relevant operator (4). Next, we construct, by means of the Riesz represntation theorem, a countably additive path space measure associated with the semigroup for the Cauchy problem (5), on a big space of paths living on the closed real half-line over each finite time interval $[0,t]$. Then it is shown that this measure is in fact concentrated on the set of continuous, piecewise smooth paths with a finite number of vertices in radial space-time. Finally, we shall show, together with this measure constructed, a path integral representation of Feynman–Kac type first for this semigroup associated with (5) and then through the procedure mentioned for the resolvent kernel and the Green function for the radial Dirac operator we are concerned with. This lecture explains some idea of T. Ichinose, *Path integral for the radial Dirac equation*, to appear in JMP, 2005 [mp-arc 04-318], with mention of a beyond.

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