General Setting of the School

Five lecturers will present a course in four lectures of one and a half hour each in the mornings. The lecturers and the topics of their courses are:

- Yann B (CNRS-Université de Nice) will lecture on *Hidden Convexity in Nonlinear PDEs from Geometry and Physics*;
- Nicolas B (Université Paris-sud) will lecture on *Large Time Dynamics for the One Dimensional Schrödinger Equation*;
- David G´ -V (Université Paris VI) will lecture on *Boundary Layer Theory for the Navier-Stokes Equation*;
- Yvan M (Université de Versailles-Saint Quentin) will lecture on *Inelastic Interaction of Solitons for the Quartic gKdV Equation*;
- Laszlo S ´ Jr. (Universität Bonn) will lecture on *From Isometric Embeddings to Turbulence*.

More detailed presentations of the courses, including prerequisites and a list of references, can be found in the following pages.

One afternoon special sessions will be organised by:

- Jean-Marc D (Université Paris XIII) on *Nonlinear Evolution Equations and Birkhoff Normal Forms*;
- Alessio F (University of Texas at Austin) on *Optimal Transport, or the Geometric Theory of Measures and the Calculus of Variations*;
- Clément M (École Normale Supérieure) on *Hypocoercivity and Cinetic Equations*;
- Fabrice P (Université Paris XIII) on *Waves on Domains: Geometry and Dispersion*;
- Olivier D (École Normale Supérieure de Lyon) on *Asymptotic Analysis for some Elliptic PDEs*;
- Laure S -R (École Normale Supérieure) on *Hydrodynamics Limits*;
- Nikolay T (Université de Cergy-Pontoise) on *Limit Properties of Randomly Forced PDEs*;
- Michael D (Cambridge University) on *The Equations of General Relativity*.

Hidden Convexity in Nonlinear PDEs from Geometry and Physics

Yann B (CNRS-Université de Nice)

The purpose of the course is to analyze several examples of nonlinear PDEs -with both strong geometric and physical features- which enjoy a hidden con- vex structure. Robust existence and uniqueness results can be unexpectedly obtained for very general data. Of course, as usual, regularity issues are left over as a hard post-process, but, at least, existence, uniqueness and stability results are obtained in a large, global, framework.

We will discuss:

- 1. The real Monge-Ampère Equation (we will show how the convex structure is related to *Optimal Transport Theory*);
- 2. The Euler Equations of Fluid Mechanics (that describe the motion of inviscid, incompressible fluids and provide the most famous example of a geodesic flow in infinite dimension) and their *hydrostatic* and *semi-geostrophic* limits;
- 3. The Born-Infeld System (a non-linear electromagnetic model introduced in 1934, playing an important role in high energy Physics since the 1990's).

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Large Time Dynamics for the One Dimensional Schrödinger Equation

Nicolas B (Université Paris-sud)

In this course I will present some recent results with L. T and N. T on the behaviour of solutions to Schrödinger equations with random initial data. The main question I want to address is the following: Are solutions to Schrödinger equations better behaved when one consider initial data randomly chosen (in some sense) than what would be predicted by the deterministic theory? To my knowledge the first result known in this direction is due to Rademacher-Kolmogorov-Paley-Zygmund, and states that random series on the torus enjoy better L^p bounds that the deterministic bounds. These lectures are somehow a natural extension on the partial differential equations field of these harmonic analysis results. We shall use some basic results from probability theory. The non linear Schrödinger I will be interested in, is the following one dimensional non linear harmonic oscillator

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u = |u|^{r-1} u, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = f(x), \end{cases} \tag{1}$$

where $r > 1$ is the order of the non linearity. On a deterministic point of view, this equation is well posed in $L^2(\mathbb{R})$ as soon as $p \leq 5$, and the assumption $p \leq 5$ is known to be optimal in some sense (see the works by Christ-Colliander-Tao and Burq-Gérard-Tzvetkov in slightly different contexts). However, we shall prove, that for all non linearities $|u|^{p-1}u$, not only is the equation well posed for a large set of initial data whose Sobolev regularity is below L^2, but also that the flows enjoys very nice large time behaviour (in a probabilistic sense).

References

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Boundary layer theory for the Navier-Stokes equation

David G´ -V (Université Paris VI)

Objectives. The aim of this course is to give mathematical insights into a central problem of fluid mechanics: the understanding of fluid flows around obstacles. This problem appears in many situations of practical interest, for instance the spreading of air around the wings of an airplane. The main difficulty comes from high speed, or low viscosity fluid flows. Mathematically, one needs to describe the asymptotics, as ν goes to zero, of the Navier-Stokes equation

$$(NS_\nu) \qquad \begin{cases} \partial_t u + u\nabla u + \nabla p - \nu\,\Delta u = 0, \ t > 0, \ x \in \Omega\,, \\ \nabla \cdot u = 0, \ t > 0, \ x \in \Omega\,, \\ u_{|t=0} = u_0\,, \ \ u_{|\partial\Omega} = 0\,. \end{cases}$$

in a domain Ω with boundary. As ν goes to zero, it is known from experiments that the velocity u_ν concentrates near $\partial\Omega$ in a thin zone near the boundary, called a *boundary layer*. The mathematical description of this layer, and its impact on the asymptotics $\nu \to 0$ is still poorly understood. In particular, it is not known in general whether or not a sequence of smooth solutions (u_ν) of (NS_ν) converges to a solution of the Euler equation.

During the course we shall describe the main mathematical results on this convergence problem, namely:

- 1. The convergence criteria of Kato [3];
- 2. The Prandtl approach for proving convergence, and the well-posedness results of Oleinik on the Prandtl model for the boundary layer [4];
- 3. The justification of the Prandtl approach in the analytic setting [5];
- 4. Instability problems in the Sobolev setting [1, 2].

Prerequisites. Acquaintance with some basic notions of mathematical fluid mechanics (local existence of strong solutions for Navier-Stokes and Euler, or global existence of weak solutions for Navier-Stokes) is recommended, but not necessary.

References

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Inelastic Interaction of Solitons for the Quartic gKdV Equation

Yvan M (Université de Versailles-Saint Quentin)

The main objective of the course is to present recents work by Yvan Martel and Frank Merle on collision of two solitons for the generalized Korteweg-de Vries equations, and in particular the quartic KdV equation. It is a non integrable equation and no explicit multi-soliton solutions can be found in this case. However, we are able to describe accuretely the interaction of two solitons in two disctinct situations: first, the case where the size of one soliton is small with respect to the other soliton, and second, the case where the two solitons have almost the same size.
Prerequisites. Only basic PDE theory.

References

[1] Y. M , F. M , *Description of Two Soliton Collision for the Quartic gKdV Equation*, arxiv.org/abs/0709.2672.

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From Isometric Embeddings to Turbulence

László S ´ Jr. (Universität Bonn)

The following dichotomy concerning isometric embeddings of the shere is well-known: whereas the only C^2 isometric embedding of S^2 into $\mathbb{R}^3$ is the standard embedding modulo rigid motion, there exist many C^1 isometric embeddings which can ”wrinkle” S^2 into arbitrarily small regions. The latter ”flexibility”, known as the Nash-Kuiper theorem [8, 7], involves an iteration scheme called convex integration which turned out to have surprisingly wide applicability.

More generally, this type of flexibility appears in a variety of different geometric contexts and is known as the ”h-principle” [6]. But one has to distinguish two contrasting cases: in problems which are formally highly undetermined, such as isometric embeddings into Euclidean space with high codimension, one might expect to find flexibility among smooth solutions. On the other hand in problems which are formally determined, like embedding a surface into $\mathbb{R}^3$, the flexibility can only be expected at very low regularity. In these lectures I will focus on this latter case and in particular show how the same ideas can be applied to the Euler equations in fluid mechanics.

After a discussion of the proof of the Nash-Kuiper theorem, we show that - at least if we relax C^1 to Lipschitz -, the ideas can be applied in a general framework originally due to L. Tartar [12], which consists of a wave-plane analysis in the phase space. We then show that with this framework at hand, the celebrated results of Scheffer and Shnirelman [10, 11] concerning the existence of weak solutions to the Euler equations with compact support in space-time, can be recovered [4, 5].

Finally, we take another look at the Nash-Kuiper theorem and analyse whether the construction can be extended to produce more regular solutions [1, 2, 3]. The motivation for this comes from Onsager's theory of turbulence [9], which predicts the existence of certain weak solutions of the Euler equations.

Prerequisites Familiarity with basic PDE theory, conservation laws and differential geometry is assumed.

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