Chromatic structure in the fundamental groups of directed graphs

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Abstract

Directed graphs are basic combinatorial objects, and discrete homotopy theory studies graphs and directed graphs using methods from algebraic topology. This extended abstract surveys a series of ideas in which homotopy and homology theories for directed graphs are linked by Hurewicz and Seifert–van Kampen-type theorems. The central objects are the r-fundamental groupoids of Di–Ivanov–Mukoseev–Zhang and the magnitude–path spectral sequence of Asao. The former form a family of fundamental groupoids that is sensitive to the quasimetric structure of a directed graph, whereas the latter interpolates between magnitude homology, path homology, and reachability homology. The main message is that the two filtrations are linked by a one-dimensional Hurewicz theorem: the $(1,0)$-entry of the spectral sequence is the abelian shadow of the corresponding r-fundamental group. I explain this through concrete examples, a combinatorial model of the groupoids in terms of paths and C_r-homotopies, a Hurewicz theorem, and a Seifert–van Kampen theorem with applications to Mayer–Vietoris sequences. I also briefly describe the higher homotopy groups of directed graphs introduced by Li–Wu–Yau–Zhang, their Hurewicz theorem with singular cubical homology, and a more general cubical-set framework that suggests future higher-groupoid and higher Seifert–van Kampen theorems.

This note is based on joint work with Daisuke Kishimoto.

1 Directed graphs, maps, and the box product

In this section, we fix notation and recall the basic definitions. A *directed graph* G consists of a set of vertices $V(G)$ and a set of directed edges

$$E(G) \subset V(G) \times V(G) \setminus \Delta,$$

where $(x, y) \in E(G)$ means an edge from x to y. In this convention there are no loop edges and no multiple edges with the same direction. Thus an edge from x back to itself and two distinct arrows from x to y are excluded. The two opposite arrows $x \to y$ and $y \to x$, however, may coexist, because they are two different ordered pairs. A subgraph is defined in the evident way. The underlying undirected graph $\widehat{G}$ is obtained by forgetting directions; if both (x, y) and (y, x) are present, then $\widehat{G}$ has two edges between x and y.

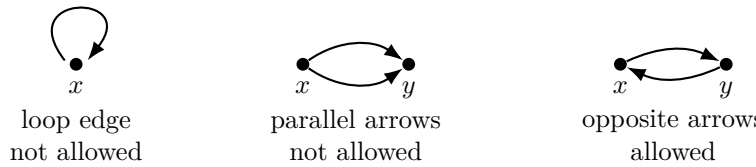


Figure 1: Our convention for directed graphs excludes loop edges and multiple edges with the same direction, but it allows a pair of opposite directed edges.

Maps of directed graphs are slightly more flexible than graph homomorphisms. A map $f\colon G \to H$ is a function on vertices such that every directed edge (x, y) of G is sent either to a directed edge $(f(x), f(y))$ of H or is collapsed to a vertex, that is, $f(x) = f(y)$. Thus maps are allowed to contract edges. This is important because homotopies of paths and maps in this theory are combinatorial analogues of deformation, and degeneracies must be allowed.

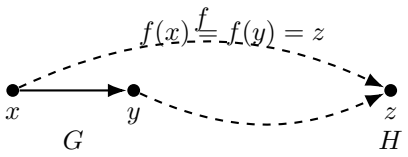


Figure 2: A map of directed graphs may collapse an edge to a vertex. This is why maps are not required to send every edge to an edge.

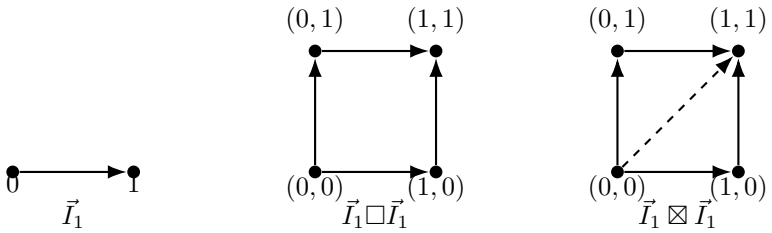


Figure 3: The directed interval, its box square, and its strong square. The strong product has the diagonal edge; the box product does not.

The *box product* $G \square H$ is the directed graph with

$$V(G \square H) = V(G) \times V(H),$$

and an edge $((g_0, h_0), (g_1, h_1))$ when exactly one coordinate moves along an edge and the other coordinate remains fixed. Thus either $g_0 = g_1$ and $(h_0, h_1) \in E(H)$, or $(g_0, g_1) \in E(G)$ and $h_0 = h_1$. The *strong product* $G \boxtimes H$ is obtained by adding the diagonal edges in which both coordinates move simultaneously. The strong product is the categorical product in DiGraph, while the box product is the cubical product used to build grid-shaped homotopies. Figure 3 shows the smallest example.

For a directed graph G, define the directed distance

$$d_G(x, y) = \min\{n \mid x = x_0 \to x_1 \to \cdots \to x_n = y\},$$

with value ∞ if no directed path exists. This is a *quasimetric*: $d_G(x, y)$ need not equal $d_G(y, x)$, but the triangle inequality holds. For a tuple $(x_0, \ldots, x_n)$, set

$$L(x_0, \ldots, x_n) = \sum_{i=0}^{n-1} d_G(x_i, x_{i+1}).$$

The length L is the bridge between the homotopical filtration and the homological spectral sequence below.

2 From path homology and fundamental groups to MPSS and filtered nerves

In this section, we introduce several algebraic-topological invariants of directed graphs and explain how they are related.

2.1 Homological invariants of a directed graph

Path homology, introduced by Grigor'yan–Lin–Muranov–Yau [5], is a homology theory of directed graphs that remembers the directions of edges. The path homology of a directed graph G is denoted by $\mathrm{PH}_*(G)$. Its construction is inspired by the simplicial homology theory: one starts with allowed paths in G and takes the alternating sum obtained by deleting vertices, but terms that are not allowed paths must be handled carefully. This produces a genuinely directed invariant. For instance, in degree one, path homology records directed cycles in a way that differs from the ordinary homology of the underlying one-dimensional simplicial complex. Path homology also has familiar formal properties, such as a Künneth formula for box products, and Mayer–Vietoris sequences under appropriate hypotheses.

Magnitude homology, introduced by Hepworth–Willerton [10], is another homological invariant of directed graphs, originally motivated by the categorification of graph magnitude. The magnitude homology of a directed graph G is denoted by $\mathrm{MH}^*_*(G)$; the extra grading records length. A third invariant is reachability homology, introduced by Hepworth–Roff [8]. The reachability homology of a directed graph G is denoted by $\mathrm{RH}_*(G)$. It is built from tuples of vertices that are mutually reachable in the directed sense, and it is the limiting homology to which the magnitude–path spectral sequence converges. Hepworth–Roff also studied the bigraded form of path homology and the MPSS from this viewpoint [9].

Asao [1] showed that magnitude homology and path homology are joined by a spectral sequence, called the *magnitude–path spectral sequence* (MPSS). For readers who do not usually use spectral sequences, the following informal picture is enough. A spectral sequence is a sequence of successive approximations of bigraded modules

$$E^1 \Rightarrow E^2 \Rightarrow E^3 \Rightarrow \cdots \Rightarrow E^\infty$$

to the homology of a filtered chain complex. Each page E^{r+1} is obtained as the homology of the preceding page E^r with respect to a differential d^r, and the limiting page E^∞ encodes the associated graded information of the desired homology.

For the MPSS, the filtration comes from directed length. The first pages have the form

$$E^1_{p,q}(G) \cong \mathrm{MH}^p_{p+q}(G), \qquad E^2_{p,0}(G) \cong \mathrm{PH}_p(G),$$

and the spectral sequence converges to reachability homology $\mathrm{RH}_*(G)$. Thus, the MPSS starts with magnitude homology, has path homology on the horizontal axis of the second page, and eventually converges to reachability homology. The term $E^r_{1,0}(G)$ may be regarded as a group of one-dimensional cycles after imposing all relations visible up to the r-th stage.

2.2 Homotopical invariants of a directed graph

Grigor'yan–Lin–Muranov–Yau also introduced a fundamental group $\pi_1(G, v)$ for a pointed directed graph (G, v) [6]. Its loops are directed-graph paths in the flexible sense explained below, and its equivalence relation is generated by C-homotopies. This construction is a directed analogue of the

fundamental group of a space: it records nonabelian information about loops before passing to homology.

Di–Ivanov–Mukoseev–Zhang later introduced a filtered homotopical model that parallels the MPSS [4]. Let $\mathrm{N}(G)$ be the nerve whose n-simplices are tuples $(x_0, \dots, x_n)$ of finite directed length

$$L(x_0, \dots, x_n) = \sum_{i=0}^{n-1} d_G(x_i, x_{i+1}).$$

The filtered pieces are

$$\mathrm{N}_n^r(G) = \{(x_0, \dots, x_n) \mid L(x_0, \dots, x_n) \le r\}, \qquad r = 0, 1, 2, \dots,$$

and they form a filtration

$$\mathrm{N}^0(G) \subset \mathrm{N}^1(G) \subset \mathrm{N}^2(G) \subset \cdots \subset \mathrm{N}(G).$$

Informally, $\mathrm{N}(G)$ remembers the directions but not the length bound, whereas the filtration records how much directed length is allowed.

For $1 \le r < \infty$, the *r-fundamental groupoid* is the edge-path groupoid of this filtered nerve:

$$\Pi_1^r(G) = \Pi_1(\mathrm{N}^r(G)),$$

and $\Pi_1^\infty(G) = \Pi_1(\mathrm{N}(G))$. These groupoids form a natural sequence

$$\Pi_1^1(G) \longrightarrow \Pi_1^2(G) \longrightarrow \Pi_1^3(G) \longrightarrow \cdots \longrightarrow \Pi_1^\infty(G),$$

which is the identity on objects and is surjective on hom-sets. At a basepoint v, we write

$$\pi_1^r(G, v) = \Pi_1^r(G)(v, v).$$

Two identifications explain why this is a genuine extension of older directed fundamental groups:

$$\Pi_1^1(G) \cong \text{the edge-path groupoid of } \widehat{G}, \qquad \pi_1^2(G, v) \cong \pi_1(G, v).$$

Thus Π_1^1 forgets the directions, while Π_1^2 recovers the GLMY fundamental group. Higher values of r retain information about the directed length scale at which loops become filled.

2.3 The Hurewicz connection

The homological and homotopical constructions above are not independent. The same length filtration produces both the MPSS and the tower of r-fundamental groupoids:

$$\text{homotopy:} \qquad \Pi_1^1(G) \longrightarrow \Pi_1^2(G) \longrightarrow \Pi_1^3(G) \longrightarrow \cdots$$

$$\text{homology:} \qquad E^1(G) \Longrightarrow E^2(G) \Longrightarrow E^3(G) \Longrightarrow \cdots.$$

The bridge is a Hurewicz theorem. In the classical GLMY case, the Hurewicz map identifies $\mathrm{PH}_1(G)$ with the abelianization of $\pi_1(G, v)$ under the usual connectedness assumption. The result discussed here refines this statement page by page: for each $r \ge 2$, the group $E^r_{1,0}(G)$ is the abelianization of $\pi_1^r(G, v)$, and the limiting statement identifies $\mathrm{RH}_1(G)$ with the abelianization of $\pi_1^\infty(G, v)$. The next section gives the combinatorial model that makes this statement visible.

3 A combinatorial description of the r-fundamental groupoids

The simplicial definition of $\Pi_1^r(G)$ is concise, but a path-based description is more useful in a talk. Let I_n be a directed line graph with vertices $0, 1, \ldots, n$, where each consecutive edge may point in either direction. A *path* in G is a map $I_n \to G$. Since the orientations in I_n may alternate, such a path is a zig-zag in G; a map $I_n \to G$ is an n-step path. A *directed path* is the special case in which the domain is $\vec{I}_n$, so all edges of the domain point forward:

$$0 \to 1 \to \cdots \to n.$$

Thus, a directed path always follows the arrows in G, whereas a general path may move forward and backward along directed edges by choosing orientations in the domain. Concatenation and reversal of paths are defined as usual.



Figure 4: A path in G may be a zig-zag, whereas a directed path follows arrows in the forward direction.

The elementary relation is controlled by a graph Γ_r. For $r \geq 2$, Γ_r has two directed paths of length r with the same initial and terminal vertices:

$$u_0 = v_0 \to u_1 \to \cdots \to u_{r-1} \to u_r = v_r,$$

$$v_0 = u_0 \to v_1 \to \cdots \to v_{r-1} \to v_r = u_r.$$

A map $\Gamma_r \to G$ therefore gives two directed paths of length r with the same endpoints. Informally, a C_r-homotopy permits the insertion or deletion, within a larger path, of a loop that goes once around such an image of Γ_r.

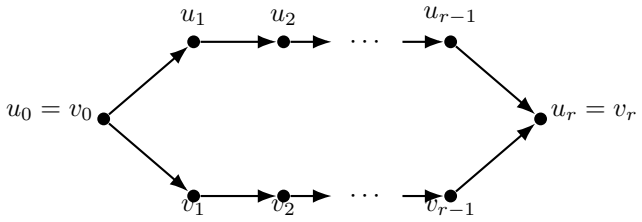


Figure 5: The graph Γ_r. A map $\Gamma_r \to G$ witnesses an elementary C_r-homotopy.

We specify the cases when $r = 0, 1, 2$. A C_0-homotopy inserts or deletes a one-step constant path. Thus, the constant path of length one at x is equivalent to the length-zero constant path at x; this is the reparameterization relation. A C_1-homotopy inserts or deletes a backtrack $x \to y \leftarrow x$. Thus it is the weakest equivalence relation under which the concatenation of a path with its inverse is identified with the constant path.

For the C_2-homotopy, one inserts or deletes the clockwise perimeter of the image of a map $\Gamma_2 \to G$. Such a map may be degenerate. Besides the backtrack already permitted at scale one, its basic nonconstant images are the directed triangle and the directed square shown in Figure 6; these are the elementary C_2 configurations described in [11, Figure 3].

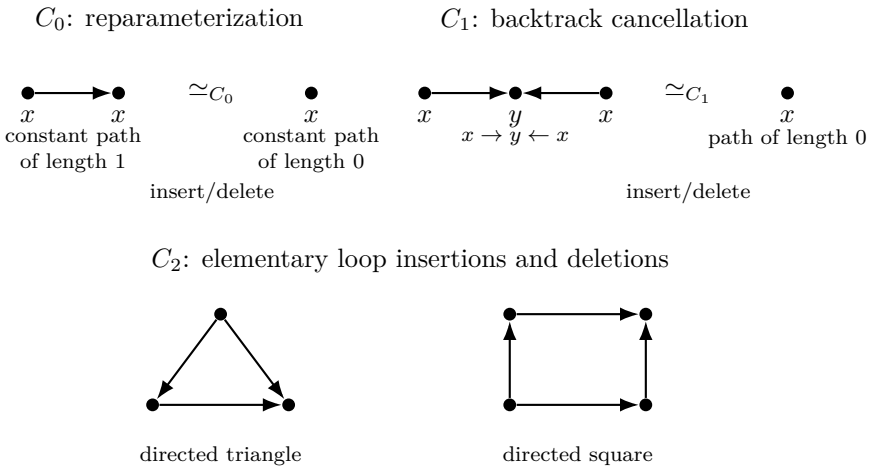


Figure 6: Elementary C_r-homotopy relations.

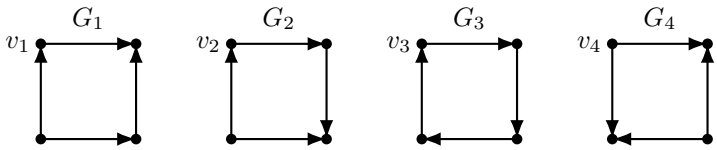
Precisely, for $1 \leq r < \infty$, one defines a groupoid $\widetilde{\Pi}_1^r(G)$ whose objects are the vertices of G and whose morphisms are C_r-homotopy classes of paths. Concatenation gives composition, and reversal gives inverses. The combinatorial theorem is

$$\widetilde{\Pi}_1^r(G) \cong \Pi_1^r(G).$$

For $r = \infty$, two paths are equivalent if they are C_r-homotopic for some finite r.

This model is the one to keep in mind. The parameter r records how long a directed "diamond" is allowed to be before its two sides are identified. Small r imposes only short local relations; large r sees larger directed shortcuts.

Example 3.1 (Four directed squares)**.** *Consider the following four orientations of a square, each based at its upper-left vertex v_i.*



Their r-fundamental groups at the chosen basepoints are

	$r=1$	$r=2$	$r=3$	$r=4$
G_1	$\mathbb{Z}$	0	0	0
G_2	$\mathbb{Z}$	$\mathbb{Z}$	0	0
G_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0
G_4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

This example, from [11, Example 2.7], shows that the whole sequence of r-fundamental groupoids can distinguish directed graphs that are not distinguished by looking at only one fixed value of r.

Example 3.2 (A loop killed at a prescribed scale)**.** *Let Γ_m itself be regarded as a directed graph, and take the basepoint $v = u_0 = v_0$. The loop that follows the upper path from v to $u_m = v_m$ and*

then returns along the lower path is visible at small scales, but it is killed by the C_m-relation. More precisely,

$$\pi_1^r(\Gamma_m, v) \cong \begin{cases} \mathbb{Z}, & r < m, \\ 0, & r \geq m. \end{cases}$$

This is the basic reason the sequence

$$\pi_1^1(G, v) \to \pi_1^2(G, v) \to \cdots$$

can contain more information than a single fundamental group: it records the directed length scale at which a loop becomes filled.

Example 3.3 (Products)**.** *For $r \geq 2$, the r-fundamental groupoid behaves well with products:*

$$\Pi_1^r(G \square H) \cong \Pi_1^r(G) \times \Pi_1^r(H) \cong \Pi_1^r(G \boxtimes H).$$

In the box square of Figure 3, the two directed paths from $(0,0)$ to $(1,1)$ differ by the standard directed square. The product theorem says that, from the viewpoint of $r \geq 2$ fundamental groupoids, the cubical and strong products have the same one-dimensional homotopy theory.

The Hurewicz theorem: statement and idea

The Hurewicz theorem compares this sequence of r-fundamental groups with the $(1,0)$-line of the MPSS.

Theorem 3.4 (Hurewicz theorem for π_1^r and MPSS)**.** *Let (G, v) be a connected pointed directed graph. There is a commutative ladder*

$$\begin{array}{ccccccc} \pi_1^2(G,v) & \longrightarrow & \pi_1^3(G,v) & \longrightarrow & \pi_1^4(G,v) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle h^2} & & \downarrow{\scriptstyle h^3} & & \downarrow{\scriptstyle h^4} & & \\ E^2_{1,0}(G) & \longrightarrow & E^3_{1,0}(G) & \longrightarrow & E^4_{1,0}(G) & \longrightarrow & \cdots, \end{array}$$

such that each h^r is the abelianization map. Moreover, there is a limiting Hurewicz map

$$h^\infty \colon \pi_1^\infty(G, v) \longrightarrow \mathrm{RH}_1(G)$$

which is likewise the abelianization map.

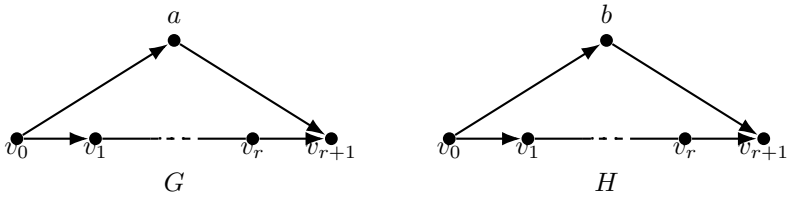
The idea of proof is conceptually simple. First, $\Pi_1^r(G)$ is the fundamental groupoid of the filtered nerve $\mathrm{N}^r(G)$. Thus, the classical Hurewicz map for $|\mathrm{N}^r(G)|$ gives a map from $\pi_1^r(G, v)$ to the first homology of the r-th filtered piece. Second, the MPSS is built from the same filtration. Its differentials impose homological versions of the C_r-relations. Therefore, after abelianization, the path-level relation measured by $\Pi_1^r(G)$ matches the homological relation measured by $E^r_{1,0}(G)$. At $r = 2$, the theorem recovers the GLMY Hurewicz theorem because $\pi_1^2(G, v) \cong \pi_1(G, v)$ and $E^2_{1,0}(G) \cong \mathrm{PH}_1(G)$.

One should not think of the theorem merely as an analogy. It says that MPSS has a genuine one-dimensional homotopy interpretation: its $(1,0)$-line is the abelianization of the corresponding stage of the fundamental group tower.

4 Seifert–van Kampen theorem and Mayer–Vietoris sequences

For ordinary spaces, the Seifert–van Kampen theorem computes the fundamental group of a union from the fundamental groups of the pieces. In the directed graph setting this is more delicate. Directed homotopies can use a filling whose edges lie partly in G and partly in H, even if neither piece contains the filling. The following example is the warning sign.

Example 4.1 (Failure of naive Seifert–van Kampen theorem)**.** *Fix $r \geq 2$ and consider the following pair of directed graphs. Each piece contains a loop based at v_0, but the directed filling that compares the two loops appears only after the pieces are united.*



In this example, $\pi_1^r(G, v_0)$ and $\pi_1^r(H, v_0)$ are generated by loops α_G and α_H, while $G \cap H$ is 1-homotopy equivalent to a point. If the usual pushout theorem held without hypotheses, the two generators would remain unrelated in the union. However, in $G \cup H$ one can form a C_2-homotopy using the vertex a from G and the vertex b from H, giving the relation $\alpha_G \alpha_H^{-1} = 1$. Thus, the naive pushout is too large.

The solution of this is a separability condition. A degenerate Γ_r in $G \cup H$ is the image of a map $\Gamma_r \to G \cup H$, allowing collapses and identifications. A pair (G, H) is *r-separable* if every degenerate Γ_r in the union can be subdivided into degenerate Γ_r's each lying entirely in G or entirely in H. In other words, every elementary directed filling in the union can be decomposed into elementary fillings in the pieces. This is exactly the condition needed to prevent new mixed C_r-relations.

Theorem 4.2 (Seifert–van Kampen theorem for r-fundamental groupoids)**.** *Let (G, H) be an r-separable pair of directed graphs, with $1 \leq r \leq \infty$. Then the square*

$$\begin{array}{ccc} \Pi_1^r(G \cap H) & \longrightarrow & \Pi_1^r(G) \\ \downarrow & & \downarrow \\ \Pi_1^r(H) & \longrightarrow & \Pi_1^r(G \cup H) \end{array}$$

is a pushout of groupoids.

The proof uses the combinatorial description above. One constructs the algebraic pushout groupoid and the natural functor to $\Pi_1^r(G \cup H)$. Surjectivity comes from the fact that every path in the union is a concatenation of edges from the two pieces. Injectivity is where r-separability enters: every relation generated by a C_r-homotopy in the union can be subdivided into relations already present in one piece or the other. Thus no mixed relation is missing from the pushout.

Passing to based groups gives the corresponding group-level statement when the intersection is connected. Combining this result with the Hurewicz theorem gives a Mayer–Vietoris sequence for the $(1, 0)$-line of the MPSS. If (G, H) is r-separable, $2 \leq r < \infty$, and G, H, and $G \cap H$ are connected, then

$$E^r_{1,0}(G \cap H) \xrightarrow{((i_G)_*, (i_H)_*)} E^r_{1,0}(G) \oplus E^r_{1,0}(H) \xrightarrow{(j_G)_* - (j_H)_*} E^r_{1,0}(G \cup H) \to 0$$

is exact. This is one of the main applications: the homotopical pushout theorem yields a homological exact sequence for MPSS terms.

There is also a pushout version. Given a pushout of directed graphs

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & G \\ \downarrow & & \downarrow \\ H & \longrightarrow & G \cup_A H, \end{array}$$

one defines an r-cofibration $i\colon A \to G$, interpolating between the cofibrations used in path homology and the long cofibrations used in reachability homology. If i is an r-cofibration, then for all $s \geq r$ the corresponding square of Π_1^s is a pushout. Consequently, for finite $s \geq r$, one obtains exact Mayer–Vietoris sequences for $E^s_{1,0}$. This gives a partial answer to the question of whether cofibration-type hypotheses imply Mayer–Vietoris sequences beyond the E^1 and E^2 pages.

5 Higher homotopy groups and singular cubical homology

The story above is one-dimensional: it concerns fundamental groupoids and the $(1,0)$-line of a spectral sequence. A natural next question is what higher homotopy theory of directed graphs should look like.

Li-Wu-Yau-Zhang introduced higher homotopy groups $\bar{\pi}_n(G, v)$ of a pointed directed graph [12]. Their representatives are spherical n-dimensional grids. A grid is a map

$$I_1 \square I_2 \square \cdots \square I_n \longrightarrow G,$$

where each I_k is a directed line graph. Spherical means that the boundary of the box is sent to the basepoint v. Homotopies are generated by subdivisions and elementary grid homotopies. For $n = 1$, this recovers the GLMY fundamental group:

$$\bar{\pi}_1(G, v) \cong \pi_1(G, v).$$

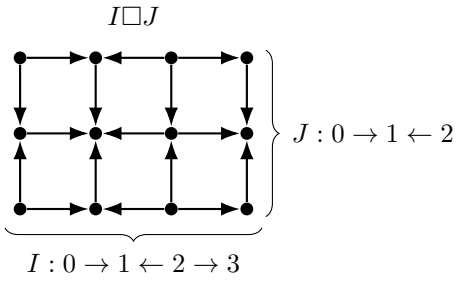


Figure 7: A two-dimensional grid directed graph built as the box product of two zig-zag directed line graphs.

The natural homology target for these higher groups is not path homology in all degrees, but singular cubical homology [7]. Let $Q_n(G)$ be the free abelian group generated by maps

$$(\vec{I})^{\square n} \longrightarrow G,$$

modulo degenerate grids, with the usual cubical boundary. The resulting homology is denoted $\mathrm{QH}_*(G)$. There is a discrete Hurewicz map [13]

$$h_*\colon \bar{\pi}_n(G,v) \longrightarrow \mathrm{QH}_n(G),$$

which sends a spherical grid to the signed sum of its elementary directed n-cubes.

Theorem 5.1 (Higher Hurewicz theorem for directed graphs)**.** *Let (G,v) be a pointed directed graph. If G is $(n-1)$-connected in the cubical-nerve sense, then*

$$h_*\colon \bar{\pi}_n(G,v) \longrightarrow \mathrm{QH}_n(G)$$

is the abelianization for $n=1$ and an isomorphism for $n \geq 2$.

This theorem extends the one-dimensional GLMY Hurewicz theorem, since the first singular cubical homology agrees with the first path homology. It also generalizes the Hurewicz theorem for undirected graphs proved in [2]. Finally, it explains why singular cubical homology is the correct higher-dimensional target: it is built from exactly the same cubical grids that represent higher directed homotopy.

The proof comes from a more general result for cubical sets. One defines discrete homotopy groups $\pi_n^\delta(K,v)$ for pointed cubical sets. For quasisymmetric cubical sets, these groups are naturally isomorphic to the ordinary homotopy groups of their geometric realizations:

$$\pi_*^\delta(K,v) \cong \pi_*(|K|,v).$$

There is a Hurewicz theorem for these discrete homotopy groups: if K is $(n-1)$-connected, then

$$\pi_n^\delta(K,v) \longrightarrow H_n(K)$$

is abelianization for $n=1$ and an isomorphism for $n \geq 2$. The directed-graph theorem follows by applying this cubical theorem to the cubical nerve $\mathrm{N}_1 G$ and identifying

$$\pi_n^\delta(\mathrm{N}_1 G, v) \cong \bar{\pi}_n(G,v), \qquad H_n(\mathrm{N}_1 G) \cong \mathrm{QH}_n(G).$$

6 Future directions

The results above suggest that directed graphs should have a richer nonabelian homotopy theory than a sequence of fundamental groupoids alone. I conclude by emphasizing three directions.

Higher groupoid models. The r-fundamental groupoid models one-dimensional directed homotopy at a fixed length scale. The LWYZ higher homotopy groups show that higher directed spheres exist and admit a Hurewicz theorem. A natural next step is to construct higher-groupoid or higher-categorical models of a directed graph whose one-dimensional truncations recover $\Pi_1^r(G)$ and whose higher homotopy groups recover the grid-based groups. Such a model should remember not only groups but also compositions of higher directed homotopies.

Higher Seifert–van Kampen theorem. The current Seifert–van Kampen theorem is a pushout theorem for Π_1^r. A higher version should describe the homotopy type of a union by a higher pushout, under a suitable higher-separability condition requiring every elementary directed n-dimensional filling in the union to decompose into fillings in the pieces. This would be the directed-graph analogue of the principle that homotopy colimits, rather than ordinary colimits, control gluing.

Applications to Mayer–Vietoris sequences. The one-dimensional Seifert–van Kampen theorem gives exact Mayer–Vietoris sequences for $E^r_{1,0}$ of the MPSS. A higher-groupoid theorem should produce higher Mayer–Vietoris sequences or exact couples for other MPSS entries, possibly explaining more of the differentials in the spectral sequence through explicit directed homotopies. In this sense, the long-term goal is not merely to compute invariants, but to understand the geometric meaning of the MPSS page by page.

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