GEOMETRY ON SPACES OF METRICS

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1. Introduction

This report provides a comprehensive overview of the author's recent research [35, 36, 30, 37] on the geometry of spaces of metrics on topological spaces, which will be compared with some well-known results on the one of spaces of continuous functions. Given a topological space X, we denote by $\mathrm{C}(X)$ the space of real-valued bounded continuous functions on X equipped with the topology of uniform convergence. The space $\mathrm{C}(X)$ is a Banach space with the sup-norm $\|\cdot\|$: for any $f \in \mathrm{C}(X)$,

$$\|f\| = \sup\{|f(x)| \mid x \in X\},$$

and it admits the sup-metric. Let $\mathrm{PM}(X)$ and $\mathrm{AM}(X)$ be the spaces consisting of bounded continuous pseudometrics and bounded admissible metrics on X, that is, metrics inducing its topology, respectively, both of which are subspaces of $\mathrm{C}(X^2)$. As is easily observed, $\mathrm{PM}(X)$ and $\mathrm{AM}(X)$ are cones in the Banach space $\mathrm{C}(X^2)$. In this report, we shall investigate the topological and metric structures of these function spaces.

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2. Recognizing topologies on spaces of metrics

Infinite-dimensional topology has developed together with the study of function spaces. Recognizing topologies on function spaces is one of the most important issues in that area. It is known that every infinite-dimensional Banach space is homeomorphic to a Hilbert space of the same density (cf. [1, 31, 52, 53]). In particular, we have the following:

Theorem 2.1. *For an infinite space X, the Banach space $\mathrm{C}(X)$ is homeomorphic to a Hilbert space of the same density.*

Research in this direction has advanced, and the topologies on many function spaces have been identified. In this section, we will decide the topological types of spaces consisting of metrics.

2.1. **The Borel complexity and the complete metrizability of spaces of metrics.** In the recognition of infinite-dimensional manifolds among function spaces, finding their Borel complexity plays a significant role. In this subsection, we shall study the Borel complexity and the complete metrizability of spaces of metrics, which summarizes the paper [36]. Recall that for a topological space X, the sup-metric is complete on $\mathrm{C}(X^2)$. The subspace $\mathrm{PM}(X)$ is closed in $\mathrm{C}(X^2)$, refer to the proof of [29, Lemma 5.1], so it is also complete. Y. Ishiki [27] showed that the space consisting of admissible metrics (not only bounded admissible metrics) on X with the topology of uniform convergence, that is, an extended space of $\mathrm{AM}(X)$, is a Baire space, and hence its open subspace $\mathrm{AM}(X)$ is also Baire. It is well-known that completely metrizable spaces are Baire (the Baire Category Theorem, cf. [44, Theorem 2.5.1]), however the inverse does not hold. For example, the space $\mathbb{R}\times(0,\infty)\cup\mathbb{Q}\times\{0\}$, where $\mathbb{R}$ is the real line and $\mathbb{Q}$ is the set of rationals, is Baire but not completely metrizable. Whether is $\mathrm{AM}(X)$ always completely metrizable or not?

Given a metrizable space Y, let $\mathfrak{A}_0(Y)$ be the family of open sets in Y, and let $\mathfrak{M}_0(Y)$ be the one of closed sets in Y. For a natural number n, define inductively the collections

$$\mathfrak{A}_n(Y)=\left\{\bigcup_{k<\aleph_0}A_k\;\middle|\;A_k\in\bigcup_{m<n}\mathfrak{M}_m(Y)\right\}\text{ and }\mathfrak{M}_n(Y)=\left\{\bigcap_{k<\aleph_0}A_k\;\middle|\;A_k\in\bigcup_{m<n}\mathfrak{A}_m(Y)\right\}.$$

Denote by $\mathfrak{A}_n$ (respectively, $\mathfrak{M}_n$) a class of spaces X such that X is belonging to $\mathfrak{A}_n(Y)$ (respectively, $\mathfrak{M}_n(Y)$) for every metrizable space Y which contains X as a subspace. We call $\mathfrak{A}_n$ and $\mathfrak{M}_n$ to be the *absolute Borel classes*. Recall the following:

- $\mathfrak{A}_0=\{\emptyset\}$ and $\mathfrak{M}_0$ is the class of compact metrizable spaces.
- $\mathfrak{M}_1$ is consisting of completely metrizable spaces (cf. [44, Theorem 2.5.2]).
- $\mathfrak{A}_1$ is consisting of σ-locally compact metrizable spaces (σ-compact metrizable spaces in the separable case) (cf. [48]).

The Borel complexity of spaces of metrics are restricted by the one of their underlying spaces as follows:

Theorem 2.2. *Let X be a metrizable space and $n\geq 1$. If $\mathrm{AM}(X)$ is in $\mathfrak{A}_n$ (respectively, $\mathfrak{M}_n$), then X is in $\mathfrak{M}_n$ (respectively, $\mathfrak{A}_n$).*

By $\mathrm{M}(X)$, denote the space of bounded continuous metrics on a space X. Note that $\mathrm{M}(X)$ coincides with $\mathrm{AM}(X)$ when X is compact, however that they are not coincident in general. For an example, define $d\in\mathrm{M}([0,1))$ by

$$d(x,y)=\min\{|x-y|,1-|x-y|\},$$

so $d\notin\mathrm{AM}([0,1))$. It follows from the similar argument as the proof of [29, Lemma 5.1] that $\mathrm{M}(X)$ is a G_δ set in $\mathrm{C}(X^2)$ if X is a σ-compact metrizable space, and that $\mathrm{AM}(X)$ is a G_δ set in $\mathrm{C}(X^2)$ if X is a separable locally compact metrizable space. Moreover, we have the following:

Proposition 2.3. *If X is a σ-compact metrizable space, then $\mathrm{AM}(X)$ is G_δ in $\mathrm{C}(X^2)$.*

Proof. Since X is σ-compact, we can write $X = \bigcup_{m<\aleph_0} X_m$, where X_m is a compact subset. Fix any admissible metric $d_X \in \mathrm{AM}(X)$. For $m, n < \aleph_0$, set

$$U(m,n) = \{d \in \mathrm{M}(X) \mid d \text{ fulfills the property } (\star)\}.$$

($\star$) There is $\delta > 0$ such that for all $x \in X_m$ and $y \in X$ with $d(x,y) < \delta$, $d_X(x,y) + \delta < 2^{-n}$.

Then we get that

$$\mathrm{AM}(X) = \bigcap_{m<\aleph_0} \bigcap_{n<\aleph_0} U(m,n),$$

and that each $U(m,n)$ is open in $\mathrm{M}(X)$. □

As a corollary of Theorem 2.2 and Proposition 2.3, we can give an equivalent condition on the complete metrizability of $\mathrm{AM}(X)$ as follows:

Corollary 2.4. *Assume that X is a separable metrizable space. Then X is σ-compact if and only if $\mathrm{AM}(X)$ is completely metrizable.*

The following example contains a good technique to prove Theorem 2.2.

Example 2.5. Let $Q = [0,1] \cap \mathbb{Q}$ and $P = [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$. Suppose that X is the topological sum $P \oplus (0,1]$, so $\mathrm{AM}(X)$ is not completely metrizable. Indeed, for each $q \in Q$, define $d_q \in \mathrm{AM}(X)$ by

$$d_q(x,y) = \begin{cases} |x-y| & \text{if } (x,y) \in P^2 \text{ or } (0,1]^2, \\ |x-q| + y & \text{if } x \in P \text{ and } y \in (0,1], \\ x + |y-q| & \text{if } x \in (0,1] \text{ and } y \in P. \end{cases}$$

Then the set

$$\{d_q \in \mathrm{AM}(X) \mid p \in Q\}$$

is a closed subset isometric to Q, which is not completely metrizable.

F. Hausdorff [18] showed the metric extension theorem, which states that for every metrizable space X and its closed set A, any $d \in \mathrm{AM}(A)$ can be extended over X. We can get its pseudometric version, where their sup-norms are preserved.

Theorem 2.6. *Suppose that X is a metrizable space and $A \subset X$ is a closed subset. For each $d \in \mathrm{PM}(A)$, there exists $\bar{d} \in \mathrm{PM}(X)$ such that $\bar{d}|_{A^2} = d$ and $\|\bar{d}\| = \|d\|$. Moreover, when $d \in \mathrm{AM}(A)$, we can choose its extension $\bar{d} \in \mathrm{AM}(X)$.*

This extension theorem has been developed due to the efforts by C. Bessaga [11, 7], T. Banakh [5, 7], O. Pikhurko [43] and M. Zarichnyi [62] (see also [9] for the variable domains case and [26] for the unbounded case), and we can obtain the following extensor on spaces of metrics.

Theorem 2.7. *Let X be a metrizable space and $A \subset X$ be a closed set. Then there exists an isometric embedding $E : \mathrm{PM}(A) \to \mathrm{PM}(Y)$ such that $E(d)|_{A^2} = d$ for all pseudometrics $d \in \mathrm{PM}(A)$ and $E(\mathrm{AM}(A)) \subset \mathrm{AM}(X)$.*

These results will play key roles in constructing "nice" metrics instead of Urysohn's lemma throughout the report.

Observe the following basic properties on absolute Borel classes.

Remark 2.8. The following hold.

(i) All closed subspaces of certain metrizable space in an absolute Borel class also belong to it.

(ii) If a metrizable space is a union of a locally finite family consisting of subspaces belonging to an absolute Borel class, then it is also belonging to the same one (cf. [16, 4.5.8 (a)]).
(iii) For a metrizable space X, $X \in \mathfrak{A}_n$, $n \geq 2$, (respectively, $X \in \mathfrak{M}_n$, $n \geq 1$,) if and only if $X \in \mathfrak{A}_n(Y)$ (respectively, $\mathfrak{M}_n(Y)$) for some completely metrizable space Y (cf. [45, Theorem 5.11.2]). If $X \notin \mathfrak{A}_1$, then there is a completely metrizable space Y such that $X \notin \mathfrak{A}_1(Y)$.

For spaces $A \subset X$, denote the interior and closure of A in X by $\mathrm{int}_X A$ and $\mathrm{cl}_X A$, respectively. Now we will provide a proof of Theorem 2.2.

Proof of Theorem 2.2. We only show the case that $\mathrm{AM}(X) \in \mathfrak{M}_n$. Suppose that X is not belonging to $\mathfrak{A}_n$, and take a bounded complete metric space $Y = (Y, \rho)$ so that $X \notin \mathfrak{A}_n(Y)$ by (iii) of Remark 2.8. Put $B = X \setminus \mathrm{int}_Y X$ and $Z = \mathrm{cl}_Y B \setminus X$, so $B \notin \mathfrak{A}_n(Y)$, $Z \notin \mathfrak{M}_n$ and $\mathrm{cl}_Y B = B \cup Z$. According to (ii) of Remark 2.8, we can find a locally finite open cover $\mathcal{U}$ of $\mathrm{cl}_Y B$ such that $Z \cap \mathrm{cl}_Y U \notin \mathfrak{M}_n$, $Z \cap U' \neq \emptyset$ and $\mathrm{cl}_Y U \cap \mathrm{cl}_Y U' = \emptyset$ for some $U, U' \in \mathcal{U}$. Fix any point $a \in Z \cap U'$ and choose a sequence $\{a_n\} \subset X \cap U'$ converging to the point a. Then $A = \{a_n\}$ and $B' = B \cap \mathrm{cl}_Y U$ are closed in X, and $Z' = Z \cap \mathrm{cl}_Y U \notin \mathfrak{M}_n$. Define $i : Z' \to \mathrm{AM}(A \oplus B')$ by

$$i(z)(x,y) = \begin{cases} \rho(x,y) & \text{if } (x,y) \in A^2 \text{ or } B'^2, \\ \rho(x,a) + \rho(y,z) & \text{if } x \in A \text{ and } y \in B', \\ \rho(x,z) + \rho(y,a) & \text{if } x \in B' \text{ and } y \in A. \end{cases}$$

Taking an extending map $E : \mathrm{AM}(A \oplus B') \to \mathrm{AM}(X)$ according to Theorem 2.7, we have that the composition $E \circ i$ is a closed embedding, which contradicts to (i) of Remark 2.8. □

2.2. **The topological types of spaces of metrics.** This subsection is devoted to determining the topological types of spaces of metrics, which provides a summary of the paper [35]. A topological space X is said to be *generalized compact* if every open cover of X has a subcover whose cardinality is less than the density of X. When X is a separable metrizable space, the compactness coincides with the generalized compactness. For an infinite cardinal κ, let $\ell_2(\kappa)$ be the Hilbert space of density κ (see [44, p. 16]):

$$\ell_2(\kappa) = \left\{ x \in \mathbb{R}^\kappa \;\middle|\; \sum_{\gamma<\kappa} x(\gamma)^2 < \infty \right\},$$

and denote $\ell_2(\aleph_0)$ by ℓ_2 for simplicity. The symbol ℓ_∞ stands for the space of bounded sequences with the sup-norm, which is homeomorphic to $\ell_2(2^{\aleph_0})$. We can classify the topologies of $\mathrm{PM}(X)$, $\mathrm{M}(X)$ and $\mathrm{AM}(X)$ as follows:

Theorem 2.9. *Let X be a metrizable space of density κ. Then $\mathrm{PM}(X)$ is homeomorphic to*
(i) $[0,1)^{\kappa(\kappa-1)/2}$ *if X is finite;*
(ii) $\ell_2(2^{<\kappa})$ *if X is infinite and generalized compact;*
(iii) $\ell_2(2^\kappa)$ *if X is not generalized compact.*

Theorem 2.10. *Suppose that X is an infinite σ-compact metrizable space. Then the spaces $\mathrm{M}(X)$ and $\mathrm{AM}(X)$ are homeomorphic to*
(i) ℓ_2 *if X is compact;*
(ii) ℓ_∞ *if X is not compact.*

Using the following criterion due to T. Dobrowolski and H. Toruńczyk [15, Theorem 2], and T. Banakh and R. Cauty [8, Theorem 2], we will prove Theorems 2.9 and 2.10, where a sufficient condition is given for G_δ convex sets in Fréchet spaces to be homeomorphic to Hilbert spaces.

Theorem 2.11. *Let C be a G_δ convex set in a Fréchet space. If the closure of C is not locally compact, then C is homeomorphic to a Hilbert space. Especially, if C is not separable, then C is homeomorphic to a non-separable Hilbert space.*

It is shown that $\mathrm{PM}(X)$ is the closure of $\mathrm{M}(X)$ and $\mathrm{AM}(X)$ in $\mathrm{C}(X^2)$.

Proposition 2.12. *For every metrizable space X, $\mathrm{AM}(X)$ is dense in $\mathrm{PM}(X)$.*

Proof. For each $d \in \mathrm{PM}(X)$ and each $\epsilon > 0$, we will find $\rho \in \mathrm{AM}(X)$ with $\|d - \rho\| \leq \epsilon$. Fix any admissible metric $d_X \in \mathrm{AM}(X)$ of norm $\leq \epsilon$, and define $\rho \in \mathrm{AM}(X)$ by

$$\rho(x, y) = \max\{d(x, y), d_X(x, y)\}$$

for every $(x, y) \in X^2$, which is the desired. □

We will estimate the densities of $\mathrm{PM}(X)$, $\mathrm{M}(X)$ and $\mathrm{AM}(X)$.

Lemma 2.13. *If a metrizable space X contains a closed discrete subset of cardinality κ, then the density of $\mathrm{PM}(X)$ is greater than or equal to 2^κ.*

Proof. Take a closed discrete subset $D = \{x_\gamma \in X \mid \gamma < \kappa\}$ and fix a point $x_0 \in D$. We can define a pseudometric $d_a \in \mathrm{PM}(D)$ for each sequence $a = (a_\gamma)_{\gamma<\kappa} \in \{0, 1\}^\kappa$ by

(1) $d_a(x_0, x_0) = 0$ and $d_a(x_\gamma, x_0) = d_a(x_0, x_\gamma) = a_\gamma$ for any $0 < \gamma < \kappa$,
(2) $d_a(x_\gamma, x_\lambda) = |d_a(x_\gamma, x_0) - d_a(x_\lambda, x_0)|$ for any $0 < \gamma \leq \lambda < \kappa$.

Note that the subset

$$\{d_a \in \mathrm{PM}(D) \mid a \in \{0, 1\}^\kappa\}$$

is closed and discrete in $\mathrm{PM}(D)$, and hence the density of $\mathrm{PM}(D)$ is greater than or equal to 2^κ. By virtue of the metric extension theorem 2.6, the restriction

$$\mathrm{PM}(X) \ni d \mapsto d|_{D^2} \in \mathrm{PM}(D)$$

is surjective, which implies that the density of $\mathrm{PM}(X)$ is also greater than or equal to 2^κ. □

In the papers [10, 14], cardinal functions of topological spaces and their function spaces are estimated by the generalized compactness of them. Combining the above lemma with several results of those papers, we have the following:

Proposition 2.14. *Suppose that X is a metrizable space of density κ. The density of $\mathrm{PM}(X)$ is equal to $2^{<\kappa}$ if X is infinite and generalized compact, and is equal to 2^κ if X is not generalized compact.*

Now Theorems 2.9 and 2.10 are shown.

Proof of Theorem 2.9. We will prove only the infinite cases. Combining the latter part of Theorem 2.11 with Proposition 2.14, we can establish (iii) immediately.

Similarly, to show (ii), it remains to verify that $\mathrm{PM}(X)$ is not locally compact. Fix any pseudometric $d \in \mathrm{PM}(X)$ and any neighborhood U of d. Take $\epsilon > 0$ such that if $\|d - d'\| \leq \epsilon$, then $d' \in U$. Since X is infinite and generalized compact, we can choose a point $x \in X$ all of whose neighborhoods are of density $\geq \kappa$ according to [10, Lemma 11]. Hence for any neighborhood V of (x, x) in X^2, there is $y \in X$ such that $x \neq y$ and $(x, y) \in V$. Applying Theorem 2.6, we can obtain a pseudometric $e \in \mathrm{PM}(X)$ so that $e(x, y) = \epsilon$ and $\|e\| = \epsilon$. Let $\rho \in \mathrm{PM}(X)$ be defined by $\rho = d + e$. Then $\|d - \rho\| \leq \epsilon$ and $\rho(x, y) \geq \rho(x, x) + \epsilon$. By virtue of Arzerà-Ascoli's Theorem (refer to [41, Lemma 45.2]), the space $\mathrm{PM}(X)$ is not locally compact. □

Proof of Theorem 2.10. Recall that $\mathrm{PM}(X)$ is the closure of $\mathrm{M}(X)$ and $\mathrm{AM}(X)$ by Proposition 2.12. As is seem in the above proof, $\mathrm{PM}(X)$ is not locally compact. Applying Theorem 2.11, we can prove Theorem 2.10. □

2.3. **Problems.** The complete topological classification of $\mathrm{PM}(X)$ is obtained. In fact, they are homeomorphic to a Hilbert space. In infinite-dimensional topology, it is known that for each absolute Borel class, there is its universal space, which is a dense subset of a Hilbert space. Their topological properties and characterizations have been studied, refer to [13, 60, 12, 47, 40, 33, 34] and [45, Section 5]. Moreover, there are a lot of research on function spaces that are homeomorphic to such spaces. In [23], the denseness and Borel hierarchy of function spaces consisting of several kinds of metrics are investigated.

Problem 2.15. *For a metrizable space X, which subspaces in $\mathrm{PM}(X)$ are homeomorphic to universal spaces for absolute Borel classes?*

3. Isometric universality of spaces of metrics

In this section, we shall study the metric structures of spaces of metrics, which is based on a joint work with Y. Ishiki [30]. Given a class $\mathcal{C}$ of metric spaces, we say that a metric space $Z = (Z, d)$ is *universal* for $\mathcal{C}$ if every $X = (X, \rho) \in \mathcal{C}$ can be isometrically embedded into Z. For an infinite cardinal κ, let $\mathcal{S}_\kappa$ be the class of bounded metric spaces of density $\leq \kappa$, and let $\mathcal{S} = \mathcal{S}_{\aleph_0}$ be the one of bounded separable metric spaces. Denote by $\mathcal{T}$ the class of totally bounded metric spaces. Note that $\mathcal{T} \subset \mathcal{S}$. There are many results on the universality of function spaces. S. Banach and S. Mazur proved the following (see [4, Theorem 10 in Chapter XI]):

Theorem 3.1 (the Banach-Mazur theorem)**.** *The space $\mathrm{C}([0,1])$ is universal for $\mathcal{S}$.*

This section is devoted to giving variants of this famous theorem. Especially, we have the following:

Theorem 3.2. *The space $\mathrm{AM}([0,1])$ is universal for $\mathcal{T}$.*

3.1. **Banach-Mazur type theorems.** The following is a key lemma, which can be proven by the method based on Example 2.5.

Lemma 3.3. *Let $X = (X, d)$ be a bounded metric space. Then there exists an isometric embedding from X into the space $\mathrm{AM}(X \oplus \{*\})$.*

Proof. We shall define $I : X \to \mathrm{AM}(X \oplus \{*\})$ as follows: for each $z \in X$,

$$I(z)(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in X; \\ d(x,z) + 1 & \text{if } x \in X \text{ and } y = *, \\ d(y,z) + 1 & \text{if } x = * \text{ and } y \in X, \\ 0 & \text{if } x = y = *. \end{cases}$$

Then the map I is an isometric embedding. □

We will investigate the universality of spaces of metrics on topological universal spaces.

Theorem 3.4. *Let $\mathcal{C}$ be a class of bounded metric spaces and Z be a metrizable space. Assume that the following condition is satisfied:*

(†) *For each $X = (X, d) \in \mathcal{C}$, there is a bounded metric space $Y = (Y, \rho)$ containing X as a subspace such that $\rho|_{X^2} = d$, and there exists a closed topological embedding from $Y \oplus \{*\}$ to Z.*

Then the space $\mathrm{AM}(Z)$ is universal for $\mathcal{C}$.

Proof. Take an extension $Y = (Y, \rho)$ of $X = (X, d) \in \mathcal{C}$ as in (†), and let $i : X \to Y$ be the inclusion map. It follows from Lemma 3.3 that there exists an isometric embedding $I : Y \to \mathrm{AM}(Y \oplus \{*\})$. By virtue of Theorem 2.7, we can obtain an isometric extensor $E : \mathrm{AM}(Y \oplus \{*\}) \to \mathrm{AM}(Z)$. Consequently, the composition $J = E \circ I \circ i$ is an isometric embedding from X to $\mathrm{AM}(Z)$. □

Let κ be an infinite cardinal. For every completely metrizable space Z of density $\leq \kappa$, there exists a topological embedding $f : Z \to \ell_2(\kappa)$ such that the image $f(Z)$ is closed in $\ell_2(\kappa)$, see [44, Theorem 6.2.4]. We denote by $\mathbf{Q} = [0,1]^{\aleph_0}$ the Hilbert cube. According to the method of the proof of the Urysohn metrization theorem (see [41, Theorem 34.1]), $\mathbf{Q}$ admits a topological embedding from any compact metrizable space.

Theorem 3.5. *The following statements are true.*

(1) *For every infinite cardinal κ, the space $\mathrm{AM}(\ell_2(\kappa))$ is universal for $\mathcal{S}_\kappa$.*
(2) *The space $\mathrm{AM}(\mathbf{Q})$ is universal for $\mathcal{T}$.*

Proof. For every bounded metric space $X = (X, d)$, take the completion $\overline{X} = (\overline{X}, \bar{d})$ of X. Then the extended space $\overline{X} \oplus \{*\}$ is a completely metrizable space of density $\leq \kappa$ if $X \in \mathcal{S}_\kappa$, and is a compact metrizable space if $X \in \mathcal{T}$. Therfore the condition (†) in Theorem 3.4 is fulfilled. □

As is easily observed, we have the following.

Lemma 3.6. *Let X be a metrizable space. For every $d \in \mathrm{PM}(X)$ and every $\rho \in \mathrm{AM}(X)$, their sum $d + \rho \in \mathrm{AM}(X)$.*

Applying the above lemma, we can prove the following:

Lemma 3.7. *Let $\mathcal{C}$ be a class of metric spaces, and let X be a metrizable space. If $\mathrm{PM}(X)$ is universal for $\mathcal{C}$, then so is $\mathrm{AM}(X)$.*

Proof. Fixing any admissible metric $d \in \mathrm{AM}(X)$, we can construct an isometric embedding $I_d : \mathrm{PM}(X) \to \mathrm{AM}(X)$ defined by $I_d(\rho) = \rho + d$ due to Lemma 3.6. □

Let X and Y be metrizable spaces, and $f : X \to Y$ be a continuous map. Then we can define an operator $J_f : \mathrm{PM}(Y) \to \mathrm{PM}(X)$ by $J_f(d)(x, y) = d(f(x), f(y))$.

Lemma 3.8. *Let X and Y be metrizable spaces, and let $f : X \to Y$ be a continuous surjection. If $\mathrm{AM}(Y)$ is universal for a class $\mathcal{C}$ of metric spaces, then so is $\mathrm{AM}(X)$.*

Proof. As is easily observed, the map $J_f : \mathrm{PM}(Y) \to \mathrm{PM}(X)$ is an isometric embedding. Let $I_d : \mathrm{PM}(X) \to \mathrm{AM}(X)$ be an isometric embedding as in Lemma 3.7 for any fixed $d \in \mathrm{AM}(X)$, so we can construct the desired isometric embedding $I_d \circ J_f|_{\mathrm{AM}(Y)}$. □

Combining Theorem 3.5(2) with the above lemma, we have the following:

Theorem 3.9. *Let X be a metrizable space such that there exists a continuous surjection from X to the Hilbert cube $\mathbf{Q}$. Then $\mathrm{AM}(X)$ is universal for $\mathcal{T}$.*

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. There exists a continuous surjection from $[0,1]$ to $\mathbf{Q}$ (the Hahn–Mazurkiewicz theorem, see [61, Theorem 31.5]). Thus the closed unit interval $[0,1]$ satisfies the assumption of Theorem 3.9. □

It is said that a metrizable space X is an *absolute retract*, briefly AR, provided that there is a retraction $r : Y \to X$ of an arbitrarily metrizable space Y that contains X as a closed subspace. Recall that X is an AR if and only if every map $f : A \to X$ from a closed set A in a metrizable space Y extends over Y (cf. [44, Theorem 6.2.5]). We can establish the following as a corollary of Theorem 3.9.

Corollary 3.10. *For every uncountable Polish space X, the space $\mathrm{AM}(X)$ is universal for $\mathcal{T}$.*

Proof. An arbitrary uncountable Polish space X contains a topological copy C of the Cantor set (the Cantor–Bendixson theorem, refer to [32, Corollary 6.5]). Then there is a continuous surjection $f : C \to \mathbf{Q}$ (cf. [61, Theorem 30.7]). Since $\mathbf{Q}$ is an AR (cf. [45, Theorem 3.8.7]), we can obtain an extension of f over X. Theorem 3.9 implies that $\mathrm{AM}(X)$ is universal for $\mathcal{T}$. □

3.2. **Kuratowski type theorems.** Due to the same argument as Theorem 3.4, we can construct some isometric embeddings from a given bounded metric space $X = (X, d)$ into $\mathrm{AM}(X)$, that is an analogue of the Kuratowski embedding theorem (cf. [19, p. 100]).

Theorem 3.11. *Let $X = (X, d)$ be a metric space satisfying either of the following:*

(1) *X contains a subset A such that A is uniformly homeomorphic to X and its closure $\mathrm{cl}_X A$ is a proper subset of X;*
(2) *X contains a closed proper subset B homeomorphic to X.*

Then there exits an isometric embedding from X into $\mathrm{AM}(X)$.

Proof. For a uniform homeomorphism $f : (X, d) \to (A, d|_{A^2})$ in the case (1), there exists an extended space $Y = (Y, \bar{d})$ of X and a homeomorphism $\bar{f} : Y \to \mathrm{cl}_X A$ such that $\bar{d}|_{X^2} = d$ and $\bar{f}|_X = f$. By the assumptions (1) and (2), we can choose points $p \in X \setminus \mathrm{cl}_X A$ and $q \in X \setminus B$, and regard the extended spaces $Y \oplus \{*\}$ and $X \oplus \{*\}$ homeomorphic to the closed subsets $\mathrm{cl}_X A \cup \{p\}$ and $B \cup \{q\}$ of X, respectively. It follows form the same argument as the proof of Theorem 3.4 that there exists an isometric embedding from X to $\mathrm{AM}(X)$. □

3.3. **Problems.** We obtain that the space $\mathrm{AM}([0, 1])$ is universal for $\mathcal{T}$. Compared with the Banach-Mazur theorem 3.1, it is natural to ask the following:

Problem 3.12. *Is $\mathrm{AM}([0, 1])$ universal for $\mathcal{S}$?*

P. Urysohn constructed a separable metric space universal for $\mathcal{S}$, which space is called the Urysohn universal metric space (see [54, 55]). It is known that this space is homeomorphic to ℓ_2 (cf. [56]) and has not only that universality but also a kind of homogeneity for finite subsets. In the category of ultrametric spaces, Y. Ishiki detected an analogue of the Urysohn universal metric space among function spaces consisting of ultrametrics (cf. [24, 25, 28]).

Problem 3.13. *Do the spaces $\mathrm{PM}(X)$ and $\mathrm{AM}(X)$ of a metrizable space X admit "nice" properties like the Urysohn universal space?*

4. Isometries between spaces of metrics

This section provides a concise overview of the paper [37]. Throughout this section, an isometry means a surjective isometry. Isometries between function spaces have been studied in functional analysis. The Banach-Stone theorem [3, 49], which asserts that the metric structures of function spaces decide the topological ones of their underlying spaces, is one of the most important results in that field.

Theorem 4.1 (the Banach-Stone theorem)**.** *Suppose that X and Y are compact Hausdorff spaces. Then X and Y are homeomorphic if and only if $\mathrm{C}(X)$ and $\mathrm{C}(Y)$ are isometric. In this case, for every isometry $T : \mathrm{C}(X) \to \mathrm{C}(Y)$, there is a homeomorphism $\phi : Y \to X$ such that for all functions $f \in \mathrm{C}(X)$ and for all points $y \in Y$,*

$$T(f)(y) = g(y)f(\phi(y)), \qquad (*)$$

where $|g(y)| = 1$.

Various Banach-Stone type theorems have been researched (see [17] as a historical note), but there seem to be very few studies addressing this issue on multivariate function spaces. In this section, we shall establish a Banach-Stone type theorem on spaces of metrics. As will be easily observed in the proof of Theorem 4.3, $\mathrm{PM}(X)$ and $\mathrm{PM}(Y)$ (respectively, $\mathrm{AM}(X)$ and $\mathrm{AM}(Y)$) are isometric if metrizable spaces X and Y are homeomorphic. However, Theorems 2.9 and 2.10 mean that X and Y are not necessarily homeomorphic even if $\mathrm{PM}(X)$ and $\mathrm{PM}(Y)$ (respectively, $\mathrm{AM}(X)$ and $\mathrm{AM}(Y)$) are homeomorphic. In the case where X and Y are compact, M.E. Shanks [46] proved the converse of it as follows:

Theorem 4.2. *Suppose that X and Y are compact metrizable spaces. The following are equivalent:*

(1) *X and Y are homeomorphic;*
(2) *$\mathrm{PM}(X)$ and $\mathrm{PM}(Y)$ are isometric;*
(3) *$\mathrm{AM}(X)$ and $\mathrm{AM}(Y)$ are isometric.*

As is seen in the Banach-Stone theorem, isometries between function spaces can be described as weighted composition operators by homeomorphisms of their underlying spaces such as $(*)$. Furthermore, D. Hirota, I. Matsuzaki and T. Miura [21] (cf. [50]) showed a Banach-Stone type theorem on non-negative cones of continuous function spaces, where the isometries are composition operators of weight $= 1$. On the other hand, Shanks did not give such a canonical formula. For a metrizable space Z, set

$$\mathrm{Pc}(Z) = \{d \in \mathrm{AM}(Z) \mid \text{there are } \{z, w\} \subset Z \text{ and a compact set } K \subset Z \text{ such that } d(z,w) = \|d\| \text{ and if } d(x,y) = \|d\|, \text{ then } x, y \in K\},$$

$$\mathrm{Pp}(Z) = \{d \in \mathrm{AM}(Z) \mid \text{ there uniquely exists } \{z, w\} \subset Z \text{ such that } d(z,w) = \|d\|\}.$$

Remark that $\mathrm{Pp}(Z) \subset \mathrm{Pc}(Z)$, and that $\mathrm{AM}(Z) = \mathrm{Pc}(Z)$ when Z is compact. Let $\mathrm{P}(Z)$ be either $\mathrm{Pc}(Z)$ or $\mathrm{Pp}(Z)$. We shall generalize Theorem 4.2 and describe isometries as composition operators as follows:

Theorem 4.3. *Let X and Y be metrizable spaces. The following are equivalent:*

(1) *X and Y are homeomorphic;*
(2) *there is an isometry $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ with $T(\mathrm{P}(X)) = \mathrm{P}(Y)$;*
(3) *there is an isometry $T : \mathrm{AM}(X) \to \mathrm{AM}(Y)$ with $T(\mathrm{P}(X)) = \mathrm{P}(Y)$;*
(4) *there is an isometry $T : \mathrm{P}(X) \to \mathrm{P}(Y)$.*

In this case, for each isometry $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ with $T(\mathrm{P}(X)) = \mathrm{P}(Y)$, there exists a homeomorphism $\phi : Y \to X$ such that for any $d \in \mathrm{PM}(X)$ and for any $x, y \in Y$,

$$T(d)(x,y) = d(\phi(x), \phi(y)). \qquad (*)$$

Except for the case that the cardinality of X or Y is equal to 2, the homeomorphism ϕ can be chosen uniquely.

4.1. **The peaking function argument.** We will prove Theorem 4.3 by using the peaking function argument, which traces its history back to Stone's method, and is different from Shanks' one. From now on, suppose that X and Y are non-degenerate metrizable spaces and $T : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ is an isometry. Then T is norm-preserving.

Proposition 4.4. *For any $d \in \mathrm{PM}(X)$, $\|T(d)\| = \|d\|$, and for any $\rho \in \mathrm{PM}(Y)$, $\|T^{-1}(\rho)\| = \|\rho\|$.*

Proof. This proposition follows from the equality $T(\mathbf{0}_{X^2}) = \mathbf{0}_{Y^2}$ immediately, where $\mathbf{0}_{X^2} \in \mathrm{PM}(X)$ and $\mathbf{0}_{Y^2} \in \mathrm{PM}(Y)$ are the zero functions on X^2 and Y^2 respectively. Assume that $T(\mathbf{0}_{X^2}) \neq \mathbf{0}_{Y^2}$,

so there exist points $x, y \in Y$ such that $T(\mathbf{0}_{X^2})(x, y) > 0$. Then

$$\max\{\|T(\mathbf{0}_{X^2})\|, \|T^{-1}(\mathbf{0}_{Y^2})\|\} < \left\| \frac{2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\|}{\|T(\mathbf{0}_{X^2})\|} T(\mathbf{0}_{X^2}) \right\|.$$

Since T is isometric, by the similar argument as in Lemmas 2.3 and 2.4 of [21],

$$\begin{aligned} 2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\| &> \left\| \frac{2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\|}{\|T(\mathbf{0}_{X^2})\|} T(\mathbf{0}_{X^2}) - T(\mathbf{0}_{X^2}) \right\| \\ &= \left\| T^{-1}\left(\frac{2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\|}{\|T(\mathbf{0}_{X^2})\|} T(\mathbf{0}_{X^2}) \right) \right\| \\ &= \left\| \frac{2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\|}{\|T(\mathbf{0}_{X^2})\|} T(\mathbf{0}_{X^2}) \right\| \\ &= 2\|T(\mathbf{0}_{X^2})\| + \|T^{-1}(\mathbf{0}_{Y^2})\|. \end{aligned}$$

This is a contradiction. □

Given a metrizable space Z, let $\mathrm{Fin}_2(Z)$ be the hyperspace consisting of non-empty finite sets in Z of cardinality ≤ 2, which is endowed with the Vietoris topology. Denote by $\mathrm{D}(Z)$ the subspace of doubletons of Z. For each $\{x, y\} \in \mathrm{Fin}_2(Z)$ and each $d \in \mathrm{PM}(Z)$, set

$$\mathcal{P}(Z, \{x, y\}) = \{\rho \in \mathrm{PM}(Z) \mid \rho(x, y) = \|\rho\|\},$$

$$\mathcal{F}(Z, d) = \{\{z, w\} \in \mathrm{Fin}_2(Z) \mid d(z, w) = \|d\|\}.$$

We can prove the following, which is a key lemma in this section.

Lemma 4.5. *Let Z be a non-degenerate metrizable space. For each $d \in \mathrm{PM}(Z)$ and each $\{x, y\} \in \mathrm{D}(Z)$, there exists $\rho \in \mathcal{P}(Z, \{x, y\})$ such that*

$$d + \rho \in \mathcal{P}(Z, \{x, y\}) \cap \mathrm{Pp}(Z).$$

Proof. Adding an admissible metric $e \in \mathcal{P}(Z, \{x, y\})$ with $\|e\| = 1$, we get that $d' = d + e \in \mathrm{AM}(Z)$ according to Lemma 3.6. Let $a = d'(x, y) > 0$ and

$$b = \min\Big\{ \sup_{z \in Z} d'(x, z), \sup_{z \in Z} d'(y, z) \Big\}.$$

Using Theorem 2.6, we can obtain $\rho_n \in \mathrm{PM}(Z)$ for each natural number n such that

(i) $\rho_n(z, w) = 4b$ if $z \in \overline{B}_{d'}(x, a/2^{n+1})$ and $w \in \overline{B}_{d'}(y, a/2^{n+1})$;
(ii) $\rho_n(z, w) = 2b$ if $z \in Z \setminus (B_{d'}(x, a/2^n) \cup B_{d'}(y, a/2^n))$ and $w \in \overline{B}_{d'}(x, a/2^{n+1}) \cup \overline{B}_{d'}(y, a/2^{n+1})$;
(iii) $\rho_n(z, w) = 0$ if $z, w \in Z \setminus (B_{d'}(x, a/2^n) \cup B_{d'}(y, a/2^n))$, if $z, w \in \overline{B}_{d'}(x, a/2^{n+1})$, or if $z, w \in \overline{B}_{d'}(y, a/2^{n+1})$;
(iv) $\rho_n(z, w) \leq 4b$ if otherwise.

Here for every $z \in Z$ and every $r > 0$, we write

$$B_{d'}(z, r) = \{w \in Z \mid d'(z, w) < r\},\ \overline{B}_{d'}(z, r) = \{w \in Z \mid d'(z, w) \leq r\}.$$

Then $\rho = e + \sum_{n=1}^{\infty} \rho_n / 2^n$ is the desired admissible metric. □

Using the finite intersection property in compact spaces (refer to [16, Theorem 3.1.1]), we can get the following:

Lemma 4.6. *Suppose that* $T(\mathrm{P}(X)) \subset \mathrm{P}(Y)$. *For every doubleton* $\{x, y\} \in \mathrm{D}(X)$,

$$\bigcap_{d \in \mathcal{P}(X,\{x,y\}) \cap \mathrm{P}(X)} \mathcal{F}(Y, T(d)) \neq \emptyset.$$

Proof. Remark that $\mathcal{P}(X, \{x, y\}) \cap \mathrm{P}(X) \neq \emptyset$ by virtue of Lemma 4.5. According to the assumption, fix any $d_0 \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{P}(X)$, so $T(d_0) \in \mathrm{P}(Y)$ and there exists a non-empty compact set $K \subset Y$ such that for all $z, w \in Y$ with $T(d_0)(z, w) = \|T(d_0)\|$, $z, w \in K$. Since K is compact, the natural surjective image $\mathrm{Fin}_2(K)$ of K^2 is also compact (cf. [58, Lemma 5.3.4]). Observe that $\mathcal{F}(Y, T(d))$ is closed in $\mathrm{Fin}_2(Y)$ for every $d \in \mathcal{P}(X, \{x, y\})$. Then the family

$$\{\mathcal{F}(Y, T(d)) \cap \mathcal{F}(Y, T(d_0)) \mid d \in \mathcal{P}(X, \{x, y\})\}$$

has the finite intersection property.

In fact, for all $d_i \in \mathcal{P}(X, \{x, y\}) \cap \mathrm{P}(X)$, $1 \leq i \leq n$, observe that

$$d = \sum_{i=1}^{n} d_i \in \mathcal{P}(X, \{x, y\}).$$

By the assumption, we can find $\{z, w\} \in \mathrm{Fin}_2(Y)$ such that $T(d)(z, w) = d(x, y)$. For each $i \in \{1, \ldots, n\}$, since T is isometric,

$$T(d)(z, w) - T(d_i)(z, w) \leq \|T(d) - T(d_i)\| = \|d - d_i\| = d(x, y) - d_i(x, y),$$

and hence according to Proposition 4.4,

$$\|T(d_i)\| = \|d_i\| = d_i(x, y) \leq T(d_i)(z, w) \leq \|T(d_i)\|.$$

This means that $\{z, w\} \in \mathcal{F}(Y, T(d_i))$. □

We can construct the following corresponding between $\mathrm{D}(X)$ and $\mathrm{D}(Y)$.

Proposition 4.7. *Assume that* $T(\mathrm{P}(X)) = \mathrm{P}(Y)$. *There is a bijection* $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$ *such that for every* $\{x, y\} \in \mathrm{D}(X)$ *and every* $\{z, w\} \in \mathrm{D}(Y)$,

$$T(\mathcal{P}(X, \{x, y\}) \cap \mathrm{P}(X)) = \mathcal{P}(Y, \Phi^{-1}(\{x, y\})) \cap \mathrm{P}(Y),$$

$$T^{-1}(\mathcal{P}(Y, \{z, w\}) \cap \mathrm{P}(Y)) = \mathcal{P}(X, \Phi(\{z, w\})) \cap \mathrm{P}(X).$$

Proof. We can define Φ by

$$\{\Phi(\{z, w\})\} = \bigcap_{\rho \in \mathcal{P}(Y,\{z,w\}) \cap \mathrm{P}(Y)} \mathcal{F}(X, T^{-1}(\rho))$$

for each $\{z, w\} \in \mathrm{D}(Y)$, since the intersection is not empty by Lemma 4.6, and it is a singleton by virtue of Lemma 4.5. □

4.2. **Composition operators.** Show that the isometry T is a composition operator by some homeomorphism $\phi : Y \to X$. From now on, assume that $T(\mathrm{P}(X)) = \mathrm{P}(Y)$, and let $\Phi : \mathrm{D}(Y) \to \mathrm{D}(X)$ be a bijection as in Proposition 4.7.

Proposition 4.8. *The equality* $T(d)(x, y) = d(\Phi(\{x, y\}))$ *holds for every* $d \in \mathrm{PM}(X)$ *and every* $\{x, y\} \in \mathrm{D}(Y)$, *where define* $d(\Phi\{x, y\}) = d(z, w)$ *with* $\Phi(\{x, y\}) = \{z, w\} \in \mathrm{D}(X)$.

Proof. We only prove that $T(d)(x,y) \leq d(\Phi(\{x,y\}))$. Using Lemma 4.5, we can take $\rho \in \mathcal{P}(Y,\{x,y\})$ so that $T(d)+\rho \in \mathcal{P}(Y,\{x,y\}) \cap \mathrm{P}(Y)$. By Propositions 4.7 and 4.4, since T is an isometry,

$$\begin{aligned} T(d)(x,y)+\rho(x,y)-d(\Phi(\{x,y\})) &= T^{-1}(T(d)+\rho)(\Phi(\{x,y\}))-d(\Phi(\{x,y\})) \\ &\leq \|T^{-1}(T(d)+\rho)-d\| = \|\rho\| = \rho(x,y), \end{aligned}$$

which implies that $T(d)(x,y) \leq d(\Phi(\{x,y\}))$. □

We shall construct a homeomorphism from Y to X which is compatible with the bijection Φ.

Proposition 4.9. *Suppose that the cardinality of X or Y is greater than 2. There exists a homeomorphism $\phi : Y \to X$ such that for any $\{x,y\} \in \mathrm{D}(Y)$ and any $\{z,w\} \in \mathrm{D}(X)$,*

$$\Phi(\{x,y\}) = \{\phi(x),\phi(y)\},\ \Phi^{-1}(\{z,w\}) = \{\phi^{-1}(z),\phi^{-1}(w)\}.$$

Proof. We can define a bijection $\phi : Y \to X$ by

$$\{\phi(y)\} = \bigcap_{z \in Y \setminus \{y\}} \Phi(\{y,z\})$$

for every $y \in Y$. To verify that ϕ is continuous, fix any point $y \in Y$ and any open neighborhood U of $\phi(y)$ in X. According to Theorem 2.6, we can find $d \in \mathrm{PM}(X)$ so that $d(\phi(y),x) = 2$ if $x \in X \setminus U$, and $d(x,x') = 0$ if $x, x' \in X \setminus U$. Then by Proposition 4.8, for every $z \in B_{T(d)}(y,1) \setminus \{y\}$,

$$d(\phi(y),\phi(z)) = d(\Phi(\{y,z\})) = T(d)(y,z) < 1,$$

which implies that $\phi(z) \in U$. □

The subspace $\mathrm{AM}(Z)$ is dense in $\mathrm{PM}(Z)$, see Proposition 2.12. Furthermore, we can obtain the denseness of $\mathrm{P}(Z) \subset \mathrm{PM}(Z)$ by the similar argument as Lemma 4.5.

Proposition 4.10. *For every metrizable space Z, the subset $\mathrm{P}(Z)$ is dense in $\mathrm{PM}(Z)$.*

Now we shall prove Theorem 4.3.

Proof of Theorem 4.3. The implications (1) ⇒ (2), (1) ⇒ (3) and (1) ⇒ (4) follow from Lemma 3.8. Indeed, take any homeomorphism $\psi : Y \to X$ and define the isometry $S : \mathrm{PM}(X) \to \mathrm{PM}(Y)$ by $S(d)(x,y) = d(\psi(x),\psi(y))$ for any $d \in \mathrm{PM}(X)$ and for any $x, y \in Y$, namely S is the composition operator by ψ. Observe that $S(\mathrm{AM}(X)) = \mathrm{AM}(Y)$ and $S(\mathrm{P}(X)) = \mathrm{P}(Y)$. By Propositions 2.12 and 4.10, $\mathrm{AM}(X)$ and $\mathrm{P}(X)$ are dense in $\mathrm{PM}(X)$, and hence each isometry in (3) and (4) can be extended to the one in (2), which means that (3) ⇒ (2) and (4) ⇒ (2) hold.

The implication (2) ⇒ (1) and the canonical formula (∗) hold from Propositions 4.8 and 4.9. To show the uniqueness of ϕ, let $\psi : Y \to X$ be a homeomorphism such that for all $d \in \mathrm{PM}(X)$ and for all $x, y \in Y$, $T(d)(x,y) = d(\psi(x),\psi(y))$, and suppose that $\phi(x) \neq \psi(x)$ for some point $x \in Y$. Fixing any $y \in Y \setminus \{x, \phi^{-1}(\psi(x))\}$, we can choose $\rho \in \mathrm{PM}(X)$ so that $\rho(\phi(x),\phi(y)) = 0$ and $\rho(\psi(x),\psi(y)) = 1$ by Theorem 2.6. Then

$$0 = \rho(\phi(x),\phi(y)) = T(\rho)(x,y) = \rho(\psi(x),\psi(y)) = 1,$$

which is a contradiction. □

4.3. **Problems.** There are many variants of the Banach-Stone theorem for subspaces of $\mathrm{C}(Z)$ on topological spaces Z, so it is interesting to consider this issue for subspaces of $\mathrm{PM}(Z)$. The author [38] showed a Banach-Stone type theorem on the subspaces consisting of uniformly continuous metrics on uniform spaces, which is an analogue of [2, 20]. Furthermore, given a metric space $Z = (Z, d_Z)$, let

$$\mathrm{LPM}(Z, d_Z) = \{d \in \mathrm{PM}(Z) \mid \mathrm{lip}(d) < \infty\},$$

where for any $d \in \mathrm{PM}(Z)$, put

$$\mathrm{lip}(d) = \sup_{x,y \in Z \text{ with } x \neq y} \frac{d(x,y)}{d_Z(x,y)}.$$

The author also established Banach-Stone type theorems for $\mathrm{LPM}(Z)$ and its subspaces. A variant of the Banach-Stone theorem for spaces of Lipschitz continuous functions with the Lipschitz norm was shown by N. Weaver [59].

Problem 4.11. *Solve a Banach-Stone type theorem for* $\mathrm{LPM}(Z)$ *equipped with the other norm:*

$$\|d\|_{lip} = \max\{\|d\|, \mathrm{lip}(d)\}.$$

Tingley's problem [51] is one of the most significant questions in functional analysis, that asks whether an isometry between the unit spheres of two Banach spaces can be extended to a real-linear isometry over the whole spaces (cf. [42] and [22, Introduction] for the recent developments). T. Banakh [6] solved the problem for 2-dimensional Banach spaces affirmatively. Although this question remains open in general case, it has been solved for specific function spaces since the study due to R. Wang [57]. The author [39] solved a variant of Tingley's problem on function spaces consisting of metrics on compact metrizable spaces.

Problem 4.12. *Solve Tingley type problem of function spaces consisting of metrics in the non-compact case.*

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