Biquandle Brackets and Quivers

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Abstract

In this brief expository article we review the background for biquandle bracket quivers – including biquandles, biquandle homsets, biquandle coloring quivers and biquandle brackets – for a talk at the 70th Topology Symposium at Nara Women's University in August 2023.

KEYWORDS: Quantum enhancements, biquandles, biquandle counting invariants, biquandle brackets, trace diagrams, quivers

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1 Introduction

Biquandles are algebraic structures with axioms motivated by the Reidemeister moves in knot theory. Every oriented knot or link L in $\mathbb{R}^2$ or S^2 has a *fundamental biquandle* $\mathcal{B}(L)$ analogous to the fundamental group of a topological space. A finite biquandle X determines an invariant of oriented knots and links called the *biquandle homset invariant* $\mathrm{Hom}(\mathcal{B}(L), X)$ consisting of biquandle homomorphisms from the fundamental biquandle of L to X. These homset elements have the property that they can be represented visually as *colorings* of a diagram of the oriented knot or link L analogously to the way linear transformations between vector spaces are represented by matrices. In particular, choosing a different diagram of L yields a different representation of the homset elements analogously to the way choosing a different basis yields a different matrix representing the same linear transformation, with the role of change-of-basis matrices played in the biquandle homset case by biquandle-colored Reidemeister moves. See [5,10] for more.

From the homset, many useful computable invariants can be defined. The simplest of these is the cardinality of the homset, a non-negative integer-valued oriented link invariant known in the literature as the *biquandle counting invariant*, denoted $\Phi_X^{\mathbb{Z}}(L)$. Any invariant ϕ of biquandle-colored diagrams determines a multiset-valued invariant of oriented knots and links called an *enhancement* of the counting invariant. The first such examples use a cohomology theory on the category of biquandles to define 2-*cocycle* and 3-*cocycle* invariants, multiset-valued invariants which can be transformed into polynomial invariants which evaluate to $\Phi_X^{\mathbb{Z}}(L)$ at zero but in general are stronger invariants; see [1]. A more recent example of an enhancement is the *biquandle coloring quiver* associated to a subset S of the automorphism group of the finite biquandle X; such a subset determines a directed graph-valued enhancement of the counting invariant. Since directed graphs (also known as *quivers*) are categories, this enhancement is actually a *categorification* of the counting invariant. Several new polynomial invariants can be obtain from these quivers via different forms of decategorification; see [2–4,6,9]

Our main interest in this manuscript is in a family of enhancements known as *biquandle brackets*. These are a type of *quantum enhancement*, i.e. quantum invariants of biquandle-colored oriented knots and links representing elements of the homset $\mathrm{Hom}(\mathcal{B}(L), X)$. In this manuscript and its associated talk, we review (in a fairly self-contained way) biquandles and biquandle quivers, leading up to biquandle brackets and biquandle bracket quivers. In Section 2 we review the basics of biquandles and the biquandle homset invariant. In Section 5 we review the quiver categorification of the biquandle homset invariant. In Section 4 we review biquandle brackets and their categorification via quivers. We conclude in Section 5 with a few words about current work in this area.

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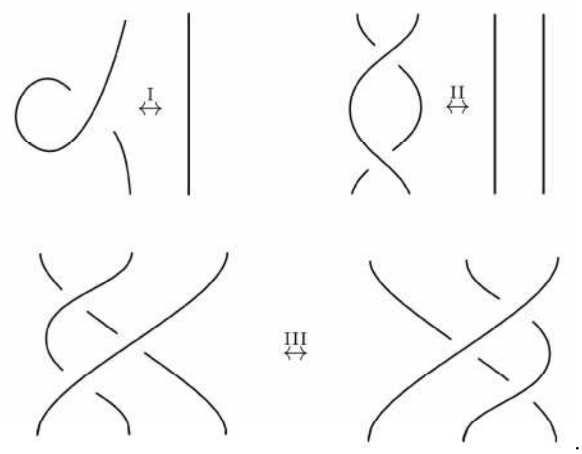
2 Knots, Biquandles and Biquandle Coloring

A *knot* is an embedding $K : S^1 \to \mathbb{R}^3$ or $K : S^1 \to S^3$; we may also consider the image of such an embedding as a knot. A *link* is a disjoint union of knots. A knot is *tame* if the embedding is piecewise-linear or (equivalently) C^∞. In the 1920s, Kurt Reidemeister, a mathematician at the Georg August Universität in Göttingen, Germany, proved that two diagrams represent ambient isotopic tame knots in S^3 iff they are related by a sequence of the following moves, now known as *Reidemeister moves*:



For the remainder of this Manuscript, we will consider only tame knots and links. See [15] for more.

Thus, to prove that two diagrams represent the same knot, it suffices to identify a sequence of Reidemeister moves changing one diagram to the other. To prove that two diagrams represent different knots, we can identify *knot invariants*, functions $f : \mathcal{D} \to X$ taking knot diagrams as inputs and giving output values in some set X (which could be integers, polynomials, graphs etc.) with the property that diagrams differing by Reidemeister moves have the same function value, i.e.,

$$D \sim D' \Rightarrow f(D) = f(D').$$

Then if two knot diagrams share the same value of an invariant, it says nothing – the two diagrams could represent the same knot or they could just coincidentally have the same value; however, if two diagrams have different values of an invariant, then they cannot be related by Reidemeister moves and hence must represent different knots.

To define invariants, we need to find things that are not changed by Reidemeister moves. One way to do this is to use the power of universal algebra, creating algebraic structures with axioms derived from the Reidemeister moves. The example of interest for this manuscript is the algebraic structure known as *biquandles* (see [5, 10] for more).

Definition 1. A *biquandle* is a set X with a pair of binary operations $\underline{\triangleright}, \overline{\triangleright} : X \times X \to X$ satisfying the axioms

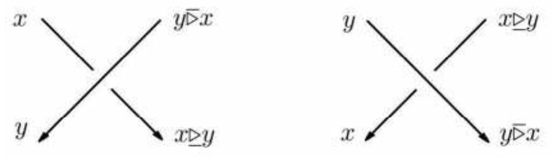
(i) For all $x \in X$, $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$,

(ii) For all y in X, the maps $\alpha_y, \beta_y : X \to X$ defined by $\alpha_y(x) = x \,\overline{\triangleright}\, y$ and $\beta_y(x) = x \,\underline{\triangleright}\, y$ and the map $S : X \times X \to X \times X$ defined by $S(x, y) = (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ are invertible, and

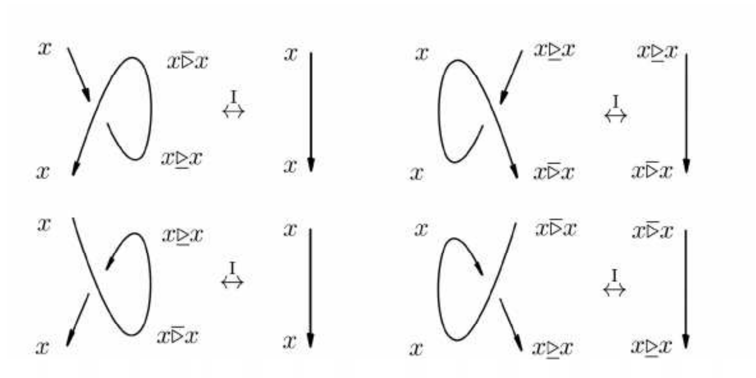
(iii) For all $x, y, z \in X$, we have

$$\begin{array}{rcl}(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) & = & (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\ (x \,\underline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\underline{\triangleright}\, y) & = & (x \,\overline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\ (x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) & = & (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z)\end{array} .$$

These axioms result from interpreting the elements of X as labels or "colors" for the semiarcs (segments between crossings) in an oriented knot diagram and interpreting the operations as over- and under-crossing rules as shown:



Then axiom (i) expresses the four oriented versions of the Reidemeister I move:



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Similarly axiom (ii) expresses the four oriented Reidemeister II moves and axiom (iii) expresses the all-positive Reidemeister III move.

Since these moves form a generating set of oriented Reidemeister moves (see [14]), we have:

Theorem 1. *If X is a biquandle and D and D' are oriented knot or link diagrams related by Reidemeister moves, there is a one-to-one correspondence between the set of X-colorings of D and the set of X-colorings of D'.*

See [5, 10] for more details.

Definition 2. A map $f : X \to Y$ between biquandles is a *homomorphism* if for all $x, y \in X$ we have

$$f(x \underline{\triangleright} y) = f(x) \underline{\triangleright} f(y) \text{ and } f(x \overline{\triangleright} y) = f(x) \overline{\triangleright} f(y).$$

A bijective homomorphism is an *isomorphism.*

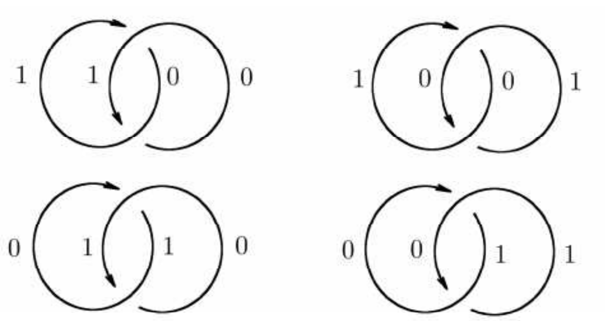
Example 1. Given a diagram representing an oriented knot or link diagram L, we define the *fundamental biquandle* $\mathcal{B}(L)$ in the following way:

- We form a set of *generators* consisting of a symbol for each semiarc in the diagram,
- We form a set of *biquandle words* including generators and expressions of the forms $u \underline{\triangleright} v$, $u \overline{\triangleright} v$, $\alpha_u^{-1}(v)$, $\beta_u^{-1}(v)$ and $S_i^{-1}(x_j, x_k)$ for $i \in \{1, 2\}$ (we interpret $S_i^{-1}(u, v)$ as the ith component of $S^{-1}(u, v)$) where u, v are biquandle words,
- Then $\mathcal{B}(L)$ is the set of equivalence classes of biquandle words under the equivalence relation generated by the biquandle axioms and the crossing relations.

The set of generators and crossing relations (taking the biquandle axioms as understood) is known as a *biquandle presentation.* Reidemeister moves on diagrams produce Tietze moves on the presentation, resulting in isomorphic biquandles; hence, the fundamental biquandle does not depend on our choice of diagram for L and is a link invariant.

An X-coloring of an oriented knot or link diagram D is an assignment of an image in X to each generator of $\mathcal{B}(L)$ satisfying the crossing relations and hence determines a unique biquandle homomorphism $f : \mathcal{B}(K) \to X$. Thus the set of biquandle colorings of a diagram D of an oriented knot or link L can be identified with the homset $\mathrm{Hom}(\mathcal{B}(L), X)$. In particular, a choice of diagram for L is analogous to a choice of basis for a vector space – just as a linear transformation $f : X \to Y$ is determined by a choice of output vector for each input basis vector, a biquandle homset element is determined by a choice of output color in X for each semiarc in our chosen diagram of L such that the crossing relations are satisfied in X. Different choices of diagram for L yield different representations of the same homset element just as different choices of basis yield different matrices representing the same linear transformation, with the role of change-of-basis matrices played by X-colored Reidemeister moves.

Example 2. Let X be the biquandle $\mathbb{Z}_2 = \{0, 1\}$ with $x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = x + 1$ and consider the Hopf link L. The homset $\mathrm{Hom}(\mathcal{B}(L), X)$ has four elements which we can represent with the set of X-colored diagrams



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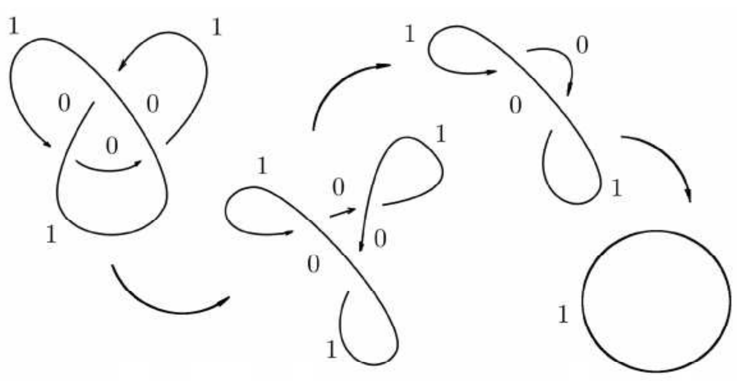
Example 3. Each of the diagrams below represents the same homset element in $\mathrm{Hom}(\mathcal{B}(L), X)$ where L is the unknot and X is the biquandle in Example 2.



See [5] for more.

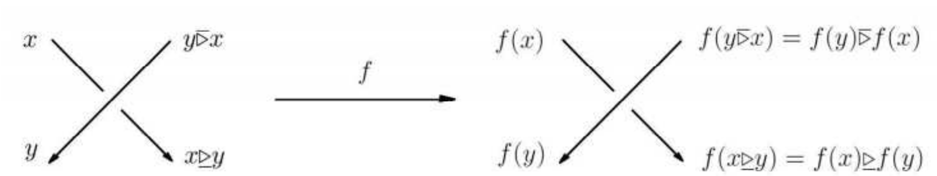
3 Quivers and Categorification

Categories are algebraic structures consisting of a collection of *objects* and for each pair of objects X, Y a set of *morphisms* denoted $\mathrm{Hom}(X, Y) = \{f : X \to Y\}$ satisfying certain axioms. Examples include the category of sets and functions, the category of vector spaces and linear transformations, the category of groups and group homomorphisms, the category of topological spaces and continuous maps, etc. Indeed, many subject areas of mathematics can be described in category-theoretic terms. Connections between these subject areas can often be formalized as maps known as *functors* between categories, e.g. the fundamental group functor π_1 which transforms topological questions into group-theoretic questions. Similarly, functors known as *homology theories* measure information about topological or algebraic structures in terms of chain complexes and homology groups.

In the late 1990s, mathematical physicists such as Louis Crane and (my former colleague) John Baez proposed a program of *categorification* wherein simpler structures are replaced with richer categorical structures. For example, we might replace natural numbers n with vector spaces of dimension n, addition with direct sum and multiplication with tensor product. As a result, we get a richer and more powerful structure – while integers can be either equal or not, vector spaces may be identical, distinct but isomorphic, or different. From these richer structures we can recover the original simpler structures and potentially get new ones via *decategorification*, e.g. taking the dimension of the vector spaces. Famous examples in recent decades include Khovanov homology (categorifying the Jones polynomial) and Knot Floer homology (categorifying the Alexander polynomial).

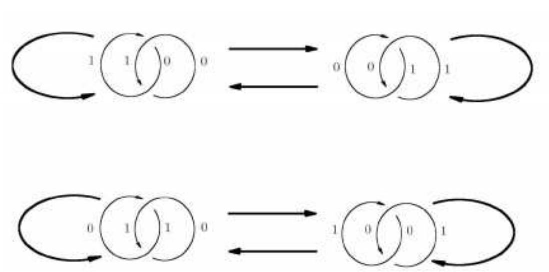
Directed graphs, sometimes known as *quivers* (as collections of arrows), form categories with vertices as objects and directed paths as morphisms. In recent years my students, professional collaborators and I have used quivers to categorify several enhancements of the biquandle counting invariant; see [2–4, 9, 11] for more.

For this section we will focus on one example, the biquandle coloring quiver. Let X be a finite biquandle and D an oriented knot or link diagram with a choice of coloring X-coloring, and let $f : X \to X$ be a biquandle endomorphism. Then applying f to each color on the diagram D yields another assignment of elements of X to generators of $\mathcal{B}(D)$, and the fact that f is a quandle endomorphism implies that this assignment is a valid X-coloring.



In particular, each endomorphism f determines an arrow from one coloring in the homset to another. Thus, a set S of endomorphisms of X determines a quiver structure on the homset which we call the *biquandle coloring quiver* associated to S. Changing the diagram D by X-colored Reidemeister moves does not change the homset or the resulting quiver, and hence the quiver is an invariant of knots and links for each S. In particular the case of $S = \emptyset$ can be identified with the original homset invariant. The case of $S = \mathrm{Hom}(X, X)$, the entire set of endomorphisms, is called the *full quiver*.

Example 4. In our previous Example 2, the identity map and the map $f(x) = x + 1$ are the only endomorphisms. Then the full biquandle coloring quiver looks like



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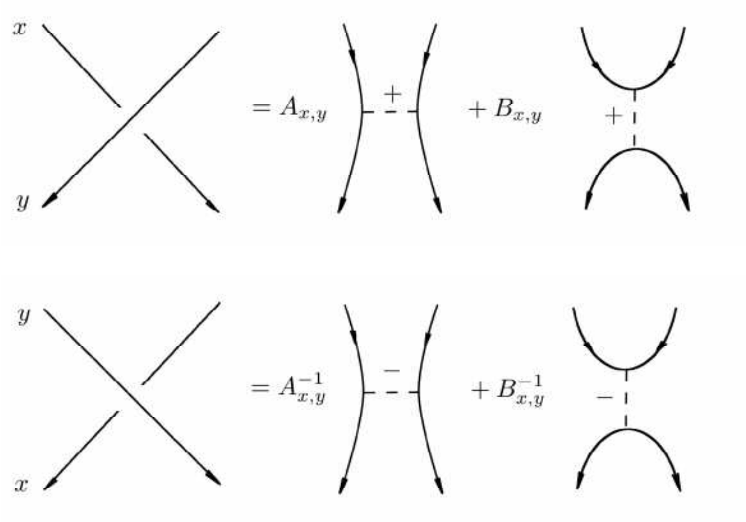
Such a quiver is a category and hence the biquandle coloring quiver is a categorification of the biquandle homset invariant; it decategorifies to the original homset by deleting the arrows. Other polynomial invariants such as the *in-degree polynomial* can be obtained as decategorifcations; see [2, 4] for more.

4 Biquandle Brackets

In the 1980s, Vaughn Jones won the Fields medal for his introduction of a powerful knot invariant known as the *Jones polynomial.* The invariant can be defined using the *Kauffman bracket skein relation* which we may understand as thinking of crossings as linear combinations of *smoothings* or diagrams with crossings removed. More precisely, applying the skein relation to a diagram with a crossing replaces the diagram with a linear combination of two diagrams with the crossing removed and the resulting endpoints connected in the two possible ways with coefficients in $\mathbb{Z}[A^{\pm 1}]$. Recursively applying this procedure to all crossings yields a linear combination of diagrams without crossings, known as *Kauffman states.* Evaluating each Kauffman state as δ^{k-1} where k is the number of components and $\delta = -A^2 - A^{-2}$ yields the *unnormalized Kauffman bracket polynomial*, and multiplying by $(-A^3)^{-wr}$ where wr is the *writhe* or number of positive crossings minus the number of negative crossings yields the Kauffman bracket polynomial, which can be easily checked to be invariant under Reidemeister moves.

For biquandle brackets we want to generalize the Kauffman bracket to the case of biquandle colored knots and links. To do this, we replace the Kauffman bracket skein relation with the skein relations



yielding *trace diagrams* with dashed signed edges called a *traces.* Traces act like crossings, enabling us to retain the biquandle colors in a homset element diagram. We now have skein coefficients $A_{x,y}$ and $B_{x,y}$ which are functions of the biquandle colors $x, y \in X$ on the left side of the crossing. Deleting the traces and colors in a fully-smoothed diagram yields the usual Kauffman states, and we require the elements $-A_{x,y}B^{-1}_{x,y} - A^{-1}_{x,y}B_{x,y}$ to be equal, with their common value denoted as δ.

The Reidemeister moves then impose some conditions on the skein coefficients:

$$\begin{aligned}
A_{x,y}A_{y,z}A_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} &= A_{x,z}A_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \\
A_{x,y}B_{y,z}B_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} &= B_{x,z}B_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \\
B_{x,y}A_{y,z}B_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} &= B_{x,z}A_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \\
A_{x,y}A_{y,z}B_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} &= A_{x,z}B_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} + A_{x,z}A_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \\
&\quad +\delta A_{x,z}B_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} + B_{x,z}B_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \\
B_{x,y}A_{y,z}A_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} + A_{x,y}B_{y,z}A_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} & \\
+\delta B_{x,y}B_{y,z}A_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} + B_{x,y}B_{y,z}B_{x\,\underline{\triangleright}\,y,z\,\overline{\triangleright}\,y} &= B_{x,z}A_{y\,\overline{\triangleright}\,x,z\,\overline{\triangleright}\,x}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}.
\end{aligned}$$

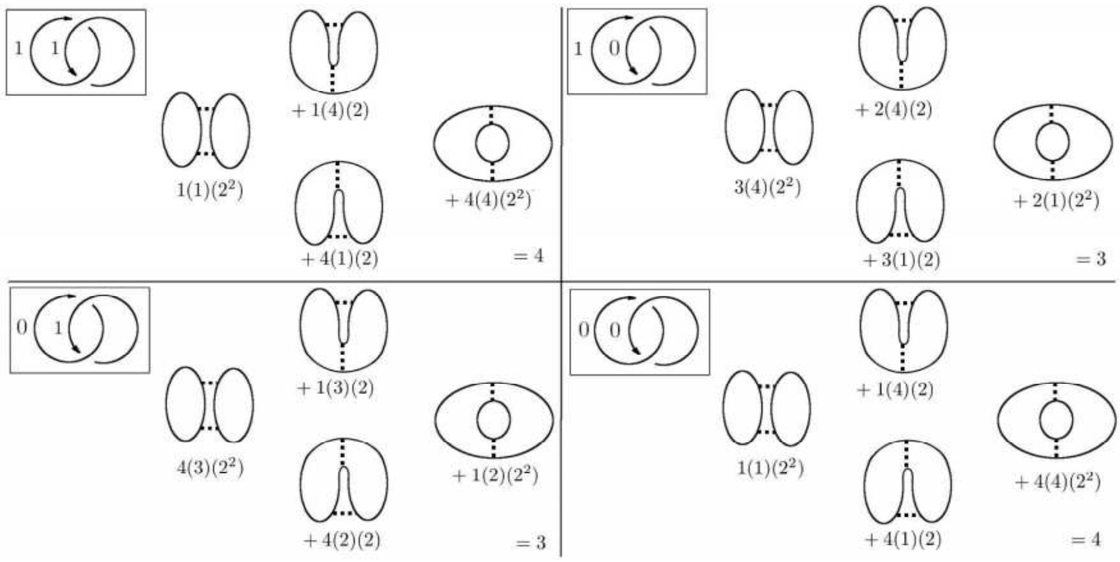
A biquandle bracket β defined on a biquandle X and a commutative unital ring R then consists of a pair of maps $A, B : X \times X \to R^{\times}$ such that the above equations are satisfied and the values $\delta = -A_{x,y}B^{-1}_{x,y} - A^{-1}_{x,y}B_{x,y}$ and $w = -A^2_{x,x}B^{-1}_{x,x}$ are the same for all $x, y \in X$. Then for each X-coloring of an oriented link diagram D, the sum of the products of state values times smoothing coefficients times writhe adjustment w^{-wr} over the set of all Kauffman states, known as the *state-sum* value, is unchanged by X-colored Reidemeister moves, and the multiset of such values over the complete homset is an invariant of oriented knots and links.

Example 5. Let X be the biquandle from our previous Example 2 and let $R = \mathbb{Z}_5$. Then one can verify that the coefficient tables

A	1	0
1	1	3
0	4	1

B	1	0
1	4	2
0	1	4

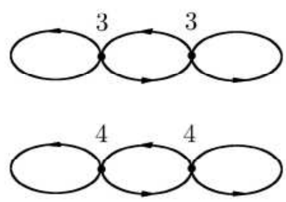
satisfy the biquandle bracket axioms with $\delta = 2$ and $w = 1$. Then the Hopf link with its four X-colorings has biquandle bracket multiset $\{3, 3, 4, 4\}$:



See [12, 13] for more details and examples.

Finally, we can categorify the biquandle bracket invariant using the biquandle coloring quiver. More precisely, where the biquandle bracket invariant is a multiset of β-values over the biquandle homset, a choice of set of biquandle endomorphisms S gives us an invariant quiver with vertices weighted with β-values which we call a *biquandle bracket quiver*.

Example 6. Continuing with the Hopf link and biquandle bracket from the previous examples, we have the following biquandle bracket quiver:



These quivers can become very large and complex very quickly, so for useful invariants it is helpful to decategorify in various ways to obtain easy-to-use polynomial invariants.

Example 7. Given a biquandle bracket quiver, we can sum over the set of arrows terms of the form $s^{S(a)}t^{T(a)}$ where $S(a)$ and $T(a)$ are the weights at the source and target of the arrow respectively. In the biquandle bracket quiver in Example 6 above, we get $4s^3t^3 + 4s^4t^4$ for this polynomial decategorification.

Example 8. Another decategorification uses the fact that while the out-degree of every vertex in a biquandle bracket quiver is the same, the in-degrees can be (and generally are) different. Thus, we can sum over the set of vertices terms of the form $u^{\beta(v)}w^{\deg_+(v)}$ where $\beta(v)$ is the weight at the vertex v and $\deg_+(v)$ is the in-degree of the vertex v. Then in our Example 6 we obtain $2u^3w^2 + 2u^4w^2$ for this decategorification.

See [6] for more.

5 Current and Future Work

I am currently working with several collaborators around the world on generalizations and extensions of these biquandle bracket quiver invariants, and other groups are also working on biquandle brackets. In [7] it is shown that many biquandle brackets are cohomologous to the Jones polynomial, meaning that biquandle bracket quivers yield a family of new categorifications of the Jones polynomial quite unlike Khovanov homology it is various forms. In [8] biquandle brackets are generalized to picture-valued invariants.

Faster algorithms for finding biquandle brackets are of great interest, as are new methods of finding biquandle brackets over infinite coefficient rings. Work currently in preparation (check `arXiv.org` soon!) includes *biquandle power brackets*, a major advancement over standard biquandle brackets in which the value δ of a Kauffman state component can vary as a function of the biquandle colors it contains, as well as *biquandle-colored Conway algebras* and extensions of the biquandle bracket to other algebraic structures.

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