The category of quasi-Polish spaces as a represented space

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1. Introduction

Quasi-Polish spaces are a class of well-behaved countably based T_0-spaces which include most of the countably based topological spaces that occur in usual mathematical practice, such as Polish spaces (used in functional analysis, topological algebra, probability theory, etc.), ω-continuous domains (used in domain theory, programming language semantics, semilattice theory, etc.), and countably based spectral spaces (used in algebraic geometry, logic, duality theory for distributive lattices, etc.). Many theoretical results for these specific subclasses of spaces naturally generalize to all quasi-Polish spaces, such as the descriptive set theory for Polish spaces [2, 4], the properties and characterizations of the upper and lower powerspaces for ω-continuous domains [8, 5], and the Stone duality and applications to logic of spectral spaces [10, 1].

Recently, there is growing interest in the effective aspects of quasi-Polish spaces [12, 9, 11, 5]. In this paper, we will go beyond individual spaces and look at the effective aspects of the whole category QPol of quasi-Polish spaces. For this purpose, we will use the characterization of quasi-Polish spaces as spaces of ideals introduced in [9] and further studied in [5] to interpret the objects of QPol as transitive binary relations on $\mathbb{N}$, and then extend this to an interpretation of QPol as a represented space. We will then show how to explicitly compute products and equalizers in QPol, and demonstrate the computability of several powerspace functors on QPol.

2. Preliminaries

Quasi-Polish spaces were introduced in [2], and were shown to have multiple equivalent characterizations. For the purposes of this paper we can define quasi-Polish spaces as follows, based on the characterization from [9] (see also [5]).

Definition 1 *Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an* ideal *(with respect to $\prec$) if and only if:*

1. $I \neq \emptyset$, *(I is non-empty)*
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, *(I is a lower set)*
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c \,\&\, b \prec c)$. *(I is directed)*

The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$. A space is quasi-Polish *if and only if it is homeomorphic to $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.* □

We often apply the above definition to other countable sets with the implicit assumption that it has been suitably encoded as a subset of $\mathbb{N}$. Spaces of the form $\mathbf{I}(\prec)$ for a computably enumerable (c.e.) relation $\prec$ on $\mathbb{N}$ provide an effective interpretation of quasi-Polish spaces, which were called *precomputable quasi-Polish spaces* in [9], and are equivalent to the *computable quasi-Polish spaces* in [12] (see also [11]).

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Let $\prec_S$ and $\prec_T$ be transitive relations on $\mathbb{N}$. Any subset $R \subseteq \mathbb{N} \times \mathbb{N}$ can be viewed as a *code* for a partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_S) \to \mathbf{I}(\prec_T)$ by defining

$$\ulcorner R \urcorner(I) = \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n\rangle \in R\}$$

for each $I \in \mathbf{I}(\prec_S)$. It was shown in [5] that a total function $f\colon \mathbf{I}(\prec_S) \to \mathbf{I}(\prec_T)$ is continuous (computable) if and only if there is a (c.e.) code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

Example: Let (X, d) be a separable metric space. Fix a countable dense subset $D \subseteq X$, and define a transitive relation $\prec$ on $D \times \mathbb{N}$ as

$$\langle x, n\rangle \prec \langle y, m\rangle \iff d(x, y) < 2^{-n} - 2^{-m}.$$

Then $\mathbf{I}(\prec)$ is homeomorphic to the completion of (X, d) (see [5]). □

Let $\mathbb{S} = \{\bot, \top\}$ be the Sierpinski space, where the singleton $\{\top\}$ is open but not closed. $\mathbb{S}$ is the simplest example of a non-Hausdorff T_0-space. It is well known that every countably based T_0-space can be embedded into the product space $\mathbb{S}^{\mathbb{N}}$.

Example: Let $\mathcal{P}_{fin}(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$, and let $\subseteq$ be the usual subset relation on $\mathcal{P}_{fin}(\mathbb{N})$. Then $\mathbf{I}(\subseteq)$ is homeomorphic to $\mathbb{S}^{\mathbb{N}}$. □

Given a topological space X, we write $\mathbf{O}(X)$ for the set of open subsets of X. We view $\mathbf{O}(X)$ as being a topological space by equipping it with the Scott-topology.

A *represented space* is a tuple (X, δ), where X is a set and $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a partial surjective function from Baire space to X. Given represented spaces (X, δ) and (Y, ρ), a function $f\colon X \to Y$ is *continuous* (*computable*) if there exists a continuous (computable) partial function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $f \circ \delta = \rho \circ F$. Every countably based space can be viewed as a represented space by equipping it with an *admissible representation*, and then a function between countably based spaces is continuous in the sense defined here if and only if it is continuous in the topological sense. In the case of a space of the form $\mathbf{I}(\prec)$, an admissible representation can be viewed as representing each ideal $I \in \mathbf{I}(\prec)$ by enumerating its elements, which is formally defined as the function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{I}(\prec)$ with

$$\delta(p) = I \iff I = \{n \in \mathbb{N} \mid (\exists m \in \mathbb{N})\, p(m) = n\} \in \mathbf{I}(\prec).$$

See [14] for more on admissible representations, and see [13] for more on represented spaces.

3. The category QPol

We represent the category of quasi-Polish spaces by the tuple $\mathsf{QPol} = (\mathsf{Obj}, \mathsf{Mor}, s, t, i, \circ)$ consisting of the following data:

- Obj (objects) is the $\mathbf{\Pi}^0_2$-subspace of $\mathbb{S}^{\mathbb{N}\times\mathbb{N}}$ of transitive relations. Each element $\prec$ of Obj is interpretted as the space of ideals $\mathbf{I}(\prec)$.

- Mor (morphisms) is the represented space constructed as follows. Let $\mathcal{M}$ be the $\mathbf{\Pi}^1_1$-subspace of $\mathbb{S}^{\mathbb{N}\times\mathbb{N}} \times \mathsf{Obj} \times \mathsf{Obj}$ of all triples $\langle R, \prec_S, \prec_T\rangle$ such that $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_S) \to \mathbf{I}(\prec_T)$ is a total function, i.e.

$$(\forall I \in \mathbf{I}(\prec_S))\, \ulcorner R \urcorner(I) \in \mathbf{I}(\prec_T).$$

Define an equivalence relation $\equiv$ on $\mathcal{M}$ as $\langle R_1, \prec_{S_1}, \prec_{T_1}\rangle \equiv \langle R_2, \prec_{S_2}, \prec_{T_2}\rangle$ if and only if $\prec_{S_1}=\prec_{S_2}$ and $\prec_{T_1}=\prec_{T_2}$ and $(\forall I \in \mathbf{I}(\prec_{S_1}))\ulcorner R_1\urcorner(I) = \ulcorner R_2\urcorner(I)$ (extensional equality of functions). Mor is then defined to be the quotient (in the category of represented spaces) of $\mathcal{M}$ by $\equiv$. For convenience, in the following our notation will treat Mor as if it is $\mathcal{M}$ since most of our constructions will respect the equivalence relation $\equiv$ (with the notable exception of equalizers; see below). However, the formal definition as a quotient is necessary when one works with universal constructions in category theory, such as products, which requires certain morphisms to be determined uniquely.

- $s\colon \mathsf{Mor} \to \mathsf{Obj}$ (source) is the projection sending $\langle R, \prec_S, \prec_T\rangle$ to $\prec_S$.
- $t\colon \mathsf{Mor} \to \mathsf{Obj}$ (target) is the projection sending $\langle R, \prec_S, \prec_T\rangle$ to $\prec_T$.
- $i\colon \mathsf{Obj} \to \mathsf{Mor}$ (identity) is the function sending $\prec$ to $\langle =_{\mathbb{N}}, \prec, \prec\rangle$.
- $\circ :\subseteq \mathsf{Mor} \times \mathsf{Mor} \to \mathsf{Mor}$ (composition) is the partial computable function with domain
$$dom(\circ) = \{\langle g, f\rangle \in \mathsf{Mor} \times \mathsf{Mor} \mid s(g) = t(f)\}$$
and which is defined for $f = \langle R_f, \prec_S, \prec\rangle$ and $g = \langle R_g, \prec, \prec_T\rangle$ as
$$\begin{aligned} R &= \{\langle m, n\rangle \mid (\exists p \in \mathbb{N})\,[\langle m, p\rangle \in R_f \,\&\, \langle p, n\rangle \in R_g]\}, \\ g \circ f &= \langle R, \prec_S, \prec_T\rangle. \end{aligned}$$
It is easy to verify that $\ulcorner R\urcorner(I) = \ulcorner R_g\urcorner(\ulcorner R_f\urcorner(I))$, hence composition of total functions yields a total function.

It is straightforward to check that QPol satisfies the axioms of a category:

- $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$,
- $s(i(\prec)) =\prec$ and $t(i(\prec)) =\prec$,
- $(h \circ g) \circ f = h \circ (g \circ f)$ when the compositions $h \circ g$ and $g \circ f$ are defined,
- if $s(f) =\prec_S$ and $t(f) =\prec_T$ then $i(\prec_T) \circ f = f = f \circ i(\prec_S)$.

See [1] for related work on topological groupoids. Note that Obj is a quasi-Polish space but Mor is not, and the fact that QPol is not cartesian closed suggests there is no natural interpretation of Mor as a quasi-Polish space. In the next two subsections we show how to compute products and equalizers in QPol.

3.1. Products and coproducts

Countable products in QPol can be defined as a computable map $\Pi\colon \mathsf{Obj}^{\mathbb{N}} \to \mathsf{Obj}$ by defining $\Pi(\varphi)$ to be the relation $\prec_\Pi$ on $\mathbb{N}^{<\mathbb{N}}$ defined as
$$\sigma \prec_\Pi \tau \iff len(\sigma) < len(\tau) \,\&\, (\forall i < len(\sigma))\, \sigma(i) \prec_i \tau(i),$$
where $\prec_i$ is the relation given by $\varphi(i)$. There is a uniform projection map $p\colon \mathsf{Obj}^{\mathbb{N}} \to \mathsf{Mor}^{\mathbb{N}}$ defined as $p(\varphi)(i) = \langle\{\langle \sigma, n\rangle \mid i < len(\sigma) \,\&\, \sigma(i) = n\}, \Pi(\varphi), \varphi(i)\rangle$, which is the projection map from $\Pi(\varphi)$ to $\varphi(i)$.

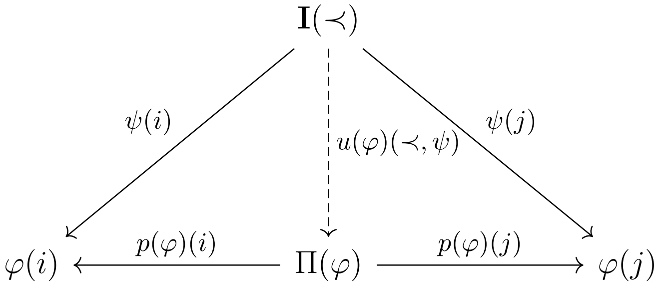
For $\varphi \in \mathsf{Obj}^{\mathbb{N}}$, there is a partial computable function $u(\varphi) :\subseteq \mathsf{Obj} \times \mathsf{Mor}^{\mathbb{N}} \to \mathsf{Mor}$ with domain

$$dom(u(\varphi)) = \{\langle \prec, \psi \rangle \mid (\forall i \in \mathbb{N})\, [s(\psi(i)) = \prec \;\&\, t(\psi(i)) = \varphi(i)]\}$$

defined as

$$u(\varphi)(\prec, \psi) = \langle \{\langle m, \sigma \rangle \mid (\forall i < len(\sigma))(\exists p \in \mathbb{N})\, [\langle p, \sigma(i) \rangle \in \psi(i) \,\&\, p \prec m]\}, \prec, \Pi(\varphi) \rangle$$

which demonstrates the universality of the product in a uniform way[1].



One can also define binary products, binary coproducts, and countable coproducts, but we leave the definitions to the reader as an exercise.

3.2. Equalizers

We can compute equalizers in QPol as a partial multivalued function $e :\subseteq \mathsf{Mor} \times \mathsf{Mor} \rightrightarrows \mathsf{Mor}$ with

$$\begin{aligned} dom(e) &= \{\langle f, g \rangle \in \mathsf{Mor} \times \mathsf{Mor} \mid \langle s(f), t(f) \rangle = \langle s(g), t(g) \rangle\} \\ e(f, g) &= \langle R_E, \prec_E, s(f) \rangle \end{aligned}$$

where

$$R_E = \{\langle \langle \{n\}, p \rangle, n \rangle \mid n, p \in \mathbb{N}\}$$

and for $F, G \in \mathcal{P}_{fin}(\mathbb{N})$ and $p, q \in \mathbb{N}$ we set $\langle F, p \rangle \prec_E \langle G, q \rangle$ if all of the following hold:

1. $p < q$
2. $F \subseteq G$
3. $G \neq \emptyset$
4. $(\forall m \leq p)\; [[(\exists n \in F)\, m \prec_S n] \Rightarrow m \in G]$
5. $(\forall a, b \in F)(\exists c \in G)\, [a \prec_S c \,\&\, b \prec_S c]$
6. $(\forall n \leq p) \left[[(\exists m_1 \in F)\, \langle m_1, n \rangle \in R_f^{(p)}] \Rightarrow (\exists m_2 \in G)\, \langle m_2, n \rangle \in R_g \right]$
7. $(\forall n \leq p) \left[[(\exists m_1 \in F)\, \langle m_1, n \rangle \in R_g^{(p)}] \Rightarrow (\exists m_2 \in G)\, \langle m_2, n \rangle \in R_f \right]$

[1] For the difficult direction of the proof that $\psi(i) = p(\varphi)(i) \circ u(\varphi)(\prec, \psi)$ for each $i \in \mathbb{N}$, if we choose any $j \in \mathbb{N}$ and $n_i \in \psi(i)(I)$ for each $i \leq j$, then there must exist $p_i \in I$ with $\langle p_i, n_i \rangle \in \psi(i)$. Let m be a $\prec$-upper bound of $\{p_i \mid i \leq j\}$ in I and set $\sigma(i) = n_i$ for $i \leq j$. Then $\langle m, \sigma \rangle \in u(\varphi)(\prec, \psi)$, hence $n_i \in p(\varphi)(i)(u(\varphi)(\prec, \psi)(I))$ for each $i \leq j$.

where $\prec_S$ is the relation corresponding to $s(f)$, R_f is a code for f, and $R_f^{(p)}$ is the set that is enumerated within the first p time steps of a given presentation of R_f (and similarly for g, R_g, and $R_g^{(p)}$). It is straightforward to check that $\prec_E$ is transitive. Since the relation $\prec_E$ in $e(f,g)$ depends on the codes R_f and R_g and their presentations, the output of e is multivalued.

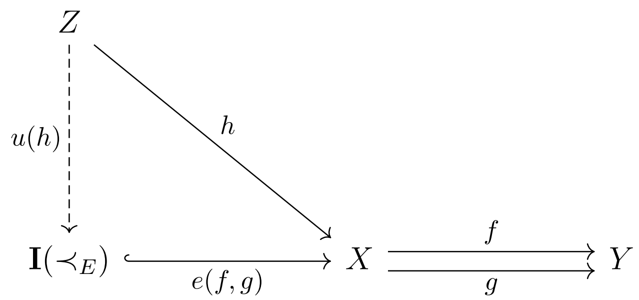
There is a partial computable function $u :\subseteq \mathsf{Mor} \to \mathsf{Mor}$ that demonstrates the universality of equalizers in a uniform way, which has domain

$$dom(u) = \{h \in \mathsf{Mor} \mid t(h) = s(f) \,\&\, f \circ h = g \circ h\}$$

and is defined as $u(h) = \langle R, s(h), \prec_E \rangle$, where

$$R = \{\langle m, \langle F, p\rangle\rangle \mid p \in \mathbb{N} \,\&\, (\forall n \in F)(\exists \langle m_0, n\rangle \in R_h)\, m_0 \prec m\}$$

and R_h is a code for h.



4. Functors

A (computable) functor on QPol is a pair $F = (F_{\mathsf{Obj}}, F_{\mathsf{Mor}})$ of (computable) functions $F_{\mathsf{Obj}} \colon \mathsf{Obj} \to \mathsf{Obj}$ and $F_{\mathsf{Mor}} \colon \mathsf{Mor} \to \mathsf{Mor}$ satisfying

- $F_{\mathsf{Obj}} \circ s = s \circ F_{\mathsf{Mor}}$,
- $F_{\mathsf{Obj}} \circ t = t \circ F_{\mathsf{Mor}}$,
- $F_{\mathsf{Mor}} \circ i = i \circ F_{\mathsf{Obj}}$, and
- $F_{\mathsf{Mor}}(g \circ f) = F_{\mathsf{Mor}}(g) \circ F_{\mathsf{Mor}}(f)$ for all composable $f, g \in \mathsf{Mor}$.

In the following subsections we show how to construct the lower, upper, and valuation powerspace functors on QPol. The double powerspace functor, which maps X to $\mathbf{O}(\mathbf{O}(X))$, is obtained by composing the lower and upper powerspace functors [8].

4.1. Lower powerspace functor

Given a topological space X, the *lower powerspace* $\mathbf{A}(X)$ is the set of all closed subsets of X with the lower Vietoris topology, which is generated by open sets of the form

$$\Diamond U = \{A \in \mathbf{A}(X) \mid A \cap U \neq \emptyset\}$$

for open $U \in \mathbf{O}(X)$. Given a continuous function $f \colon X \to Y$, define $\mathbf{A}(f) \colon \mathbf{A}(X) \to \mathbf{A}(Y)$ as

$$\mathbf{A}(f)(A) = Cl_Y(\{f(x) \mid x \in A\})$$

for each $A \in \mathbf{A}(X)$, where $Cl_Y(\cdot)$ is the closure operator of Y. It was shown in [8] that $\mathbf{A}(\cdot)$ preserves quasi-Polish spaces, hence it is an endofunctor on the category of quasi-Polish spaces.

We represent the lower powerspace functor as a computable functor $(\mathbf{A}_{\mathsf{Obj}}, \mathbf{A}_{\mathsf{Mor}})$ on QPol as follows. For each element $\prec$ of Obj, define $\prec_L$ on $\mathcal{P}_{fin}(\mathbb{N})$ as

$$A \prec_L B \iff (\forall a \in A)(\exists b \in B)\, a \prec b.$$

For each element $\langle R, \prec_S, \prec_T\rangle$ of Mor, define

$$R_L = \{\langle F, G\rangle \mid (\forall n \in G)(\exists m \in F)\, \langle m, n\rangle \in R\}.$$

Finally, define the functor $(\mathbf{A}_{\mathsf{Obj}}, \mathbf{A}_{\mathsf{Mor}})$ on QPol as

$$\begin{aligned} \mathbf{A}_{\mathsf{Obj}}(\prec) &= \prec_L \\ \mathbf{A}_{\mathsf{Mor}}(\langle R, \prec_S, \prec_T\rangle) &= \langle R_L, \mathbf{A}_{\mathsf{Obj}}(\prec_S), \mathbf{A}_{\mathsf{Obj}}(\prec_T)\rangle. \end{aligned}$$

We briefly show that $(\mathbf{A}_{\mathsf{Obj}}, \mathbf{A}_{\mathsf{Mor}})$ is equivalent to the lower powerspace functor. It was shown in [5] that $\mathbf{I}(\prec_L)$ and $\mathbf{A}(\mathbf{I}(\prec))$ are computably homeomorphic for every transitive relation $\prec$ on $\mathbb{N}$, which proves that $\mathbf{A}_{\mathsf{Obj}}$ behaves properly on objects. For $F \in \mathcal{P}_{fin}(\mathbb{N})$, the basic open subset $[F]_{\prec_L}$ of $\mathbf{I}(\prec_L)$ corresponds to the basic open subset $\bigcap_{m\in F} \Diamond[m]_\prec$ of $\mathbf{A}(\mathbf{I}(\prec))$. Explicitly, there are homeomorphisms $f_L \colon \mathbf{A}(\mathbf{I}(\prec)) \to \mathbf{I}(\prec_L)$ and $g_L \colon \mathbf{I}(\prec_L) \to \mathbf{A}(\mathbf{I}(\prec))$ defined as

$$\begin{aligned} f_L(A) &= \{G \in \mathcal{P}_{fin}(\mathbb{N}) \mid (\forall n \in G)(\exists I \in A)\, n \in I\} \\ g_L(J) &= \{I \in \mathbf{I}(\prec) \mid (\forall m \in I)(\exists F \in J)\, m \in F\}. \end{aligned}$$

To show that $\mathbf{A}_{\mathsf{Mor}}$ behaves properly on morphisms, fix a code R for a total function $\ulcorner R\urcorner \colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$, and we will prove $\ulcorner R_L\urcorner = f_L \circ \mathbf{A}(\ulcorner R\urcorner) \circ g_L$. Given $J \in \mathbf{I}(\prec_L)$, we clearly have $G \in \ulcorner R_L\urcorner(J)$ if and only if

$$(\exists F \in J)(\forall n \in G)(\exists m \in F)\, \langle m, n\rangle \in R.$$

On the other hand, $G \in f_L(\mathbf{A}(\ulcorner R\urcorner)(g_L(J)))$

$$\begin{aligned} &\iff (\forall n \in G)(\exists I \in \mathbf{A}(\ulcorner R\urcorner)(g_L(J)))\, n \in I \\ &\iff (\forall n \in G)(\exists I \in g_L(J))\, n \in \ulcorner R\urcorner(I) \\ &\iff (\forall n \in G)(\exists I \in g_L(J))\, (\exists m \in I)\, \langle m, n\rangle \in R \\ &\iff (\forall n \in G)(\exists m \in \mathbb{N})\, [g_L(J) \cap [m]_\prec \neq \emptyset \,\&\, \langle m, n\rangle \in R] \\ &\iff (\exists F \in \mathcal{P}_{fin}(\mathbb{N}))(\forall n \in G)(\exists m \in F)\, [g_L(J) \cap [m]_\prec \neq \emptyset \,\&\, \langle m, n\rangle \in R]. \end{aligned}$$

It follows that $\ulcorner R_L\urcorner(J) \subseteq f_L(\mathbf{A}(\ulcorner R\urcorner)(g_L(J)))$. Conversely, if $G \in f_L(\mathbf{A}(\ulcorner R\urcorner)(g_L(J)))$, then there is $H \in \mathcal{P}_{fin}(\mathbb{N}))$ and $h \colon G \to H$ such that

$$(\forall n \in G)\, [g_L(J) \cap [h(n)]_\prec \neq \emptyset \,\&\, \langle h(n), n\rangle \in R].$$

Set $F = \{h(n) \mid n \in H\}$. Then $F \in J$ by Lemma 7 of [5], and

$$(\forall n \in G)(\exists m \in F)\, \langle m, n\rangle \in R,$$

hence $G \in \ulcorner R_L\urcorner(J)$. Therefore, $\ulcorner R_L\urcorner = f_L \circ \mathbf{A}(\ulcorner R\urcorner) \circ g_L$.

4.2. Upper powerspace functor

Given a topological space X, the *upper powerspace* $\mathbf{K}(X)$ is the set of all saturated compact subsets of X with the upper Vietoris topology, which is generated by open sets of the form

$$\Box U = \{K \in \mathbf{K}(X) \mid K \subseteq U\}$$

for $U \in \mathbf{O}(X)$. Given a continuous function $f\colon X \to Y$, define $\mathbf{K}(f)\colon \mathbf{K}(X) \to \mathbf{K}(Y)$ as

$$\mathbf{K}(f)(K) = Sat_Y(\{f(x) \mid x \in K\})$$

for each $K \in \mathbf{K}(X)$, where $Sat_Y(\cdot)$ is the saturation operator of Y (i.e., $Sat_Y(S) = \bigcap\{U \in \mathbf{O}(Y) \mid S \subseteq U\}$ for each $S \subseteq Y$). It was shown in [8] that $\mathbf{K}(\cdot)$ preserves quasi-Polish spaces, hence it is an endofunctor on the category of quasi-Polish spaces.

We represent the upper powerspace functor as a computable functor $(\mathbf{K}_{\mathsf{Obj}}, \mathbf{K}_{\mathsf{Mor}})$ on QPol as follows. For each element $\prec$ of Obj, define $\prec_U$ on $\mathcal{P}_{fin}(\mathbb{N})$ as

$$A \prec_U B \iff (\forall b \in B)(\exists a \in A)\, a \prec b.$$

For each element $\langle R, \prec_S, \prec_T\rangle$ of Mor, define

$$R_U = \{\langle F, G\rangle \mid (\forall m \in F)(\exists n \in G)\, \langle m, n\rangle \in R\}.$$

Finally, define the functor $(\mathbf{K}_{\mathsf{Obj}}, \mathbf{K}_{\mathsf{Mor}})$ on QPol as

$$\begin{aligned} \mathbf{K}_{\mathsf{Obj}}(\prec) &= \prec_U \\ \mathbf{K}_{\mathsf{Mor}}(\langle R, \prec_S, \prec_T\rangle) &= \langle R_U, \mathbf{K}_{\mathsf{Obj}}(\prec_S), \mathbf{K}_{\mathsf{Obj}}(\prec_T)\rangle. \end{aligned}$$

We briefly show that $(\mathbf{K}_{\mathsf{Obj}}, \mathbf{K}_{\mathsf{Mor}})$ is equivalent to the upper powerspace functor. It was shown in [5] that $\mathbf{I}(\prec_U)$ and $\mathbf{K}(\mathbf{I}(\prec))$ are computably homeomorphic for every transitive relation $\prec$ on $\mathbb{N}$, which proves that $\mathbf{K}_{\mathsf{Obj}}$ behaves properly on objects. For $F \in \mathcal{P}_{fin}(\mathbb{N})$, the basic open subset $[F]_{\prec_U}$ of $\mathbf{I}(\prec_U)$ corresponds to the basic open subset $\Box \bigcup_{m\in F}[m]_\prec$ of $\mathbf{K}(\mathbf{I}(\prec))$. Explicitly, there are homeomorphisms $f_U\colon \mathbf{K}(\mathbf{I}(\prec)) \to \mathbf{I}(\prec_U)$ and $g_U\colon \mathbf{I}(\prec_U) \to \mathbf{K}(\mathbf{I}(\prec))$ defined as

$$\begin{aligned} f_U(K) &= \{G \in \mathcal{P}_{fin}(\mathbb{N}) \mid (\forall I \in K)(\exists n \in G)\, n \in I\} \\ g_U(J) &= \{I \in \mathbf{I}(\prec) \mid (\forall F \in J)(\exists m \in I)\, m \in F\}. \end{aligned}$$

To show that $\mathbf{K}_{\mathsf{Mor}}$ behaves properly on morphisms, fix a code R for a total function $\ulcorner R\urcorner\colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$, and we will prove $\ulcorner R_U\urcorner = f_U \circ \mathbf{K}(\ulcorner R\urcorner) \circ g_U$. Given $J \in \mathbf{I}(\prec_U)$, we clearly have $G \in \ulcorner R_U\urcorner(J)$ if and only if

$$(\exists F \in J)(\forall m \in F)(\exists n \in G)\, \langle m, n\rangle \in R.$$

On the other hand, $G \in f_U(\mathbf{K}(\ulcorner R\urcorner)(g_U(J)))$

$$\begin{aligned} &\iff (\forall I \in \mathbf{K}(\ulcorner R\urcorner)(g_U(J)))(\exists n \in G)\, n \in I \\ &\iff (\forall I \in g_U(J))(\exists n \in G)\, n \in \ulcorner R\urcorner(I) \\ &\iff (\forall I \in g_U(J))(\exists n \in G)(\exists m \in I)\, \langle m, n\rangle \in R \\ &\iff g_U(J) \subseteq \bigcup_{m\in S}[m]_\prec, \text{ where } S = \{m \in \mathbb{N} \mid (\exists n \in G)\, \langle m, n\rangle \in R\} \\ &\iff (\exists F \in J)\, F \subseteq \{m \in \mathbb{N} \mid (\exists n \in G)\, \langle m, n\rangle \in R\} \\ &\iff (\exists F \in J)(\forall m \in F)(\exists n \in G)\, \langle m, n\rangle \in R, \end{aligned}$$

where the fifth equivalence follows from Lemma 9 of [5]. Therefore, $\ulcorner R_U\urcorner = f_U \circ \mathbf{K}(\ulcorner R\urcorner) \circ g_U$.

4.3. Valuation powerspace functor

Let $\overline{\mathbb{R}}_+$ denote the positive extended reals (i.e., $[0,\infty]$) with the Scott-topology induced by the usual order. A *valuation* on a topological space X is a continuous function $\nu\colon \mathbf{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:

1. $\nu(\emptyset) = 0$, and (*strictness*)
2. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$. (*modularity*)

The *valuation powerspace* on X is the set $\mathbf{V}(X)$ of all valuations on X with the *weak topology*, which is generated by subbasic opens of the form

$$\langle U, q\rangle := \{\nu \in \mathbf{V}(X) \mid \nu(U) > q\}$$

with $U \in \mathbf{O}(X)$ and $q \in \overline{\mathbb{R}}_+ \setminus \{\infty\}$. Given a continuous function $f\colon X \to Y$, define $\mathbf{V}(f)\colon \mathbf{V}(X) \to \mathbf{V}(Y)$ as

$$\mathbf{V}(f)(\nu) = \lambda U \in \mathbf{O}(Y).\nu(f^{-1}(U))$$

for each $\nu \in \mathbf{V}(X)$.

$\mathbf{V}(\cdot)$ preserves quasi-Polish spaces (see [6]), hence it is an endofunctor on the category of quasi-Polish spaces. Every valuation on a quasi-Polish space can be extended to a Borel measure [7], and this extension is unique if the valuation is locally finite [3]. Conversely, it clear that the restriction of a Borel measure to the open sets is a valuation. In particular, there is a bijection between probabilistic valuations (i.e., valuations satisfying $\nu(X) = 1$) and probabilistic Borel measures on quasi-Polish spaces.

We represent the valuation powerspace functor as a computable functor $(\mathbf{V}_{\mathsf{Obj}}, \mathbf{V}_{\mathsf{Mor}})$ on QPol as follows. Let $\mathcal{B}$ be the (countable) set of all partial functions $r :\subseteq \mathbb{N} \to \mathbb{Q}_{>0}$ such that $dom(r)$ is finite, where $\mathbb{Q}_{>0}$ is the set of rational numbers strictly larger than zero. For each element $\prec$ of Obj, define $\prec_V$ on $\mathcal{B}$ as $r \prec_V s$ if and only if

$$\sum_{b\in F} r(b) < \sum_{c \in \uparrow F \cap dom(s)} s(c)$$

for every non-empty $F \subseteq dom(r)$, where $\uparrow F = \{c \in \mathbb{N} \mid (\exists b \in F)\, b \prec c\}$.

For each element $\langle R, \prec_S, \prec_T\rangle$ of Mor, define

$$R_V = \left\{ \langle r, s\rangle \;\middle|\; (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow \sum_{a \in A^R_{G,r}} r(a) > \sum_{b\in G} s(b) \right] \right\}$$

where

$$A^R_{G,r} = \{a \in dom(r) \mid (\exists a_0 \in \mathbb{N})(\exists b \in G)\, [a_0 \prec a \,\&\, \langle a_0, b\rangle \in R]\}.$$

Finally, define the functor $(\mathbf{V}_{\mathsf{Obj}}, \mathbf{V}_{\mathsf{Mor}})$ on QPol as

$$\begin{aligned} \mathbf{V}_{\mathsf{Obj}}(\prec) &= \prec_V \\ \mathbf{V}_{\mathsf{Mor}}(\langle R, \prec_S, \prec_T\rangle) &= \langle R_V, \mathbf{V}_{\mathsf{Obj}}(\prec_S), \mathbf{V}_{\mathsf{Obj}}(\prec_T)\rangle. \end{aligned}$$

We briefly show that $(\mathbf{V}_{\mathsf{Obj}}, \mathbf{V}_{\mathsf{Mor}})$ is equivalent to the valuations powerspace functor. It was shown in [6] that $\mathbf{I}(\prec_V)$ and $\mathbf{V}(\mathbf{I}(\prec))$ are computably homeomorphic for every transitive relation $\prec$ on $\mathbb{N}$, which proves that $\mathbf{V}_{\mathsf{Obj}}$ behaves properly on objects. Explicitly, there are homeomorphisms $f_V\colon \mathbf{V}(\mathbf{I}(\prec)) \to \mathbf{I}(\prec_V)$ and $g_V\colon \mathbf{I}(\prec_V) \to \mathbf{V}(\mathbf{I}(\prec))$

defined as

$$f_V(\nu) = \left\{ s \in \mathcal{B} \,\middle|\, (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow \nu(\bigcup_{b \in G}[b]_{\prec}) > \sum_{b \in G} s(b) \right] \right\},$$

$$g_V(I) = \lambda U. \bigvee \left\{ \sum_{a \in dom(r)} r(a) \,\middle|\, r \in I \text{ and } \bigcup_{a \in dom(r)} [a]_{\prec} \subseteq U \right\}.$$

To show that $\mathbf{V}_{\mathsf{Mor}}$ behaves properly on morphisms, fix a code R for a total function $\ulcorner R \urcorner \colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$, and we will prove $\ulcorner R_V \urcorner = f_V \circ \mathbf{V}(\ulcorner R \urcorner) \circ g_V$. Given $I \in \mathbf{I}(\prec_V)$, we clearly have $s \in \ulcorner R_V \urcorner(I)$ if and only if

$$(\exists r \in I)(\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow \sum_{a \in A^R_{G,r}} r(a) > \sum_{b \in G} s(b) \right].$$

Next we consider $f_V(\mathbf{V}(\ulcorner R \urcorner)(g_V(I)))$. As mentioned after the proof of Theorem 13 in [6], if $S \subseteq \mathbb{N}$ then

$$g_V(I)\left(\bigcup_{a \in S} [a]_{\prec} \right) = \bigvee \left\{ \sum_{a \in dom(r)} r(a) \,\middle|\, r \in I \text{ and } (\forall a \in dom(r))(\exists a_0 \in S) a_0 \prec a \right\}.$$

It follows that for any $q \in \mathbb{R}$, we have $g_V(I)\left(\bigcup_{\substack{b \in G\,\&\\ \langle a,b \rangle \in R}} [a]_{\prec} \right) > q$ if and only if there is $r \in I$ such that $\sum_{a \in dom(r)} r(a) > q$ and

$$(\forall a \in dom(r))(\exists a_0 \in \mathbb{N})(\exists b \in G)\, [a_0 \prec a \,\&\, \langle a_0, b \rangle \in R]. \tag{1}$$

As shown in Lemma 5 of [6], if $r \in I$ and $A \subseteq dom(r)$, then the restriction $r|_A$ is also in I. In particular, for any $r \in I$, the resriction $r' = r|_{A^R_{G,r}}$ is also in I, and r' automatically satisfies (1) with r' in place of r. Therefore,

$$g_V(I)\left(\bigcup_{\substack{b \in G\,\&\\ \langle a,b \rangle \in R}} [a]_{\prec} \right) > q \iff (\exists r \in I) \sum_{a \in A^R_{G,r}} r(a) > q.$$

Thus $s \in f_V(\mathbf{V}(\ulcorner R \urcorner)(g_V(I)))$

$$\begin{aligned}
&\iff (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow \mathbf{V}(\ulcorner R \urcorner)(g_V(I))(\bigcup_{b \in G}[b]_{\prec}) > \sum_{b \in G} s(b) \right] \\
&\iff (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow g_V(I)\left(\ulcorner R \urcorner^{-1} \left(\bigcup_{b \in G}[b]_{\prec} \right) \right) > \sum_{b \in G} s(b) \right] \\
&\iff (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow g_V(I)\left(\bigcup_{\substack{b \in G\,\&\\ \langle a,b \rangle \in R}} [a]_{\prec} \right) > \sum_{b \in G} s(b) \right] \\
&\iff (\forall G \subseteq dom(s)) \left[G \neq \emptyset \Rightarrow (\exists r \in I) \sum_{a \in A^R_{G,r}} r(a) > \sum_{b \in G} s(b) \right]
\end{aligned}$$

It immediately follows that $\ulcorner R_V \urcorner(I) \subseteq f_V(\mathbf{V}(\ulcorner R \urcorner)(g_V(I)))$.

For the other inclusion, assume $s \in f_V(\mathbf{V}(\ulcorner R \urcorner)(g_V(I)))$, and for each non-empty $G \subseteq dom(s)$ fix $r_G \in I$ with $\sum_{a \in A^R_{G,r_G}} r_G(a) > \sum_{b \in G} s(b)$. Let r be an $\prec_V$-upper bound of the r_G in I. Let $G \subseteq dom(s)$ be non-empty. Then the choice of r_G and assumption $r_G \prec_V r$ implies

$$\sum_{b \in G} s(b) < \sum_{a \in A^R_{G,r_G}} r_G(a) < \sum_{a \in \uparrow A^R_{G,r_G} \cap dom(r)} r(a).$$

Since $\uparrow A^R_{G,r_G} \cap dom(r) \subseteq A^R_{G,r}$, we obtain

$$\sum_{b \in G} s(b) < \sum_{a \in \uparrow A^R_{G,r}} r(a),$$

hence $s \in \ulcorner R_V \urcorner(I)$. Therefore, $\ulcorner R_V \urcorner = f_V \circ \mathbf{V}(\ulcorner R \urcorner) \circ g_V$.

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