Topological groups described by their continuous homomorphisms or small subgroups

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Abstract

Continuing the classical work of Bohr on almost periodic functions, in 1940 von Neumann introduced the concept of a *minimally almost periodic* (MinAP) group. A topological group is MinAP if all its non-trivial homomorphisms to a compact group are discontinuous. Around the same time, the 5th problem of Hilbert was resolved in the works of Gleason, Montgomery-Zippin and Yamabe who proved that Lie groups are precisely the locally compact NSS groups. A topological group is NSS if it has a neighbourhood of its identity containing no non-trivial subgroup. We examine the history of MinAP groups, from von Neumann to the recent progress of Dikranjan and Shakhmatov. By looking for a connection between NSS and MinAP topological groups, we are led to compare groups described by their small subgroups, with those described by their continuous homomorphisms.

1. Notations and preliminaries

We assume that every topological space is Hausdorff. If X is a topological space, given $A \subseteq X$ we denote by $cl_X(A)$ the *closure* of A in X. If G is a group and $g \in G$ is an element of G we denote by $\langle g \rangle$ the smallest subgroup of G containing g. A group is said to be *Abelian* if its operation is commutative.

For a group G with an operation $\cdot_G$, there is a natural product mapping $m_G : G \times G \to G$ such that $m_G(x, y) = x \cdot_G y$ for all $x, y \in G$. Similarly, there is an *inversion* mapping $in_G : G \to G$ such that $in_G(x) = x^{-1}$ for all $x \in G$.

A topology τ defined on a group G is a *group topology* on G if the product mapping m_G and the inversion mapping i_G are continuous in the topological product $(G, \tau) \times (G, \tau)$ and in the topology τ for G respectively. The pair (G, τ) of a group with such a topology is called a *topological group*. In what follows, when we refer to a group G as being a topological group (without specifying τ), we assume the group G to be equipped with some Hausdorff group topology.

Symbols $\mathbb{Z}$ and $\mathbb{C}$ denote the groups of integer and complex numbers, respectively.

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$\mathbb{N}$ denotes the set of natural numbers, and $\mathbf{P}$ its subset of prime numbers.

Definition 1.1. Let G be an Abelian group and $p \in \mathbf{P} \cup \{0\}$.

(i) A subset X of G is said to be *p-independent* provided that for every $n \in \mathbb{N}$, each pairwise distinct elements $x_1, \ldots, x_n \in X$ and arbitrary integers $m_1, \ldots, m_n \in \mathbb{Z}$, the equation $\sum_{i=1}^{n} m_i x_i = 0$ implies that $m_i \equiv 0 \pmod{p}$ for all $i = 1, \ldots, n$.

(ii) The symbol $r_p(G)$ denotes the maximal cardinality of a p-independent subset of G (which exists by Zorn's lemma).

(iii) The cardinal $r_p(G)$ is called the *p-rank* of G.

(iv) The cardinal $r(G) = r_0(G) + \sum_{p \in \mathbf{P}} r_p(G)$ is called the *rank* of G.

2. Hilbert's Fifth Problem and topological groups without small subgroups (NSS groups)

Definition 2.1. Let G be a topological group. We say that G has *no small subgroups* (commonly abbreviated to NSS) if there exists an open neighbourhood of the identity of G containing no non-trivial subgroups of G.

The class of NSS groups played a fundamental role for the solution of the classical 5th problem of Hilbert from the 1900s. This classical problem was in regard to a characterization for locally compact groups which are simultaneously topological manifolds (known commonly as *Lie groups*). The solution, which is due to Gleason [11], Montgomery-Zippin [15] and Yamabe [26], involved the class of groups from Definition 2.1:

Theorem 2.2. *A topological group is Lie if and only if it is both locally compact and* NSS.

3. Topological groups described by their continuous homomorphisms

3.1. Von Neumann's work on almost periodic functions

For a topological space X, the set $B(X)$ denotes the family of all bounded complex-valued continuous functions on X equipped with the topology of uniform convergence. Given a topological group G, an element $g \in G$ and a complex-valued function f on G, we define the translation of f by g as the function $f_g : G \to \mathbb{C}$ satisfying $f_g(x) = f(xg)$ for all $x \in G$.

Definition 3.1. Let G be a topological group. A function $f \in B(G)$ is almost periodic if every sequence $\{f_{g_n} : n \in \mathbb{N}\}$ of translations of f by elements $g_n \in G$ $(n \in \mathbb{N})$ has a subsequence which is uniformly convergent in $B(G)$.

Real-valued almost periodic functions play a central role in the works of Bohr pertaining to harmonic analysis, and years later the same concept was considered by von Neumann in the context of complex-valued functions. In [17, Theorem 36(i)], von Neumann proved that the family of all almost periodic functions of a topological group G separate its points when G is either compact or locally compact Abelian (and separable). This result motivated the following two concepts:

Definition 3.2 ([17, Definition 16])**.** A topological group G is called:

(a) *maximally almost periodic* (MAP) if its family of almost periodic functions separates its points.

(b) *minimally almost periodic* (MinAP) if its family of almost periodic functions is comprised of only the constant functions.

The above mentioned result of von Neumann implies that compact groups and (separable) locally compact Abelian groups are MAP. von Neumann also showed that almost periodic functions can be replaced by continuous homomorphisms in Definition 3.2:

Theorem 3.3 ([17, Theorem 3](i))**.** *Let G be a topological group.*

(i) G is MinAP *if and only if it admits no non-trivial continuous homomorphism to a unitary group.*

(ii) G is MAP *if and only if the family of continuous homomorphisms to unitary groups separate its points.*

By the classical Peter-Weyl-van Kampen theorem, every compact group is isomorphic to a closed subgroup of a product of unitary groups, so Theorem 3.3 can be reformulated as follows:

Corollary 3.4. *Let G be a topological group.*

(i) G is MinAP *if and only if it admits no non-trivial continuous homomorphism to a compact group.*

(ii) G is MAP *if and only if the family of continuous homomorphisms to compact groups separate its points.*

In [18], von Neumann and Wigner focus on the class of minimally almost periodic groups. In their paper, they construct a handful of examples of minimally almost periodic groups [18, Section 5] via linear transformations. They note, however, that constructing groups in this class is not a trivial effort.

The class of minimally almost periodic groups gained a great deal of attention from experts in topological group theory thanks to two high-profile open problems which we shall describe in the next two subsections.

3.2. The connection of MinAP groups to extreme amenability

The first problem is related to the concept of *extremely amenable* groups.

Definition 3.5. A topological group is *extremely amenable* (or satisfies the *fixed point in compacta property*) if every continuous action of it on a compact space admits a fixed point.

Extremely amenable groups appeared in the context of Harmonic Analysis and Dynamical Systems (see [10, 19]). These groups are intimately connected to the class of MinAP groups by the following fact:

Fact 3.6. Every extremely amenable group is minimally almost periodic.

It is known that the converse implication does not hold in general. However, whether the converse implication holds or not in the realm of Abelian groups remains as a major open problem to this day:

Problem 3.7 (Pestov, 1998)**.** Is every Abelian MinAP topological group extremely amenable?

A topological group is *monothetic* if it contains a dense subgroup which is isomorphic to the group of integers. Every monothetic group is Abelian. The following particular version of Problem 3.7 was posed by Glasner as far back as 1988:

Problem 3.8 (Glasner, 1988)**.** Must every monothetic MinAP topological group be extremely amenable?

This particular version of Glasner has important implications in number theory. A negative answer to Problem 3.8 of Glasner would provide an answer to the following ancient problem (see [24]) of combinatoric number theory:

Problem 3.9. If S is a big set of the integers, is it true that the difference $S-S$ is a Bohr neighbourhood of 0?

Problems 3.7, 3.8 and 3.9 are still open.

3.3. Algebraic structure of MinAP groups

The difficulty in the construction of MinAP groups sparked a great deal of interest in regards to their algebraic structure. First examples of MinAP groups were the additive groups of some topological vector spaces [3], as explained in [14]. Nienhuys [16] constructed a connected monothetic group of cardinality at most continuum which is minimally almost periodic. This implies the existence of a MinAP group topology on the group $\mathbb{Z}$ of integers.

In 1984 Protasov posed the question of whether *every* Abelian group admits a minimally almost periodic group topology. In 1989 Remus provided an example of a bounded Abelian group which does not admit a MinAP group topology, so Comfort proposed the following modification of the original question of Protasov:

Problem 3.10 (Comfort, 1990 [1, Question 521])**.** Does every Abelian group which is not of bounded order admit a minimally almost periodic group topology?

The bounded case was resolved by Gabriyelyan [9] who showed that a bounded Abelian group admits a minimally almost periodic group topology if and only if all of its leading Ulm-Kaplansky invariants are infinite. The general case was resolved by Dikranjan and Shakhmatov [6] in 2014:

Theorem 3.11 (Dikranjan-Shakhmatov [6, Theorem 3.3])**.** *For an Abelian group G, the following conditions are equivalent:*

(i) G admits a minimally almost periodic group topology;

(ii) G is connected with respect to its Markov-Zariski group topology [5]*;*

(iii) for every $n \in \mathbb{N}$, the subgroup $nG = \{ng : g \in G\}$ of G is either trivial or infinite.

3.4. Classes $\mathrm{MinAP}(\mathcal{C})$ for various classes $\mathcal{C}$ of topological groups

The author proposed the following terminology in [25]:

Definition 3.12. Let $\mathcal{C}$ denote a class of topological groups. We say that a topological group is $\mathrm{MinAP}(\mathcal{C})$ (or satisfies the $\mathrm{MinAP}(\mathcal{C})$ property) if the only continuous homomorphism to a group contained in the class $\mathcal{C}$ is the trivial homomorphism.

The following remark is obtained from Corollary 3.4 in this new terminology:

Remark 3.13. The class of MinAP(Compact) topological groups coincides with the class of MinAP topological groups of von Neumann.

We are interested in finding natural classes $\mathcal{C}$ of topological groups for which the class of $\mathrm{MinAP}(\mathcal{C})$ groups becomes a proper subclass of MinAP groups. A necessary condition for this is that $\mathcal{C}$ has at least one group which is not MAP.

Naturally, if a class $\mathcal{D}$ is a subclass of $\mathcal{C}$, then

$$\mathrm{MinAP}(\mathcal{C}) \to \mathrm{MinAP}(\mathcal{D}).$$

Theorem 3.14 ([25])**.** *The following diagram describes implications for all topological groups.* In Figure 1 below LC stands for "locally compact".

$$\begin{tikzcd}
& \mathrm{MinAP(LC)} + \mathrm{MinAP(NSS)} \arrow[dl, "8"'] \arrow[d, "2"] \arrow[dr, "7"] & \\
\mathrm{MinAP(LC)} \arrow[r, "1"] \arrow[dr, "6"'] & \mathrm{MinAP(Lie)} \arrow[d, "5"] & \mathrm{MinAP(NSS)} \arrow[l, "3"'] \arrow[dl, "4"] \\
& \mathrm{MinAP} \arrow[d, equal] & \\
& \mathrm{MinAP(Compact)} &
\end{tikzcd}$$

Figure 1: Diagram of implications between $\mathrm{MinAP}(\mathcal{C})$ properties

Arrows 2–7 are not reversible in general. The reversibility of arrow 1 is unclear. For Abelian groups, arrows 1 and 5 of Figure 1 become reversible:

Theorem 3.15 ([25])**.** *For an Abelian topological group G, properties* MinAP*,* MinAP(*Locally compact*) *and* MinAP(*Lie*) *are equivalent.*

4. Topological groups with many small subgroups

4.1. The property DW of Dierolf and Warken

Definition 4.1 (Implicitly used in [4])**.** Let G be a topological group. We say that G satisfies property DW if for every open neighbourhood U of the identity of G, every element $g \in G$ can be written as $g = \prod_{i=1}^{n} x_i$ for some $x_1, \ldots, x_n \in G$ such that $\langle x_i \rangle \subseteq U$ for $i = 1, \ldots, n$.

In [4, Proof of Theorem 1.1], Dierolf and Warken essentially prove the following

Proposition 4.2. *A topological group with property* DW *is* MinAP.

The original theorem of Dierolf and Warken [4] can thus be stated as follows:

Theorem 4.3 ([4, Theorem 1.1])**.** *Every topological group G is topologically isomorphic to a closed subgroup of some topological group H_G (depending on G) which satisfies property* DW*. As a consequence, the group H_G is* MinAP*.*

4.2. The SSGP property of Gould

Gould [12] isolated Proposition 4.2 from the result of Dierolf and Warken and considered the following class of topological groups:

Definition 4.4 (Originally by Gould [12])**.** A topological group G has the *small subgroup generating property* (SSGP) if and only if for every open neighbourhood U of the identity of G, the set of all $g \in G$ such that $g = \prod_{i=1}^{n} x_i$ for some elements $x_1, \ldots, x_n \in U$ satisfying $\langle x_i \rangle \subseteq U$ $(i = 1, \ldots, n)$, is dense in G.

The difference between the SSGP property of Gould and property DW used by Dierolf and Warken is subtle, but none the less non-trivial. Property DW is an algebraic expression for all elements of the group depending on the neighbourhoods of the identity. Meanwhile, in Definition 4.4, the requirement is that the set of elements which can be represented in the way proposed in property DW is topologically dense. Gould [12] proved that every SSGP topological group is MinAP, so

$$\mathrm{DW} \rightarrow \mathrm{SSGP} \rightarrow \mathrm{MinAP}. \tag{1}$$

Theorem 4.5 ([22])**.** *For groups of bounded order, properties* SSGP *and* DW *coincide.*

4.3. A family of $\mathtt{SSGP}(\alpha)$ properties of Dikranjan and Shakhmatov

Dikranjan and Shakhmatov invented an operator-based approach to define an entire series of SSGP-type properties in [8]. This operator (denoted by $\mathbf{S}_G$ for a topological group G) was designed along with a series of very carefully crafted iterations (which are denoted by $\mathbf{S}_G^{(\alpha)}$ for every ordinal α). The iterations of this operator are monotone with respect to subsets and with respect to ordinals:

Proposition 4.6 ([8, Lemma 4.5, Lemma 4.8])**.** *The following hold:*

(i) The operator $\mathbf{S}_G^{(\alpha)}$ is monotone with respect to subsets*. So $X \subseteq Y \subseteq G$ implies that $\mathbf{S}_G^{(\alpha)}(X) \subseteq \mathbf{S}_G^{(\alpha)}(Y) \subseteq \mathbf{S}_G^{(\alpha)}(G)$.*

(ii) The operators $\mathbf{S}_G^{(\alpha)}$ are monotone with respect to ordinals*. So $\beta \leq \alpha$ implies that $\mathbf{S}_G^{(\beta)}(X) \subseteq \mathbf{S}_G^{(\alpha)}(X)$ for all $X \subseteq G$.*

With this iterated operator, they define the following:

Definition 4.7 (Dikranjan-Shakhmatov [8])**.** A topological group G is $\mathtt{SSGP}(\alpha)$ (or satisfies the $\mathtt{SSGP}(\alpha)$ property) for some ordinal α if and only if $\mathbf{S}_G^{(\alpha)}(U) = G$ is satisfied for every neighbourhood of the identity of G.

The above definition is an extension of the SSGP property, as it was shown in [8] that the $\mathtt{SSGP}(1)$ property coincides with the original SSGP property of Gould. Comfort and Gould had pioneered a concept of a countable number of SSGP(n) properties (for every integer $n \in \mathbb{N}$) in [2]. The series of properties defined by Dikranjan and Shakhmatov, however, are substantially more general as they are defined for every ordinal (including infinite ones). Moreover, these two concepts were shown to coincide in the realm of Abelian groups ([8, Corollary 6.3]) when the index is a natural number.

Dikranjan and Shakhmatov prove in [8, Proposition 5.3(ii)] that every $\mathtt{SSGP}(\alpha)$ group admits no non-trivial continuous homomorphism to an NSS group, i.e. it is MinAP(NSS) in the terminology of Definition 3.12. Combining this with the hierarchy described in [8, Proposition 5.1] and arrow 4 of Figure 1, we obtain the following transfinite chain of implications:

$$\begin{aligned} \mathrm{SSGP} = \mathtt{SSGP}(1) \to \cdots \to \mathtt{SSGP}(n) \to \mathtt{SSGP}(n+1) \cdots \to \mathtt{SSGP}(\alpha) \to \dots \\ \cdots \to \mathtt{SSGP}(\alpha+1) \to \cdots \to \mathrm{MinAP(NSS)} \to \mathrm{MinAP}. \end{aligned} \tag{2}$$

Definition 4.8 ([25])**.** We say that a topological group is $\mathtt{SSGP}(\infty)$ if and only if it is an $\mathtt{SSGP}(\alpha)$ group for some ordinal α.

As an application of a theorem of Yamabe [26], we proved the following

Theorem 4.9 ([25])**.** *Every* $\mathtt{SSGP}(\infty)$ *group is* MinAP(*Locally compact*)*.*

Combining this with (1), (2) and terminology from Definition 4.8, we get

$$\mathrm{DW} \to \mathrm{SSGP} \to \mathtt{SSGP}(\infty) \to \mathrm{MinAP(NSS)} + \mathrm{MinAP(Locally\ compact)}. \tag{3}$$

Therefore, the implications in (3) can be added "on top" of Figure 1.

4.4. Abelian MinAP(NSS) groups are precisely $\mathtt{SSGP}(\infty)$

The following theorem bridges topological groups described by continuous homomorphisms with those described by abundance of small subgroups.

Theorem 4.10 ([25])**.** *Properties* $\mathtt{SSGP}(\infty)$ *and* MinAP(NSS) *are equivalent for abelian topological groups.*

Theorems 3.15 and 4.10 lead to the following diagram for Abelian groups:

$$\begin{tikzcd}
 & \mathrm{MinAP(NSS)} \arrow[r, leftrightarrow, "7", "Abelian"'] \arrow[d, "3"] & \mathtt{SSGP}(\infty) \arrow[d, "9"] \\
\mathrm{MinAP} \arrow[r, leftrightarrow, "Abelian"'] & \mathrm{MinAP(Lie)} \arrow[r, leftrightarrow, "Abelian"'] & \mathrm{MinAP(Locally\ compact)}
\end{tikzcd}$$

Figure 2: Simplified diagram of implications in Abelian topological groups

Corollary 4.11 ([25])**.** *An Abelian topological group G satisfies the equivalence*

$$\mathrm{MinAP}(\textit{Locally compact}) + \mathrm{MinAP(NSS)} \iff \mathrm{MinAP}(\textit{Lie})$$

if and only if G has the $\mathtt{SSGP}(\infty)$ *property.*

The following example shows that the converse of Theorem 4.9 does not hold, and Theorem 4.10 fails for non-Abelian topological groups.

Example 4.12 (Shakhmatov)**.** Let $S(\mathbb{N})$ be the group of all bijections of $\mathbb{N}$ with the composition of maps as its group operation, equipped with the subspace topology it inherits from the Tychonoff product $\mathbb{N}^{\mathbb{N}}$ when $\mathbb{N}$ is considered with its discrete topology. Then, $S(\mathbb{N})$ *is a complete separable metric (Polish) group which is both* MinAP(NSS) *and* MinAP(*Locally compact*) *but is not* $\mathtt{SSGP}(\infty)$.

4.5. Algebraic structure of SSGP-type groups and MinAP(NSS) groups

The following problem is a natural heir of Problem 3.10 of Comfort and Protasov:

Problem 4.13 ([2, Comfort and Gould])**.** Which Abelian groups admit an SSGP group topology?

In the series of papers [12, 13, 2] Gould constructs a variety of SSGP groups. He also provides several examples of groups which can never be equipped with an SSGP group topology. One aspect of note, however, is that the results of Gould required very careful manual manipulation of group metrics, highlighting the difficulty of obtaining a full answer of Problem 4.13.

Definition 4.14 ([7, Definition 7.2])**.** For an Abelian group the cardinal $r_d(G) = \min\{r(nG) : n \in \mathbb{N}^+\}$ is called the *divisible rank* of G.

Problem 4.13 of Comfort and Gould was solved even for the wider class of $\mathtt{SSGP}(\infty)$ topological groups. The result was achieved in three steps. The first step concerning torsion groups was made by Comfort and Gould themselves, by showing that a torsion group admits an SSGP group topology if and only if it admits a MinAP group topology. The second step concerning groups of infinite divisible rank was done by Dikranjan and Shakhmatov in [8]. The remaining case of positive finite divisible rank was then reduced by them to a very specific question, which in turn was recently resolved in [20].

Theorem 4.15 ([8, 20])**.** *The following are equivalent for an Abelian group G:*

(a) G admits an SSGP *group topology,*

(b) G admits an $\mathtt{SSGP}(\infty)$ *group topology, and*

(c) one of the two conditions is satisfied:

  (i) G is of infinite divisible rank, or

  (ii) the quotient $H = G/t(G)$ of G by its torsion part $t(G)$ has finite free rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

Combining this with Theorem 4.10, we obtain a complete description of Abelian groups which admit a MinAP(NSS) group topology:

Corollary 4.16 ([25])**.** *An Abelian group G admits a* MinAP(NSS) *group topology if and only if G satisfies condition (c) of Theorem 4.15.*

4.6. Algebraic structure of DW groups

Thanks to Theorems 3.11 and 4.5, it only remains to describe which unbounded groups admit property DW.

Theorem 4.17 ([22]). *A torsion Abelian group G admits a group topology with property* DW *if and only if every p-component of G admits a group topology with property* DW*.*

For Abelian groups of positive finite 0-rank we have the following necessary condition:

Theorem 4.18 ([22]). *Let G be an Abelian group such that $0 < r_0(G) < \infty$. If G admits a group topology with property* DW*, then there exists a prime number p such that the p-rank $r_p(G)$ of G is infinite.*

As for the construction of these topologies on Abelian groups, we have the following sufficient condition for countable groups:

Theorem 4.19 ([23]). *If a countable Abelian group G possesses one of the following properties, then it admits a metric group topology with property* DW*:*

(i) G has infinite $r_0(G)$ rank,

(ii) G is torsion and every non-trivial p-component G_p of G is either bounded with all of its leading Ulm-Kaplansky invariants infinite, or it has infinite divisible rank $r_d(G_p)$.

Theorem 4.20 ([23]). *Let G be an Abelian torsion group which is either divisible or countable. Then G admits a group topology with property* DW *if and only if each of its non-trivial p-components G_p admits an* SSGP *group topology.*

For non-Abelian groups, we have the following results for free groups:

Theorem 4.21 ([21, Theorem 2.2 and 2.3]). *The following hold:*

(i) A free group with a countably infinite set of generators admits a metric *group topology with property* DW*.*

(ii) A free group with infinitely many generators admits a group topology with property DW*.*

By contrast, the symmetric groups $S(X)$ with the topology of pointwise convergence do not admit an $\mathtt{SSGP}(\infty)$ group topology by [8, Example 5.4(d)], so they cannot have a group topology with property DW by (3).

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