Braids, polynomials and real algebraic links

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1. Introduction

In the last chapter of his seminal book [8] Milnor discusses the real analogue of links of isolated singularities of complex plane curves, later termed real algebraic links.

Definition 1.1. A link \( L \) is real algebraic if there exists a polynomial \( f : \mathbb{R}^4 \to \mathbb{R}^2 \) such that

- \( f((0,0,0,0)) = (0,0) \),
- \( \nabla f((0,0,0,0)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \),
- there is a neighbourhood \( B \) of \((0,0,0,0)\) such that \((0,0,0,0)\) is the only point in \( B \) where the rank of \( \nabla f \) is not full,
- \( f^{-1}((0,0)) \cap S^3_\rho = L \) for all small enough radii \( \rho \).

The first three conditions state that \( f \) has an isolated singularity at the origin in \( \mathbb{R}^4 \).

Milnor himself points out that it is highly challenging to construct examples of such links that do not come from complex plane curves \( f : \mathbb{C}^2 \to \mathbb{C} \). Indeed, the family of links that is known to be real algebraic is still comparatively small [5, 7, 9, 10].

One difference between the complex and the real polynomials is that in general the argument of a real polynomial as in Definition 1.1 (\( \arg f : S^3_\rho \to S^1 \)) is not a fibration. However, Milnor established that the following is still true.

Theorem 1.2 (Milnor [8]). If a link \( L \) is real algebraic, then \( L \) is fibred.

According to Benedetti and Shiota, this implication should be an equivalence.

Conjecture 1.3 (Benedetti-Shiota [2]). A link \( L \) is real algebraic if and only if \( L \) is fibred.

In this short note we discuss a construction of real polynomial maps with isolated singularities as in Definition 1.1, following [5]. Section 2 reviews this construction in a quite general setting, while Section 3 focuses on the class of homogeneous braids.

Definition 1.4. A braid \( B \) on \( s \) strands is called homogeneous if for every \( i = 1, 2, \ldots, s - 1 \) the generator \( \sigma_i \) appears in the word \( B \) if and only if \( \sigma_i^{-1} \) does not appear.

The 3-strand braid \((\sigma_1 \sigma_2^{-1})^2\) for example is homogeneous, because \( \sigma_1 \) always comes with a positive sign and \( \sigma_2 \) always comes with a negative sign. If we change one of the signs however, or if we consider the same braid word as a 4-strand braid, we obtain an inhomogeneous braid. Note that the homogeneous braids contain all alternating braids that do not close to split links.

Theorem 1.5 (Bode [5]). Let \( B \) be a homogeneous braid. Then the closure of \( B^2 \) is real algebraic.
The proof that homogeneous braids close to fibred links is due to Stallings [12]. An important aspect of the proof in [5] is that for a loop in the space of complex polynomials, we can establish a close relation between the braid that is formed by the roots of the polynomials and the braid that is formed by their critical values. Here, we give more details on this relation and illustrate how this could be used to generalize the construction to a larger family of links.

Section 4 gives an example of the outlined construction.

2. Constructing real algebraic links

Let $B$ be a braid on $s$ strands. We denote by $\mathcal{C}$ the set of connected components of the closure of $B$ or equivalently the set of cycles of the image of $B$ under the permutation representation. For every $C \in \mathcal{C}$ we write $s_C$ for the number of strands that make up the component $C$ or equivalently the length of that corresponding cycle. Then the total number of strands $s$ equals $\sum_{C \in \mathcal{C}} s_C$. Suppose we have a parametrisation of $B$ in $\mathbb{C} \times [0, 2\pi]$ given by

$$
\bigcup_{C \in \mathcal{C}} \bigcup_{j=1}^{s_C} \left( F_C \left( \frac{t + 2\pi j}{s_C} \right) + \i G_C \left( \frac{t + 2\pi j}{s_C} \right), t \right), \quad t \in [0, 2\pi],
$$

where $F_C, G_C : [0, 2\pi] \to \mathbb{R}$ are trigonometric polynomials. Such a parametrisation exists for every braid and in fact there are even some bounds on the Fourier degree of $F_C$ and $G_C$ [4].

Then we can define the polynomial $g_\lambda : \mathbb{C} \times [0, 2\pi] \to \mathbb{C}$,

$$
g_\lambda(u, t) = \prod_{C \in \mathcal{C}} \prod_{j=1}^{s_C} \left( u - \lambda \left( F_C \left( \frac{t + 2\pi j}{s_C} \right) + \i G_C \left( \frac{t + 2\pi j}{s_C} \right) \right) \right)
$$

with $\lambda > 0$ and the nodal set $g_\lambda^{-1}(0)$ is $B$ for all values of $\lambda$.

Furthermore, expanding the product in Equation (2) results in a polynomial not only in the complex variable $u$, but also in $e^{it}$ and $e^{-it}$.

We now replace every instance of $e^{it}$ in the polynomial expression of $g_\lambda$ by another complex variable $v$ and every instance of $e^{-it}$ by its conjugate $\overline{v}$. This identifies the variable $t$ with the angular coordinate of $v$. We thus obtain a polynomial $f_\lambda : \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \cong \mathbb{R}^2$ in $u, v$ and $\overline{v}$. In general, $f_\lambda$ does not have an isolated singularity. However, it is an easy calculation to show the following.

**Proposition 2.1 (Bode [5]).** Let $k \geq (\text{deg } f_\lambda)/(2s)$. Then for all small enough $\lambda > 0$ the map $p_\lambda : \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \cong \mathbb{R}^2$ given by

$$
p_\lambda(u, v) = (v \overline{v})^k f_\lambda \left( \frac{u}{(v \overline{v})^k}, \frac{v}{\sqrt{v \overline{v}}} \right),
$$

or equivalently

$$
p_\lambda(u, re^{it}) = \begin{cases} r^{2^k} g_\lambda \left( \frac{u}{r^{2^k}}, t \right) & \text{if } r > 0, \\ u & \text{if } r = 0, \end{cases}
$$

has an isolated singularity at the origin if and only if $\arg g_\lambda : (\mathbb{C} \times [0, 2\pi]) \setminus g_\lambda^{-1}(0) \to S^1$ is a fibration. The link of the singularity is the closure of $B$.

There are a couple of things to note here. Firstly, the condition on $\arg g_\lambda$ does not depend on $\lambda$. Secondly and very importantly, the map $p_\lambda$ is in general not a polynomial because we have introduced square root terms. However if Equation (1) is a $\pi$-periodic parametrisation, then all of the exponents with non-zero coefficients in the trigonometric polynomials $F_C$ and
$G_C$ are even. This can be arranged if and only if $B$ is a square, i.e., $B = A^2$ for some braid $A$. It follows that in this case all exponents of $v$ and $\nu$ in $f^\lambda$ can be taken to be even, so that all square roots in Equation (3) cancel.

**Proposition 2.2.** If a braid $B$ can be parametrised as in Eq. (1) such that $\arg g^\lambda$ gives a fibration over $S^1$, then $B^2$ closes to a real algebraic link.

We now want to investigate which braids satisfy the condition in Proposition 2.2 and study in particular the case of homogeneous braids.

### 3. Homogeneous real algebraic links

For every $t \in [0, 2\pi]$ the map $u \mapsto g^\lambda(u, t)$ is a monic complex polynomial. We denote by $v_i(t), i = 1, 2, \ldots, s - 1$, the critical values of $g^\lambda(u, t)$, i.e., the images $g^\lambda(c_i, t)$ of the critical points $c_i$ of $u \mapsto g^\lambda(u, t)$, determined by the condition $\frac{\partial g^\lambda}{\partial u}(c_i, t) = 0$. It is again a simple calculation to show that $\arg g^\lambda$ is a fibration over $S^1$ if and only if for all $i = 1, 2, \ldots, s - 1$ the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. This has a nice geometric interpretation in terms of the movement of the critical values in the complex plane as $t$ varies. Since $g^\lambda(u, t)$ has distinct roots, the critical values $v_i(t)$ are always non-zero. The inequality states that no critical value $v_i(t)$ ever changes the orientation in which it twists around the origin $0 \in \mathbb{C}$ as $t$ increases from 0 to $2\pi$. Every critical value moves either always clockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} < 0\right)$ or always counterclockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} > 0\right)$.

Proposition 2.2 can therefore be updated and rewritten in terms of polynomials and critical values. Let $\tilde{X}_s$ be the space of monic complex polynomials of degree $s$ with distinct roots. The fundamental theorem of algebra gives a straightforward identification of a polynomial $f = \prod_{j=1}^s (u - x_j) \in \tilde{X}_s$ with its unordered set of roots $\{x_1, x_2, \ldots, x_s\}$. This allows us to identify a loop $f_t$ in $\tilde{X}_s$ with the (closed) braid that is formed by the roots of the polynomials

$$\bigcup_{j=1}^s (x_j(t), t) \subset \mathbb{C} \times S^1. \tag{5}$$

**Proposition 3.1.** Let $f_t, t \in S^1$, be a loop in $\tilde{X}_s$ such that for all $i = 1, 2, \ldots, s - 1$ the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. Let $B$ be the braid that is formed by the roots of $f_t$. Then the closure of $B^2$ is real algebraic.

In Proposition 3.1 we do not require explicitly that the parametrisation of $B$ is given in terms of trigonometric polynomials. Since these are $C^1$-dense in the space of $2\pi$-periodic real $C^1$-functions, we can always approximate a parametrisation as in Proposition 3.1 without losing the property that $\frac{\partial \arg v_i(t)}{\partial t}$ does not vanish.

We can also assume that the critical values are distinct. This means that

$$(0, t) \cup \bigcup_{j=1}^{s-1} (v_j(t), t) \subset \mathbb{C} \times S^1 \tag{6}$$

forms a closed braid. Let $X_s \subset \tilde{X}_s$ be the space of those polynomials in $\tilde{X}_s$ that have distinct critical values. Then the space of possible sets of critical values of a polynomial in $X_s$ is given by

$$V_s = \{(v_1, v_2, \ldots, v_{s-1}) \in (\mathbb{C}\setminus\{0\})^{s-1} : v_i \neq v_j \text{ if } i \neq j\}/S_{s-1}, \tag{7}$$

where $S_{s-1}$ is the permutation group on $s - 1$ elements. Then the braid in Equation (6) can be interpreted as a loop in $V_s$. Another interpretation of Proposition 3.1 is therefore asking
Figure 1: If the critical values $v_j(t)$ move on ellipses around the origin in the complex plane without ever changing direction (from clockwise to counterclockwise or vice versa) as $t$ varies, the derivatives $\frac{\partial \arg v_j(t)}{\partial t}$ never vanishes.

which braids can be parametrized as loops in $V_s$ satisfying the condition on $\frac{\partial \arg v_i(t)}{\partial t}$ and such that this loop is the image of a loop in $X_s$ under the map that sends a polynomial to the set of its critical values.

The words

$$A_{i,j} = \sigma_j \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \ldots \sigma_{j-1} \sigma_j^{-1}$$

with $i \leq j < s$ (8)

are a set of generators for the braid group $\mathbb{B}_s$. The generator $A_{i,j}$ takes the strand $j+1$ and twists it around the strand $i$.

**Lemma 3.2.** The $s$-strand braid $\prod_{j=1}^n A_{1,i_j}$ can be parametrised as in Eq. (6) such that $\frac{\partial \arg v_i(t)}{\partial t}$, $i = 1, 2, \ldots, s-1$, never vanishes if for every $k = 1, 2, \ldots, s-1$ there is a $j$ such that $i_j = k$.

**Proof.** The parametrisation can be achieved if all $v_i(t)$ move on ellipses in the complex plane as $t$ varies as indicated in Figure 1. Then for every $j$ the generator $A_{1,j}$ can be parametrised such that $\frac{\partial \arg v_{j-1}(t)}{\partial t}$ does not vanish and all other strands are stationary. We can now concatenate the parametrisations for the $A_{1,j}$ to obtain the desired braid word.

The fact that the generator $A_{1,j}$ appears in the braid word if and only if $A_{1,j}^{-1}$ does not appear implies that none of the $v_i$ ever turns around on its ellipse. The condition that for every $k = 1, 2, \ldots, s-1$ there is a $j$ such that $i_j = k$ means that every $v_i$ moves at some point. We can thus slightly perturb the parametrisation of each $A_{1,j}$ such that none of the $v_i$ is non-stationary. For example, for the parametrisation of $A_{1,j}$ every $v_i$ with $i \neq j - 1$ moves an $\varepsilon$-amount on its ellipse.

The braids in Lemma 3.2 allow for the desired kind of parametrisation. What we need to check now is that there is a loop in the space of polynomials $X_s$, whose critical values form that parametrisation. Then we want to know which braid is formed by the roots of these polynomials.

The following theorem is very useful for this. Recall that $X_s$ is the space of monic polynomials with distinct roots and distinct critical values.

**Theorem 3.3** (Beardon-Carne-Ng [1]). Let $X_s^0 \subset X_s$ be the space of those polynomials in $X_s$ that have constant term equal to 0. Then the map $\theta_s : X_s^0 \to V_s$ that sends a polynomial $f \in X_s^0$ to the set of its critical values $(v_1, v_2, \ldots, v_{s-1})$ is a covering map of degree $s^{s-1}$. 
The map $\theta_j$ is well-defined even if the constant term is not equal to zero.

**Corollary 3.4.** Let $\hat{X}_s \subset X_s$ be the space of those polynomials in $X_s$ whose critical values are not equal to their constant terms, i.e. $v_i \neq f(0)$ for all $i = 1, 2, \ldots, s-1$. Let $\gamma$ be a loop in $V_s$ and $\tilde{\gamma}$ a path in $\hat{X}_s$ such that $\theta_s(\tilde{\gamma}) = \gamma$. Then any homotopy of $\gamma$ in $V_s$ lifts to a homotopy of $\tilde{\gamma}$ in $\hat{X}_s$.

The homotopy lifting property in Corollary 3.4 means that the properties that we want to check do not depend on the particular braid parametrisation, but only on the braid type. We are going to show that there is a braid that is conjugate to the braid in Lemma 3.2 for which it is relatively easy to determine that if it is interpreted as a loop in $V_s$, then it lifts to a loop in $\hat{X}_s$. Furthermore, determining the braid $B$ that is formed by the roots of the corresponding polynomials is straightforward. Corollary 3.4 then implies that the braids in Lemma 3.2 also lift to a loop in $\hat{X}_s$ and the braid that corresponds to the roots of the polynomials is conjugate to $B$.

We start with a polynomial $f$, whose roots are real. Then all of its critical points and critical values are real as well. This is depicted in Figure 2. Between each pair of roots there is exactly one critical point. We label the critical points $\{c_i\}_{i=1,2,\ldots,s-1}$ such that $c_i$ is the critical point between the $i$th and $i+1$th smallest root. We set $v_i = f(c_i)$. We now consider a particular loop $\gamma_i : [0,2\pi] \to \mathbb{C}$ in the target complex plane based at the origin. The loop stays close to the real line and encircles the critical value $v_i$ in a counterclockwise direction. The loop $\gamma_i$ does not intersect any critical values. When it is about to encounter a critical value $v_j$, it avoids it by moving into the upper or the lower half plane. If $j > i$ and $i \equiv s+1 \mod 2$ or if $j < i$ and $i \equiv s \mod 2$, then $\gamma_i$ moves into the upper half plane. Otherwise, it avoids $v_j$ by moving into the lower half plane. At the moment this choice might seem arbitrary (and to some degree it is), but we will come back to why this turns out to be a good rule. An example of $\gamma_i$ is shown in Figure 2b).

Now we look at the preimage $f^{-1}(\gamma_i)$. These are $s-2$ distinct loops and two paths that exchange the $i$th and $i+1$th smallest root. The braid that is formed by $f^{-1}(\gamma_i)$ forms the generator $\sigma_i$ if we choose the convention that the overpassing strand corresponds to the root with the smaller imaginary part. Note that this braid is given by $(f^{-1}(\gamma_i(t)),t) = ((f - \gamma_i(t))^{-1}(0), t) \subset \mathbb{C} \times [0,2\pi]$.

This gives us therefore a parametrisation of the generator $\sigma_i$ as a loop in $\hat{X}_s$, whose critical values form the braid

$$\left(0,t\right) \cup \bigcup_{j=1}^{s-1} (v_j - \gamma_i(t), t) \subset \mathbb{C} \times [0,2\pi], \quad (9)$$

depicted in Figure 3. We can consider the path $\gamma = \prod_{j=1}^{\ell} \gamma_{ij}$, which is the concatenation of $\gamma_i$. The path $f - \gamma(t)$ is a loop in the space of polynomials $\hat{X}_s$, whose roots form the $s$-strand braid $B = \prod_{j=1}^{\ell} \sigma_{ij}$ and whose critical values form a braid that is easily seen to be conjugate to $\prod_{j=1}^{\ell} Y_{ij}$, where

$$Y_j = \begin{cases} A_{1, \frac{j+1}{2}} & \text{if } j \text{ is odd}, \\ A_{1, \frac{j+\frac{3}{2}}{2}} & \text{if } j \text{ is even.} \end{cases} \quad (10)$$

This is illustrated in an example in Figure 4.

We have shown the following.

**Lemma 3.5.** For every $s$-strand braid $B = \prod_{j=1}^{\ell} \sigma_{ij}$ there is a conjugate of $B$ that can be parametrised as a loop in $\hat{X}_s$ whose image in $V_s$ corresponds to the braid $\prod_{j=1}^{\ell} Y_{ij}$. Equiv-
Figure 2: The definition of the loop $\gamma_i$. a) A real polynomial $f$ has a critical point $c_i$ between each pair of roots and the corresponding critical values $v_i = f(c_i)$ must be real too. Points with the same shape have the same image under $f$. E.g., the circles are the roots of $f$, all squares get mapped to $v_1$ and so on. The lines in the domain are the preimage set of the real line. b) The loop $\gamma_1$ encircles $v_1$. Its preimage set under $f$ consists of two loops and two paths exchanging the first and second root.
Figure 3: a) The relation between the braid of roots and the braid of critical values. a) The preimage set of $\gamma_1(t)$ under $f$ forms the generator $\sigma_1$, while $\gamma_1(t)$ itself forms the braid $\sigma_2\sigma_1^2\sigma_2^{-1}$. (We read braid words from the bottom to the top.) b) We can interpret $\sigma_1$ as the roots of the loop of polynomials $f - \gamma_1(t)$. The braid of its critical values is isotopic to $\sigma_1^{-1}\sigma_2^2\sigma_1$.

\[
f = v_1 v_2 v_3 0 = v_1 v_3 0 v_2
\]

alently, every parametrisation of any braid that is conjugate to $\prod_{j=1}^n Y_{ij}^{e_j}$ can be seen as a loop in $V_s$, which lifts to a loop in $\hat{X}_s$, which corresponds to a conjugate of $B$.

Note that if $B$ is homogeneous, then the braid $\prod_{j=1}^n Y_{ij}^{e_j}$ is as in Lemma 3.2. Recall the rule in the construction of the loop $\gamma$ that determines if $\gamma$ avoids a critical value $v_j$ by moving into the upper half of the complex plane or the lower half. This rule is quite arbitrary. The only thing it changes is which generator $Y_j$ corresponds to which generator $\sigma_j$. With our rule we have the nice correspondence that $Y_j$ is directly related to $\sigma_j$.

**Corollary 3.6.** Every parametrisation of $\prod_{j=1}^n A_{1,ij}^{e_{ij}}$ lifts to a loop in $\hat{X}_s$ and the corresponding braid is conjugate to a homogeneous braid. Conversely, for every homogeneous braid there is such a braid of critical values $\prod_{j=1}^n A_{1,ij}^{e_{ij}}$.

The construction that establishes this relation between the braid that is formed by the roots of a loop of polynomials and the braid that is formed by their critical values takes a lot of inspiration from work by Rudolph [11].

The main theorem now follows from Lemma 3.2 and Proposition 3.1:

**Theorem 3.7.** If $B$ is a homogeneous braid, then the closure of $B^2$ is real algebraic.

It can also be shown that on 3-spheres of small radius $\arg p_\lambda$ is a fibration of the link complement of the circle, exactly as in the complex case [5].

**4. Examples**

In this section we give an example of the explicit construction of real polynomials with isolated singularities. If we wanted to strictly follow the procedure outlined in the proof of Theorem 1.5, we would first have to write down the parametrisation for the braid of critical values and then lift this loop in $V_s$ to a loop in $\hat{X}_s$. This corresponds to solving a system of polynomial equations for every $t \in [0, 2\pi]$. In practice, this will be done for a discrete set in $[0, 2\pi]$ of sufficiently many data points. The resulting interpolating function gives a braid
Figure 4: a) Concatenation of the loops $\gamma_i$ allows us to construct any braid as the roots of a loop of polynomials in $\hat{X}_s$. Here we see the braid $\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2$ as the roots of $f - \gamma(t)$. The corresponding critical values form a braid that is conjugate to $Y_1 Y_2 Y_1^{-1} Y_2$. The generator $Y_j$ describes a movement of the strand labelled by the critical value $v_j$ around the 0-strand.

b) The resulting braid $Y_1 Y_2 Y_1^{-1} Y_2$ can be parametrised such that $\frac{\partial \arg v_i}{\partial t}$ never vanishes. This is possible because $\sigma_1 \sigma_2 \sigma_1 \sigma_3^{-2} \sigma_2$ is homogeneous.
form the braid $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})$.  

We set $a = 1$ and $b = \frac{1}{2}$ and obtain

$$g_\lambda(u, t) = u^4 + \lambda^2 u^2 \left( \frac{1}{2} e^{-it} - \frac{3}{8} e^{it} \right)$$
$$+ \lambda^4 \left( -\frac{1}{256} e^{-3it} + \frac{1}{16} e^{-2it} - \frac{1}{4} e^{-it} + \frac{1}{128} - \frac{1}{16} e^{it} - \frac{1}{256} e^{3it} \right).$$  

Figure 5 shows a plot $\arg v_1(t)$, $\arg v_2(t)$ and $\arg v_3(t)$, the critical values of $g_1(u, t)$. We see immediately that there are no stationary points. Therefore $\arg g_\lambda(u, t)$ is a fibration and this means that $(u, t) \mapsto \arg g_\lambda(u, nt)$ is also fibration for all $n$. This justifies our choice of $a$ and $b$. Note that the nodal set of $(u, t) \mapsto \arg g_\lambda(u, nt)$ is $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})^n$. For even exponents we obtain the following polynomials:

$$f_\lambda(u, v) = u^4 + \lambda^2 u^2 \left( \frac{1}{2} v^{2n} - \frac{3}{8} - \frac{1}{2} v^{2n} \right)$$
$$+ \lambda^4 \left( -\frac{1}{256} v^{6n} + \frac{1}{16} v^{4n} - \frac{1}{4} v^{2n} + \frac{1}{128} - v^{2n} - \frac{1}{256} v^{6n} \right),$$

$$p_\lambda(u, v) = (v\bar{v})^{4 \times 2n} f_\lambda \left( \frac{u}{(v\bar{v})^{2n}}, \frac{v}{(v\bar{v})^{2n}} \right)$$
$$= u^4 + \lambda^2 u^2 \left( \frac{1}{2} v^{7n} - \frac{3}{8} (v\bar{v})^{8n} - \frac{1}{2} v^{9n} \right)$$
$$+ \lambda^4 \left( -\frac{1}{256} v^{5n} - v^{11n} + \frac{1}{16} v^{6n} - \frac{1}{4} v^{10n} + \frac{1}{128} (v\bar{v})^{8n} - v^{9n} - \frac{1}{256} v^{11n} \right).$$

Here we have chosen $k = 2n$, which is larger than $\deg f/s = 6n/4$. Note that all exponents of $v$ and $\bar{v}$ in $f_\lambda$ are even, so that there are no square root terms in $p_\lambda$.

Therefore the braids $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})^{2n}$ close to real algebraic links for all $n$. They belong to the family of lemniscate links which where studied in [3]. Note that the constructed functions are semiholomorphic, i.e., holomorphic in one complex variable $u$ and $\deg_u p_\lambda = s$, the number of strands.

If we want to construct real algebraic links that are not in the family of closures of squares of homogeneous braids, it is important to recall that in our construction we only considered one lift of the braid of critical values $\prod_{j=1}^{t} A_{ij}$. If any of its other lifts in $\hat{X}_s$ are also loops, then the closure of the squares of the corresponding braids are also real algebraic. Recently, this was studied in more detail [6], but since it is challenging to check if a given link is the closure of a homogeneous braid it remains unclear if the constructed families in [6] lead to new real algebraic links.
Figure 5: Plots of graphs of the arguments of the 3 critical values of \( g \) as functions of \( t \).
Because of symmetries of the braid parametrisations two of the three functions are identical.
None of the functions have any stationary points.

References