THE SPACE OF SHORT ROPES AND THE CLASSIFYING SPACE OF THE SPACE OF LONG KNOTS

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Abstract. An overview on recent progress in the study of the spaces of long embeddings is given. Main focus is put on joint work with Syunji Moriya (Osaka prefecture University); the classifying space of the topological monoid of long knots is shown to be weakly equivalent to the space of short (or reducible) ropes.

1. An overview

An embedding $f\colon \mathbb{R}^j \hookrightarrow \mathbb{R}^n$ is said to be a *long j-embedding* in $\mathbb{R}^n$ if f coincides with the standard inclusion $\mathbb{R}^j \hookrightarrow \mathbb{R}^j \times \{\mathbf{0}\} \subset \mathbb{R}^n$ outside $[0,1]^{\times j}$. Let $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ denote the space of long j-embeddings in $\mathbb{R}^n$ endowed with C^∞-topology. The main Theorem 1.5 describes the classifying space of $B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. The author would like to firstly review recent trends in the study of such embedding spaces that motivate us to consider the classifying spaces. The author also hopes that this survey might get any young researchers interested in this topic.

The set $\pi_0(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n))$ can be seen as the set of isotopy classes of embeddings. In fact the knot theory, the study of isotopy classes of embeddings $S^1 \hookrightarrow S^3$, is equivalent to the study of $\pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$:

Fact 1.1. The 1-point compactification $\mathbb{R}^n \to S^n$ gives an isomorphism

$$\pi_0(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)) \xrightarrow{\cong} \pi_0(\mathrm{Emb}(S^j, S^n))$$

of monoids, where the monoid structure is given by the *connected sum*.

The connected sum on $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ is defined as a concatenation of two embeddings in the x_1-direction. This makes $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ a topological monoid, whereas the monoid structure for $S^j \hookrightarrow S^n$ is defined only on π_0. This monoid is commutative, and in particular $\pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ is freely generated by the isotopy classes of prime (long) knots.

1.1. Cohomology of $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$.

It is straightforward to see that $H^0(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n); A)$ is the space of isotopy invariants of long j-embeddings in $\mathbb{R}^n$ (or equivalently $S^j \hookrightarrow S^n$ by Fact 1.1) with values in A. Among them are the *Vassiliev type invariants*. From the knot-theoretic viewpoint it might be interesting what the positive-degree cohomology classes of $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ mean. Computation of cohomology of such mapping spaces (or H-spaces) as $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ has also been one of the central problems in algebraic topology. We review some approaches to $H^*(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n))$, with relations to Vassiliev invariants in mind.

1.1.1. *Vassiliev's approach to* $H^*(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n))$ [35]. Regard $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ as a subspace of the contractible space $\mathrm{Map}(\mathbb{R}^1, \mathbb{R}^n)$ of "long maps". By the "infinite dimensional Alexander duality" we may alternatively compute $H_*(\Sigma_n)$, where Σ_n (called the *discriminant*) is the complement of $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ in $\mathrm{Map}(\mathbb{R}^1, \mathbb{R}^n)$. The information of multipoints and singularities of maps induces a natural filtration on $\Sigma_{n,1}$. Roughly speaking the Alexander duals of strata of maps with k transverse double points are the Vassiliev invariants of order exactly

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k (a combinatorial characterization is given in [5]), and they appear as elements of the diagonal part $E_1^{-k,k}$ of the induced spectral sequence. When $n \geq 4$, this spectral sequence in fact converges to $H^*(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n))$ and moreover collapses at E_1 over rationals (see §1.1.3 below). This also collapses at E_1 over rationals even if $n = 3$; see [23, 33].

1.1.2. *Chern-Simons perturbative theoretic approach.* One way to produce all the Vassiliev invariants of (long) knots is the integration over *configuration spaces* associated with *graphs*.

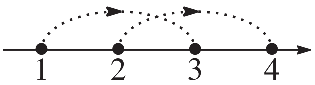
Example 1.2 (see [7, 20, 34, 37])**.** The order two invariant is essentially unique (up to constant multiplications) and is given by

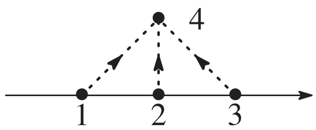
$$f \mapsto \int_{C_X(f)} \varphi_{X,f}^* \mathrm{vol}_{S^2}^{\times 2} - \int_{C_Y(f)} \varphi_{Y,f}^* \mathrm{vol}_{S^2}^{\times 3},$$

where

$$C_X(f) := \{(x_1, \ldots, x_4) \in (\mathbb{R}^1)^{\times 4} \mid x_1 < \cdots < x_4\} = \mathrm{Conf}_4(\mathbb{R}^1),$$
$$C_Y(f) := \{(x_1, x_2, x_3; y_4) \in \mathrm{Conf}_3(\mathbb{R}^1) \times \mathbb{R}^3 \mid f(x_i) \neq y_4 \text{ for } i = 1, 2, 3\} \subset \mathrm{Conf}_3(\mathbb{R}^1) \times \mathbb{R}^3$$

are configuration spaces associated with (the vertices of) the graphs

$X =$ 

$Y =$ 

and $\varphi_{X,f} \colon C_X(f) \to (S^2)^{\times 2}$ and $\varphi_{Y,f} \colon C_Y(f) \to (S^2)^{\times 3}$ are defined by

$$\varphi_{X,f}(x_1, \ldots, x_4) := \Big(\frac{f(x_3) - f(x_1)}{|f(x_3) - f(x_1)|}, \frac{f(x_4) - f(x_2)}{|f(x_4) - f(x_2)|}\Big) \in (S^2)^{\times 2},$$
$$\varphi_{Y,f}(x_1, x_2, x_3; y_4) := \Big(\frac{y_4 - f(x_i)}{|y_4 - f(x_i)|}\Big)_{i=1,2,3} \in (S^2)^{\times 3}.$$

In general, for any weight system W of order k (see [3]), a formal sum of trivalent graphs $\sum_{\Gamma;\ |V(\Gamma)|=k} W(\Gamma)\Gamma$ (added by some "correction terms") gives an order k invariant via the similar integrations over configuration spaces C_Γ as above.

Generalizing the above construction, we have a linear map $I \colon \mathcal{G}_{n,j}^* \to \Omega_{DR}^*(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n))$, where $\mathcal{G}_{n,j}^*$ is the cochain complex consisting of (not necessarily trivalent) graphs and Ω_{DR}^* is the de Rham complex functor. Moreover the map I restricted to particular graphs is a cochain map in some nice dimensions;

- $n > 3$, $j = 1$ [14],
- both n, j are odd and $n > j > 1$; $I|_{\text{1-loop graphs}}$ is a cochain map [15, 27, 30, 38],
- both n, j even and $n > j \geq 2$; $I|_{\text{tree graphs}}$ is a cochain map [27].

In other cases, the map I gives an element of $H^1(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ [28] and a nontrivial element of $H_{DR}^{2n-3j-3}(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n))$ for $2n - 3j - 3 \geq 0$ with $j > 1$, $n - j$ odd [27]. In particular if $2n - 3j - 3 = 0$ (then $n = 6k$, $n = 4k - 1$ for some $k \geq 1$), the latter coincides with the Haefliger invariant [19]. These cohomology classes generalize the Vassiliev invariants in some sense.

1.1.3. *Embedding calculus.* As we have seen in the above, configuration spaces often play important roles in the study of embedding spaces. One of the reason is that there exist evaluation maps

$$\mathrm{ev}_k \colon \mathrm{Conf}_k(\mathbb{R}^j) \times \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \mathrm{Conf}_k(\mathbb{R}^n), \quad \mathrm{ev}_k(x_1, \ldots, x_k; f) := (f(x_1), \ldots, f(x_k)).$$

These maps approximate embeddings by a finite number of points, and the adjoint map $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \mathrm{Map}(\mathrm{Conf}_k(\mathbb{R}^j), \mathrm{Conf}_k(\mathbb{R}^n))$ may be expected to become highly connected as k increases. This may be thought of as a motivative idea of *manifold calculus* [39, 18]. In fact there exists a (homotopy) commutative diagram

$$\begin{tikzcd} T_\infty\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n) \arrow[r] & \dots \arrow[r] & T_2\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n) \arrow[r] & T_1\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n) \\ & & & \mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n) \arrow[lllu] \arrow[lu] \arrow[u] \end{tikzcd}$$

where the horizontal arrows form a tower of fibrations (called the *Taylor tower* since it has some similarities to the Taylor expansion of functions), and the connectivity of the map $\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n) \to T_k\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$ is approximately $k(n-j-2)$. The space $T_k\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$ is given by the homotopy limit of the diagram consisting of the spaces of punctured embeddings with at most k holes (with some tangential data). The tower enables us to describe (a variant of) $\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$ as a derived mapping space of operads [2]. Such descriptions provide certain graph complexes that compute $H^*(\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n))$ for $n \geq 2j+2$ and that prove the collapse of the Vassiliev spectral sequence. They look very similar to that appeared in §1.1.2.

When $(n, j) = (3, 1)$, it is not known whether $T_\infty\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3)$ recovers $\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3)$ or not. But the diagram of spaces that is used to define $T_{k+1}\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$ consists of 2^{k+1} spaces, each of which is the space of embeddings with l holes, $0 \leq l \leq k+1$. This looks similar to the combinatorial characterization of Vassiliev invariants of order $\leq k$. Indeed it is known that the order two invariant factors through $T_3\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3)$ [13], and for general $k \geq 2$ the $k+1$-st stage produces order k invariants (see [12, 26, 29, 36]).

1.2. **Embedding spaces as $\mathcal{D}$-algebras.** The *little m-disks operad* is the collection $\mathcal{D}^m := \{\mathcal{D}^m_k\}_{k\geq 0}$ of spaces, where $\mathcal{D}^m_k$ is the space of configurations of k disjoint m-disks in the unit m-disk. Compositions of embeddings give maps $\mathcal{D}^m_k \times (\mathcal{D}^m_{l_1} \times \dots \times \mathcal{D}^m_{l_k}) \to \mathcal{D}^m_{l_1+\dots+l_k}$ that encode higher commutativity of m-fold loop spaces [6, 21]. The little j-disks operad acts on $\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$, namely we have maps $\mathcal{D}^j_k \times (\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n))^{\times k} \to \mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)$ that encodes all the possible ways to take connected sums of k long embeddings. Considering the space of "framed" long embeddings, this action is extended to that of $\mathcal{D}^{j+1}$ [8]. If $n-j \geq 3$, then $\pi_0(\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^j,\mathbb{R}^n))$ is a group and hence the loop space recognition theorem [21] deduces that $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^j,\mathbb{R}^n)$ is a $(j+1)$-fold loop space. Although $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^1,\mathbb{R}^3)$ is not a two-fold loop space because $\pi_0(\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^1,\mathbb{R}^3))$ is not a group, it is a "free $\mathcal{D}^2$-space" and the homotopy type of each path component of $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^1,\mathbb{R}^3)$ can be computed in principle [10, 11].

1.3. **The group completion of the long knot space; the main theorem.** As we have seen in §1.1, in the (meta)stable range of dimensions ($n-j \geq 3$, $n \geq 2j+2$, or $2n-3j-3 \geq 0$ and $j \geq 2$), $\pi_*(\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n)) \otimes \mathbb{Q}$ and $H^*(\mathrm{Emb}(\mathbb{R}^j,\mathbb{R}^n);\mathbb{Q})$ can in principle be computed by using algebraic models. On the other hand,in particular the codimension two cases, these spaces are interesting (from knot theoretic viewpoint) but we can obtain only the information of $\pi_0(\mathrm{Emb}(\mathbb{R}^{n-2},\mathbb{R}^n))$ via homotopy-theoretic methods. For example $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^{n-2},\mathbb{R}^n)$ admits an action of little 2-disks operad, but it is not homotopy equivalent to any two-fold loop space since $\pi_0(\mathrm{Emb}(\mathbb{R}^{n-2},\mathbb{R}^n))$ can never be a group [9]. The group completion $\Omega B\mathrm{Emb}(\mathbb{R}^{n-2},\mathbb{R}^n)$ would be better from the homotopy-theoretic viewpoint (in fact $\Omega B\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^{n-2},\mathbb{R}^n)$ is homotopy equivalent to a two-fold loop space).

The following result and the conjecture of Mostovoy [25] are thus quite curious;

Theorem 1.3 ([25])**.** The fundamental group of the space B_2 of "short ropes" (see Definition 1.10 below) is isomorphic to $\widehat{\pi}_0(\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3))$, the group completion of $\pi_0(\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3))$.

Conjecture 1.4 ([25])**.** B_2 would be the classifying space $B\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3)$ of $\mathrm{Emb}(\mathbb{R}^1,\mathbb{R}^3)$.

The conjecture 1.4 has been solved affirmatively in joint work [24] with Moriya. In fact it can be proved in a slightly generalized form. Outline of the proof is given in §2.

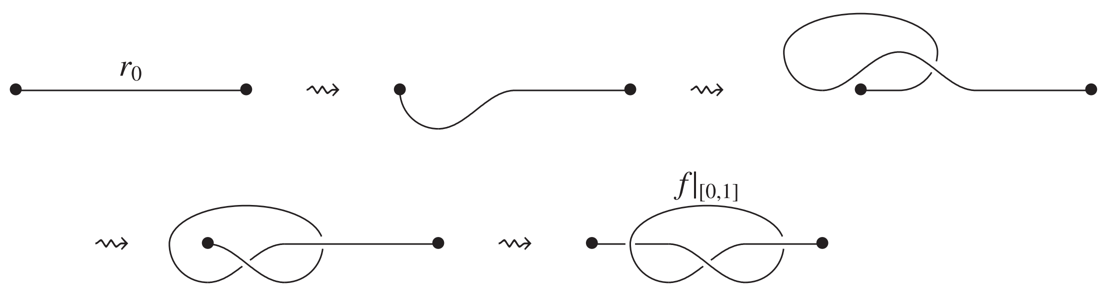


FIGURE 1.1. Tying the tight rope to get a (part of) long knot

Theorem 1.5 ([24])**.** For $n \geq 3$, the space B_2 of short ropes in $\mathbb{R}^n$ is weakly homotopy equivalent to $B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$.

1.4. **Questions.** If $n > 3$, then $\pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n))$ is the trivial group and hence B_2 is a *delooping* of $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ (see [22]):

$$\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n) \sim \Omega B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n) \sim \Omega B_2.$$

In fact P. Salvatore [31] proved that $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ $(n > 3)$ is weakly equivalent to a double loop space, and we can expect that B_2 has a further delooping.

As mentioned in §1.2, $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^1, \mathbb{R}^n)$ is acted on by the little 2-disks operad. Theorem 1.5 is valid for framed cases, and thus the space of *framed* short ropes in $\mathbb{R}^n$ $(n \geq 3)$ is expected to have a delooping.

Question 1.6. What are deloopings of the spaces of (framed) short ropes in $\mathbb{R}^n$?

For $n = 3$, Theorem 1.5 gives a partial information of $H_*(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$:

Theorem 1.7 ([4])**.** Let M be a topological monoid. If $\pi_0(M)$ is included in the center of the Pontrjagin ring $H_*(M)$, then there exists a ring isomorphism $H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega BM)$.

Question 1.8. Compute $H_*(\Omega B_2)$. Which Vassiliev invariants come from short ropes?

Question 1.9. What is $B\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ for $j \geq 2$?

1.5. **Mostovoy's map $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to \Omega B_2$.** Mostovoy's Theorem 1.3 implies that the classification of 1-parameter families of short ropes generalizes knot theory, and it would be worth describing the isomorphism $\pi_1(B_2) \to \widehat{\pi}_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ explicitly.

Definition 1.10 ([25])**.** A *rope* is an embedding $r\colon [0,1] \hookrightarrow \mathbb{R}^3$ such that $r(i) = (i, 0, 0)$ for $i = 0, 1$. A rope is said to be *short* if its arc-length is not greater than 3. The space of short ropes equipped with the C^∞-topology is denoted by B_2.

The *tight rope* is the short rope r_0 defined by $r_0(t) := (t, 0, 0)$ for $0 \leq t \leq 1$.

Clearly the length of any rope is not less than 1. The subscript "2" indicates that B_2 is the space of ropes whose lengths are $\leq 1 + 2$.

After reducing the size of $f \in \mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ enough, $f|_{[0,1]}$ gives a short rope. This rope can be joined to the tight rope in B_2 in the following two ways (see Figure 1.1):

(1) "tying rope around $(0, 0, 0)$" to get $f|_{[0,1]}$,
(2) "tying rope around $(1, 0, 0)$" to get $f|_{[0,1]}$.

Gluing the isotopy (1) and the inverse of (2), we obtain a loop in B_2 based at r_0. Therefore we get a map $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to \Omega B_2$. In other words, a natural map $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to B_2$ given by $f \mapsto f|_{[0,1]}$ is null-homotopic and there exist two null-homotopies, and hence we have a map $\widetilde{\Sigma}\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to B_2$. Mostovoy has proved that this map induces an isomorphism on π_1. Because $\pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ is a free commutative monoid, we have an injective homomorphism $\pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)) \hookrightarrow \widehat{\pi}_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)) \cong \pi_1(B_2)$ of monoids. Thus for example an invariant of the homotopy classes of loops of short ropes restricts to a knot invariant.

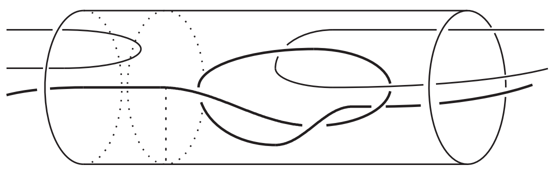


FIGURE 2.1. An element of ψ (the long component is drawn with a thick curve)

2. Outline of the proof of Theorem 1.5

Below let D^m denote the *open* unit m-disk. The proof of Theorem 1.5 goes as follows;

(1) Define a space ψ of certain 1-manifolds in $\mathbb{R}^1 \times D^2$ (Definition 2.2) and a topological category $\mathcal{K}$ of long knots (Definition 2.4), and prove that $B\mathcal{K} \sim \psi$ (Theorem 2.8).
(2) Introduce the notion of *reducible ropes* (Definition 2.9) and show that B_2 is weakly equivalent to the space $\mathcal{R}$ of reducible ropes (Theorem 2.10; the proof will be omitted).
(3) Define the *cutting-off map* $c\colon \mathcal{R} \to \psi$ and prove that this is a weak equivalence (§2.3).

Step (1) looks similar to the argument of [16]; the homotopy types of the classifying spaces of some cobordism categories have been studied in [16], and the methods in [16] work well for long knots because long knots can be considered as a kind of cobordisms. We need step (2) because reducible ropes fit better into the framework of step (1) and perhaps $\mathcal{R}$ might be easier to deal with than B_2. The map c defined in step (3) is geometric and hence might be useful in applying the "rope theory" to the knot theory.

Below we only consider the case of $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. For details see [24].

2.1. The space of reducible 1-manifolds.

For a submanifold $M \subset \mathbb{R}^1 \times D^2$ and a subset $A \subset \mathbb{R}^1$, we denote $M|_A := M \cap (A \times D^2)$. If $A = \{T\}$ then we abbreviate it as $M|_T$ and think of $M|_T \subset D^2$ in a natural way.

Definition 2.1. A 1-manifold $M \subset \mathbb{R}^1 \times D^2$ is said to be *reducible*[1] *at* T if it intersects $\{T\} \times D^2$ transversely at exactly one point. If moreover $M|_{(T-\epsilon, T+\epsilon)} = (T-\epsilon, T+\epsilon) \times \{M|_T\}$ for some $\epsilon > 0$, then we say M is *strongly reducible at* T. See Figure 2.1.

Definition 2.2. Define the set ψ as consisting of 1-manifolds $M \subset \mathbb{R}^1 \times D^2$ such that

(i) $\partial M = \emptyset$,
(ii) each path component of M is a closed, non-compact *subset* of $\mathbb{R}^3$, and
(iii) there exists T such that M is reducible at T (see Figure 2.1).

By the above conditions (i)-(iii), we see that for any $M \in \psi$ has exactly one path component M_0 satisfying $M|_T \neq \emptyset$ for any $T \in \mathbb{R}^1$. We say such a component is *long*. Other components are (if they exist) *long on exactly one side*; we say M_1 is *long in the left* (resp. *right*) if there exists $T \in \mathbb{R}^1$ such that $M_1|_s \neq \emptyset$ for any $s \leq T$ but $M_1|_{(T,\infty)} = \emptyset$ (resp. $M_1|_s \neq \emptyset$ for any $s \geq T$ but $M_1|_{(-\infty,T)} = \emptyset$).

We topologize ψ as in [16, §2.1]. Roughly speaking $M, N \in \psi$ are "close to each other if they are close in a compact set". This topology looks very similar to the "weak C^∞-topology" [1, §I-4 D].

Example 2.3. Let $\alpha\colon [0,1) \to [0,\infty)$ be a monotonically increasing function with $\alpha(t) \xrightarrow{t\to 1} \infty$. Let $M(t) \in \psi$ $(0 \leq t < 1)$ be a family of 1-manifolds satisfying $M(t)|_{[-\alpha(t),\alpha(t)]} = [-\alpha(t), \alpha(t)] \times \{0\}$. This family converges to the trivial long knot $\mathbb{R}^1 \times \{0\} \in \psi$ as $t \to 1$. See also [16, Example 2.2].

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[1]This word comes from the knot theory; if M is reducible at T, then M can be decomposed into a "connected sum" of $M|_{(-\infty,T]}$ and $M|_{[T,\infty)}$. But the author does not think it is the best terminology, because $M|_{(-\infty,T]}$ and $M|_{[T,\infty)}$ are not necessarily "non-trivial". The author would like to ask readers to suggest any better terminology.

2.2. **The category of long knots.**

Definition 2.4. Define the *topological category $\mathcal{K}$ of long knots* as follows. Define $\mathrm{Ob}(\mathcal{K}) = D^2$ equipped with the usual topology. For $p, q \in D^2$, non-identity morphisms from p to q are "long knots from p to q," namely pairs $(T, M) \in \mathbb{R}^1_{>0} \times \psi$ such that

(a) M is connected (and hence long by the conditions in Definition 2.2),
(b) there exists $\epsilon > 0$ such that $M|_{(-\infty,\epsilon)} = (-\infty, \epsilon) \times \{p\}$ and $M|_{(T-\epsilon,\infty)} = (T - \epsilon, \infty) \times \{q\}$.

We define the identity morphism $\mathrm{id}\colon p \to p$ to be $(0, \mathbb{R}^1 \times \{p\})$. We topologize the set of all morphisms $\bigcup_{p,q} \mathrm{Map}_{\mathcal{K}}(p, q)$ as a subspace of $(\{0\} \sqcup \mathbb{R}_{>0}) \times \psi$, where $\sqcup$ stands for the disjoint union. The composition $\circ\colon \mathrm{Map}_{\mathcal{K}}(q, r) \times \mathrm{Map}_{\mathcal{K}}(p, q) \to \mathrm{Map}_{\mathcal{K}}(p, r)$ is given by the connected sum

$$(T', M') \circ (T, M) := (T + T', M|_{(-\infty,T]} \cup (M'|_{[0,\infty)} + T\boldsymbol{e}_1)),$$

where $\boldsymbol{e}_1 = (1, 0, 0) \in \mathbb{R}^3$ and $+T\boldsymbol{e}_1$ is the parallel translation by T in the $\mathbb{R}^1$-direction.

The category $\mathcal{K}$ has a contractible object space and its morphism space is homotopy equivalent to $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \sqcup \{\mathrm{id}\}$. One of the reasons why we topologize the morphism space so that the identity maps are separated is that it makes a proof of "goodness" of the nerve $N_*\mathcal{K}$ of $\mathcal{K}$ (see [17, 32]) easier; see below.

The *nerve* of $\mathcal{K}$ is by definition a *simplicial space* $N_*\mathcal{K} = \{N_l\mathcal{K}\}_{l\geq 0}$ where $N_l\mathcal{K}$ is the space of composable l morphisms in $\mathcal{K}$. Thus $N_l\mathcal{K}$ is the space of long knots that are connected sums of at least l long knots;

$$N_l\mathcal{K} = \{(T_1 \leq \cdots \leq T_l; M) \in (\mathbb{R}^1_{\geq 0})^{\times l} \times \psi \mid \\ M \text{ is a long knot that is strongly reducible at each } T_i\}.$$

The *classifying space* $B\mathcal{K}$ of $\mathcal{K}$ is by definition the *geometric realization* of $N_*\mathcal{K}$. To show $B\mathcal{K} \sim \psi$, we introduce two intermediate posets $\mathcal{D}$ and $\mathcal{D}^{\perp}$ and find a sequence of (weak) homotopy equivalences $B\mathcal{K} \overset{\simeq}{\leftarrow} B\mathcal{D}^{\perp} \overset{\simeq}{\to} B\mathcal{D} \overset{\sim}{\to} \psi$.

Definition 2.5. Define spaces $\mathcal{D}^{\perp} \subset \mathcal{D} \subset \mathbb{R}^1 \times \psi$ by respectively

$$\mathcal{D}^{(\perp)} := \{(T, M) \in \mathbb{R}^1 \times \psi \mid M \text{ is (strongly) reducible at } T\}$$

and define a partial order $\leq$on $\mathcal{D}$ so that $(T, M) \leq (T', M')$ if $M = M'$ and $T \leq T'$. We regard $\mathcal{D}^{(\perp)}$ as a topological category in a natural way; $\mathrm{Ob}(\mathcal{D}^{(\perp)}) = \mathcal{D}^{(\perp)}$ and $\mathrm{Map}_{\mathcal{D}^{(\perp)}}(x, y) = \{(x, y)\}$ if $x \leq y$ and $\emptyset$ otherwise. We topologize $\bigcup_{x,y\in\mathcal{D}} \mathrm{Map}_{\mathcal{D}}(x, y)$ as a subspace of $(\Delta \sqcup (\mathbb{R} \times \mathbb{R} \setminus \Delta)) \times \psi$, where $\Delta := \{(x, x) \in \mathbb{R} \times \mathbb{R}\}$ is the diagonal set.

Remark 2.6. For $(T, M) \in \mathcal{D}$, M is not necessarily connected, but any path components of M that are "one-sided long" are separated, namely all the left-sided (resp. right-sided) long components are contained in $(-\infty, T) \times D^2$ (resp. $(T, \infty) \times D^2$).

Remark that the l-th space of the nerve of $\mathcal{D}^{(\perp)}$ is

$$N_l\mathcal{D}^{(\perp)} = \{(T_0 \leq \cdots \leq T_l; M) \mid M \in \psi \text{ is (strongly) reducible at each } T_i\}.$$

By the definition of the topologies of $\mathcal{K}$ and $\mathcal{D}^{(\perp)}$, the identity morphisms form disjoint path components, thus their nerves are *good* simplicial spaces (see [32, Appendix A]).

Proposition 2.7. There exists a sequence of simplicial maps $N_*\mathcal{K} \leftarrow N_*\mathcal{D}^{\perp} \to N_*\mathcal{D}$ each of which is a degreewise homotopy equivalence. Since they are good simplicial spaces, this induces a sequence of homotopy equivalences $B\mathcal{K} \overset{\simeq}{\leftarrow} B\mathcal{D}^{\perp} \overset{\simeq}{\to} B\mathcal{D}$ (see [32, Appendix A]).

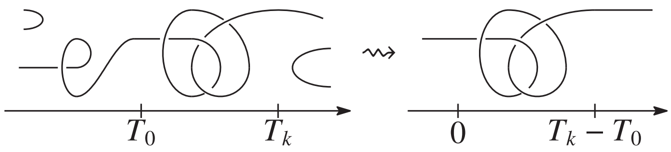

FIGURE 2.2. Cutting-off and long-extension

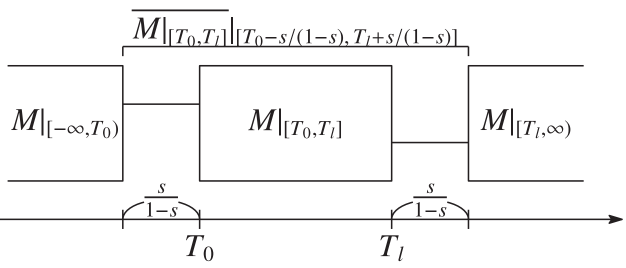

FIGURE 2.3. The homotopy from $G \circ F$ to id in the proof of Proposition 2.7

Proof. Firstly the natural inclusion induces a simplicial map $N_*\mathcal{D}^\perp \to N_*\mathcal{D}$. It is a degreewise homotopy equivalent because, for any $(T_0 \le \cdots \le T_l; M) \in N_l\mathcal{D}$, we can canonically transform M so that M becomes strongly reducible at each T_i [16, Lemma 3.4].

Define a functor $F\colon \mathcal{D}^\perp \to \mathcal{K}$ on objects by $(T, M) \mapsto M|_T$ and on morphisms by

$$F(T_0 \le \cdots \le T_l; M) := (0 \le T_1 - T_0 \le \cdots \le T_l - T_0; \overline{M|_{[T_0,T_l]}} - T_0\boldsymbol{e}_1),$$

where

$$\overline{M|_{[T_0,T_l]}} := ((-\infty, T_0] \times M|_{T_0}) \cup M|_{[T_0,T_l]} \cup ([T_l, \infty) \times M|_{T_l}) \tag{2.1}$$

(see Figure 2.2) is obtained by cutting $M|_{(-\infty,T_0]} \sqcup M|_{[T_l,\infty)}$ off and adding two half-lines. We call it the *long-extension* of $(T_0 \le \cdots \le T_l; M)$. Remark 2.6 confirms that $\overline{M|_{[T_0,T_l]}}$ is a morphism in $\mathcal{K}$. This induces a simplicial map $F\colon N_*\mathcal{D}^\perp \to N_*\mathcal{K}$.

The "inverse" $G\colon N_*\mathcal{K} \to N_*\mathcal{D}^\perp$ is defined by $G(p) := (0, \mathbb{R}^1 \times \{p\})$ on level zero and by the natural inclusion on positive levels. It is not induced by any functor, but is a degreewise homotopy inverse to F. Indeed $F \circ G$ is the identity, and

$$G \circ F(T_0 \le \cdots \le T_l; M) = (0 \le T_1 - T_0 \le \cdots \le T_l - T_0; \overline{M_0|_{[T_0,T_l]}})$$

is a result of a homotopy that "throws $M|_{(-\infty,T_0]}$ and $M|_{[T_l,\infty)}$ away to $\pm\infty$" (see Figure 2.3), and is homotopic to the identity by the definition of the topology of ψ; see Example 2.3. □

Theorem 2.8. The forgetful map $N_l\mathcal{D} \to \psi$ given by $(T_1 \le \cdots \le T_l; M) \mapsto M$ induces a weak equivalence $u\colon B\mathcal{D} \xrightarrow{\sim} \psi$.

Outline of proof. This is proved by showing the relative homotopy group $\pi_m(\psi'_s, B\mathcal{D})$ (where ψ' is the mapping cone of u) vanishes for all m. This holds essentially because the fiber of $N_l\mathcal{K} \to \psi$ is a union of l-simplices. See also [16, Theorem 3.10]. □

2.3. **The space of reducible ropes.** We first extend the meaning of ropes (compare the following Definition 2.9 with Definition 1.10).

Definition 2.9 ([25]). A *rope* is a compact connected 1-submanifold $r \subset \mathbb{R}^1 \times D^2$ with nonempty boundary $\partial r = \{r_0, r_1\}$ satisfying $r_i \in \{i\} \times D^2$. A *reducible rope* is a rope that is reducible at some $t \in (0, 1)$ in the sense of Definition 2.1. Denote by $\mathcal{R}$ the space of reducible ropes equipped with the same topology as ψ.

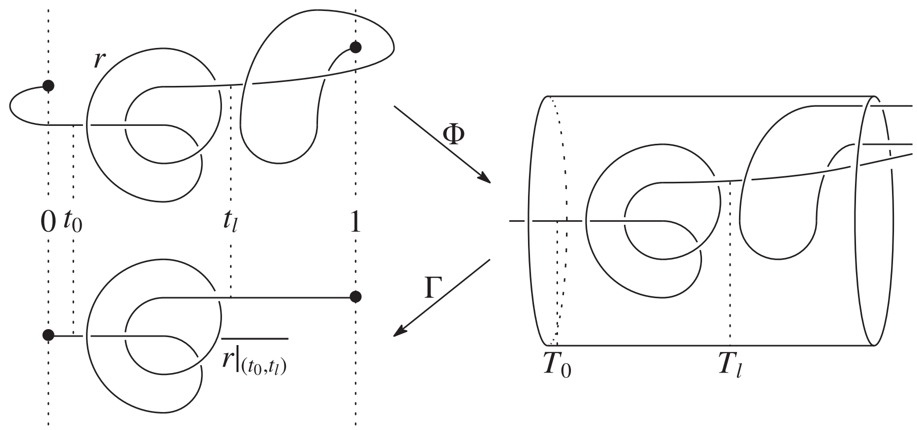

Figure 2.4. The maps Φ and Γ

By inspection we see that Mostovoy's short rope (Definition 1.10) must be reducible at some $t \in (0,1)$. Thus there exists an injective map $B_2/\mathrm{Diff}_+[0,1] \to \mathcal{R}$, where $\mathrm{Diff}_+[0,1]$ acts on B_2 as parameter changes. Composing the natural projection $B_2 \to B_2/\mathrm{Diff}_+[0,1]$ (this is a homotopy equivalence because $\mathrm{Diff}_+[0,1]$ is contractible), we have a map $B_2 \to \mathcal{R}$.

Theorem 2.10. The above map $B_2 \to \mathcal{R}$ is a weak homotopy equivalence.

The proof is not difficult but technical. See [24, §3].

Choose and fix an orientation preserving diffeomorphism $f\colon (0,1) \xrightarrow{\cong} \mathbb{R}$, and define the *cutting-off map* $c\colon \mathcal{R} \to \psi$ by

$$c(r) := (f \times \mathrm{id}_{D^2})(r|_{(0,1)}).$$

See also Figure 2.4 below. Notice the similarity between c and the "long-extension" (2.1).

The rest of this article is devoted to showing that c is a weak equivalence. As we have done in §2.2, we introduce posets $\mathcal{E}^{(\perp)}$ that intermediate $\mathcal{R}$ and ψ.

Definition 2.11. Define the spaces $\mathcal{E}$ and $\mathcal{E}^{\perp}$ by respectively

$$\mathcal{E}^{(\perp)} := \{(t,r) \in (0,1) \times \mathcal{R} \mid r \text{ is strongly reducible at } t\},$$

and define partial order $\le$ on $\mathcal{E}$ and $\mathcal{E}^{\perp}$ so that $(t,r) \le (t',r')$ if $r = r'$ and $t \le t'$. We equip $\mathcal{E}$ and $\mathcal{E}^{\perp}$ with a structure of topological categories in the same way as $\mathcal{D}$; $\bigcup_{x,y} \mathrm{Map}_{\mathcal{E}^{(\perp)}}(x,y)$ is topologized as subspaces of $(\Delta \sqcup ((0,1) \times (0,1) \setminus \Delta)) \times \mathcal{R}$.

Notice that the nerve of $\mathcal{E}^{(\perp)}$ is the space of "connected sums of ropes";

$$N_l\mathcal{E}^{(\perp)} = \{(t_0 \le \cdots \le t_l; r) \in (0,1)^{\times(l+1)} \times \mathcal{R} \mid r \text{ is (strongly) reducible at each } t_i\}.$$

The nerves $N_*\mathcal{E}^{(\perp)}$ are good simplicial spaces; the reason is the same as for $N_*\mathcal{K}$.

Proposition 2.12. There exists a sequence of simplicial maps $N_*\mathcal{E} \xleftarrow{\simeq} N_*\mathcal{E}^{\perp} \xrightarrow{\simeq} N_*\mathcal{D}^{\perp}$ each of which is a degreewise homotopy equivalence. Since they are good simplicial spaces, this induces a sequence of homotopy equivalences $B\mathcal{E} \xleftarrow{\simeq} B\mathcal{E}^{\perp} \xrightarrow{\simeq} B\mathcal{D}$ $(\simeq B\mathcal{K})$.

Proof. The inclusion functor $\mathcal{E}^{\perp} \to \mathcal{E}$ induces a levelwise homotopy equivalence $N_*\mathcal{E}^{\perp} \xrightarrow{\simeq} N_*\mathcal{E}$. The reason is the same as for $\mathcal{D}^{\perp} \hookrightarrow \mathcal{D}$ (compare this with Proposition 2.7).

Recalling $f\colon (0,1) \xrightarrow{\approx} \mathbb{R}$ and c from the above, define a functor $\Phi\colon \mathcal{E}^{\perp} \to \mathcal{D}^{\perp}$ by

$$\Phi(t;r) := (f(t); c(r)),$$

and for any $l \ge 0$ define a map $\Gamma\colon N_l\mathcal{D}^{\perp} \to N_l\mathcal{E}^{\perp}$ by

$$\Gamma(T_0 \le \cdots \le T_l; M) := (t_0 \le \cdots \le t_l; (f^{-1} \times \mathrm{id}_{D^2})(\overline{M|_{[T_0,T_l]}})),$$

(recall $\overline{M|_{[T_0,T_l]}}$ from (2.1)), where $t_i := f^{-1}(T_i) \in (0,1)$. See Figure 2.4.

Notice that, if N is "knotted" outside any compact set, then possibly $r := (f^{-1} \times \mathrm{id}_{D^2})(N)$ might not be a regular (or tame) submanifold of $\mathbb{R}^1 \times D^2$. But if $N = \overline{M|_{[T_0,T_l]}}$ then it is a union of straight lines outside $[T_0, T_l] \times D^2$ and r is indeed a regular submanifold.

The functor Φ induces a simplicial map $\Phi\colon N_*\mathcal{E}^\perp \to \mathcal{D}^\perp$. It is in fact a levelwise homotopy equivalence with homotopy inverse Γ. Indeed $\Phi \circ \Gamma$ is given by

$$\Phi \circ \Gamma(T_0 \le \cdots \le T_l; M) = (T_0 \le \cdots \le T_l; \overline{M|_{[T_0,T_l]}})$$

and it is homotopic to the identity by the similar argument to $G \circ F \simeq \mathrm{id}$ in the proof of Proposition 2.7. Next $\Gamma \circ \Phi$ is given by

$$\Gamma \circ \Phi(t_0 \le \cdots \le t_l; r) := (t_0 \le \cdots \le t_l; \overline{r|_{(t_0,t_l)}}),$$

where

$$\overline{r|_{(t_0,t_l)}} := ([0, t_0] \times r|_{t_0}) \cup r|_{(t_0,t_l)} \cup ([t_l, 1] \times r|_{t_l}) \in \mathcal{R}$$

is the "long-extension" of $r|_{(t_0,t_l)}$, namely we replace $r|_{(-\infty,t_0]} \cup r|_{[t_l,\infty)}$ with straight segments (see Figure 2.4). Thus to show $\Gamma \circ \Phi \simeq \mathrm{id}$, we have to show that $r|_{(-\infty,t_0]} \cup r|_{[t_l,\infty)}$ can be unknotted in a canonical way.

One way to do this has been given in [25, Lemma 10]. The outline is as follows. Notice that $r|_{(-\infty,t_0]}$ can be seen as a rope that is strongly reducible at the endpoint $r|_{t_0}$. Parametrize $r|_{(-\infty,t_0]}$ by some $\rho\colon [0,1] \to \mathbb{R}^3$ and consider a family of "truncated ropes" $\rho|_{[s,1]}$ $(0 \le s \le 1)$. Using some rotation-like homotopy centered at $r|_{t_0}$, the truncated rope $\rho|_{[s,1]}$ can be transformed to a rope whose endpoints are $(0,0,0)$ and $r|_{t_0}$. This homotopy gives a way to unknot $r|_{(-\infty,t_0]}$ keeping r strongly reducible at each t_i. Similarly $r|_{[t_l,\infty)}$ can be unknotted and hence $\Gamma \circ \Phi$ is homotopic to the identity. □

Theorem 2.13. The forgetful map $N_l\mathcal{E} \to \mathcal{R}$ given by $(t_0 \le \cdots \le t_l; r) \mapsto r$ induces a weak equivalence $v\colon B\mathcal{E} \xrightarrow{\sim} \mathcal{R}$.

The proof is almost the same as that of Theorem 2.8. Thus we have $\mathcal{R} \sim B\mathcal{K}$:

Corollary 2.14. There exists a homotopy commutative diagram consisting of (weak) homotopy equivalences, where u', v' are the composite of u, v with the inclusions:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "c", "\sim"'] & \psi & \\
B\mathcal{E}^\perp \arrow[u, "v'", "\sim"'] \arrow[r, "\Phi", "\sim"'] & B\mathcal{D}^\perp \arrow[u, "u'", "\sim"'] \arrow[r, "F", "\sim"'] & B\mathcal{K}
\end{tikzcd}$$

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