A CONSTRUCTION OF COUNTABLE SIMPLE ORDERABLE GROUPS

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Abstract. We provide a uncountable family of countable simple orderable groups. For this we consider a certain generalization of Thompson's group F_n. More precisely, we consider a *chain group*, which is a group generated by real line homeomorphisms whose supports form a chain of intervals. We investigate its dynamical property and abundance of the possible isomorphism types. The main reference is [4].

1. Introduction

Recall a group G is *orderable* (or, *left-orderable*) if there exists a linear order $\leqslant$ on G that is invariant under the left G–multiplication. Let us denote by $\mathrm{Homeo}^+(\mathbb{R})$ the group of the orientation preserving homeomorphisms on the real line $\mathbb{R}$. A group order can be *dynamically realized* in the following sense:

Lemma 1.1 ([3]). *For a countable group G, the following are equivalent.*

(1) G is orderable.
(2) G embeds into $\mathrm{Homeo}^+(\mathbb{R})$.

One has another criterion for orderability. If

$$1 \to A \to B \to C \to 1.$$

is a short exact sequence of groups and if A and C are orderable, then so is B. Note that B is non-simple if B is different from A or C.

Many known nontrivial constructions of orderable groups rely on this criterion and Lemma 1.1. On the other hand, finding a *simple* orderable group is usually a much trickier business. It is an outstanding question whether or not there exists a finitely generated infinite simple orderable group, for instance.

The commutator group of the Thompson's group F is one of the earliest example of countable simple orderable group; see Section 3. In this talk, we address the following question

Question 1.2. *Are there uncountably many countable simple orderable groups?*

Key words and phrases. homeomorphism; Thompson's group F; simple group; smoothing; chain group; orderable group.

2. Thompson's group F

The Thompson's group F is defined as the group of piecewise linear homeomorphisms of the unit interval $[0, 1]$ such that the breakpoints are in $\mathbb{Z}[1/2]$ and such that the slopes are in $2^{\mathbb{Z}}$. Here, we allow only finitely many breakpoints.

The group F enjoys several intriguing group theoretic properties. To list a few, we have:

- F is finitely presented, orderable and infinite with trivial center.
- F has exponential growth.
- F does not contain a rank-two free group.
- The commutator group $F' = [F, F]$ is simple. That is, every non-trivial normal subgroup of F contains F'.
- F contains the infinite direct sum of $\mathbb{Z}$.
- Every proper quotient of F is abelian.

See [1] as a standard reference in this field. Let us consider the following maps in F.

$$a(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ x + \frac{1}{4} & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ \frac{x+1}{2} & \text{otherwise.} \end{cases} \qquad b(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 2x - \frac{1}{2} & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{5}{8} \\ x + \frac{1}{8} & \text{if } \frac{5}{8} \leqslant x \leqslant \frac{3}{4} \\ \frac{x+1}{2} & \text{otherwise.} \end{cases}$$

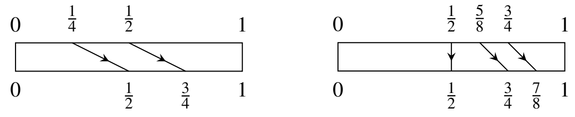
See Figure 1.

Lemma 2.1 ([1]). *We have the presentation*

$$F = \langle a, b \mid [a^k b a^{-k}, b^{-1} a] = 1 \textit{ for } k = 1, 2 \rangle.$$

Let us investigate the meaning of the the relators of this presentation. For $f \in \mathrm{Homeo}^+(\mathbb{R})$, we write

$$\operatorname{supp} f = \mathbb{R} \setminus \operatorname{Fix} f = \{x \in \mathbb{R} \colon f(x) \neq x\}.$$



(a) a (b) b

Figure 1. Elements a, b of Thompson's group F.

Let us write $u = b^{-1}a$. Then we have

$$u(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{x}{2} + \frac{3}{8} & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ x & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{supp} u = (0, 3/4).$$

Moreover,

$$bu \operatorname{supp} b = bu[1/2, 1] = b[5/8, 1] = [3/4, 1].$$

Since

$$(bu)^k \operatorname{supp} b \cap \operatorname{supp} u = \varnothing$$

for all $k \geqslant 1$, we have

$$[(bu)^k b(bu)^{-k}, u] = 1.$$

By finding a normal form in F, one can actually show that

$$F = \langle u, b \mid [(bu)^k b(bu)^{-k}, u] = 1 \text{ for } k = 1, 2\rangle.$$

See [1]. We note that $\operatorname{supp} u$ and $\operatorname{supp} b$ form a "chain of intervals", as shown in Figure 2.

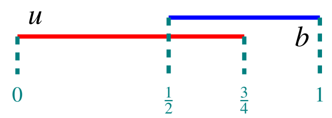


FIGURE 2. A chains of two intervals.

For an interval $J \subseteq \mathbb{R}$, let us denote the left– and the right–endpoints of J by $\partial^- J$ and $\partial^+ J$, respectively. Suppose $\mathscr{J} = \{J_1, \ldots, J_n\}$ is a collection of nonempty open subintervals of $\mathbb{R}$. We call $\mathscr{J}$ a *chain of intervals* (or an *n–chain of intervals* if the cardinality of $\mathscr{J}$ is important) if $J_i \cap J_k = \varnothing$ if $|i - k| > 1$, and if $J_i \cap J_{i+1}$ is a proper nonempty subinterval of J_i and J_{i+1} for $1 \leqslant i \leqslant n - 1$.

Setting 2.1. *We let $n \geqslant 2$ and let $\mathscr{J} = \{J_1, \ldots, J_n\}$ be a chain of intervals such that $\partial^- J_i < \partial^- J_{i+1}$ for each $i < n$. We consider a collection of homeomorphisms $\mathscr{F} = \{f_1, \ldots, f_n\}$ such that* $\operatorname{supp} f_i = J_i$ *and such that $f_i(t) \geqslant t$ for each $t \in \mathbb{R}$. We set $G_{\mathscr{F}} = \langle \mathscr{F} \rangle \leqslant \operatorname{Homeo}^+(\mathbb{R})$.*

We call the group $G = G_{\mathscr{F}}$ a *prechain group*. Note that Thompson's group $F = \langle u, b \rangle$ is a prechain group generated by u and b; see Figure 2.

Let us prove the first *stabilization* result.

Lemma 2.2. *Let $n = 2$ in Setting 2.1. Then for all sufficiently large $N > 0$ we have*

$$\langle f_1^N, f_2^N \rangle \cong F.$$

Proof. Put $U = f_1^N$ and $B = f_2^N$. From the consideration on the supports, we have

$$\left[(BU)^k \cdot B \cdot (BU)^{-k}, U\right] = 1$$

for all $k \geqslant 1$ and for all sufficiently large N. In particular, F naturally surjects onto

$$\langle f_1^N, f_2^N \rangle.$$

Since every proper quotient of F is abelian, we see this surjection is an injection as well. □

3. Main Result

Motivated by Lemma 2.2, we propose the following definition.

Definition 3.1. Let $G = G_{\mathscr{F}}$ be as in Setting 2.1. If $\langle f_i, f_{i+1} \rangle \cong F$ for each $1 \leqslant i < n$, then we say G is a *chain group*.

Lemma 2.2 implies that

$$\langle f_1^N, f_2^N, \ldots, f_n^N \rangle$$

is a chain group for all sufficiently large N.

Convention. From now on, whenever a chain group G is given, we assume that generators of G are as in Setting 2.1 and that

$$\operatorname{supp} G = \bigcup_{g \in G} \operatorname{supp} g = \mathbb{R}.$$

The main result of this talk is the following.

Theorem 3.2 (Main Theorem)**.** *There exists a uncountable collection $\mathscr{H}$ of pairwise non-isomorphic countable simple orderable groups. Moreover, each group H in $\mathscr{H}$ can be written as $H = G'$ for some 3–chain group G.*

We will establish Theorem 3.2 through the following results. These results are of its own interest regarding fundamental properties of chain groups.

A topological group action is called *minimal* if every orbit is dense. An *open Cantor set* is the Cantor set minus the two endpoints of the unit interval.

Theorem 3.3 (Dynamical Dichotomy)**.** *For a chain group G, exactly one of the following holds:*

(i) G is minimal; in this case, G' is simple.

(ii) G leaves invariant an open Cantor set; in this case, G surjects onto a minimal chain group.

Theorem 3.4 (Diversity)**.** *If $n \geqslant 3$, then there exists a uncountable family*

$$\{G_i \mid i \in I\}$$

of minimal n–chain groups such that

$$G_i' \not\cong G_j'$$

for all $i \neq j$ in the index set I.

Theorems 3.3 and 3.4 immediately imply the Main Theorem. We also note a certain uniformity in the isomorphism types of chain groups:

Theorem 3.5 (Stability)**.** *For all sufficiently large N, the chain group*

$$\langle f_1^N, \ldots, f_n^N \rangle$$

is isomorphic to the n–adic Thompson's group F_n.

Here F_n is the group of piecewise linear homeomorphisms on the unit interval such that each breakpoint is in $\mathbb{Z}[1/n]$ and such that each slope is $n^{\mathbb{Z}}$. The proof of Theorem 3.5 for $n = 2$ is given in Lemma 2.2. The general case is very similar, using the fact that every proper quotient of F_n is abelian. We omit the details.

Remark 3.6. We can easily choose each prechain generator f_i to be C^∞. This choice gives a smooth realization of F_n.

4. Dynamics

If a finitely generated group G acts on $\mathbb{R}$, then there exists a minimal closed nonempty G–invariant set C. Moreover, we have

(i) $C' = \varnothing$, i.e. C is discrete, or
(ii) $\partial C = \varnothing$, i.e. G acts minimally, or
(iii) $C' = \partial C = C$, i.e. C is an open Cantor set.

Since every orbit is accumulated at $\partial^+ \operatorname{supp} f_1$, we see that the case (i) does not occur. Moreover, if (iii) occurs, then one can contract the complement of C by the Cantor function

$$h \colon \mathbb{R} \to \mathbb{R},$$

which is obtained after replacing the closure of each open interval in $\mathbb{R} \backslash C$ by a single point. More precisely, we have the following lemma.

Lemma 4.1. *Let $G \leqslant \mathrm{Homeo}^+(\mathbb{R})$ be a group such that $\operatorname{supp} G = \mathbb{R}$ and G leaves invariant an open Cantor set. Then there exists a monotone continuous surjective map $h \colon \mathbb{R} \to \mathbb{R}$ and a (possibly non-faithful) representation*

$\Phi\colon G \to \mathrm{Homeo}^+(\mathbb{R})$ *such that* $\Phi(G)$ *acts minimally on* $\mathbb{R}$ *and such that* $hg = \Phi(g)h$ *for each* $g \in G$.

The lemma asserts that for each $g \in G$, the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[d, "h"] & \mathbb{R} \arrow[d, "h"] \\ \mathbb{R} \arrow[r, "\Phi(g)"] & \mathbb{R} \end{tikzcd}$$

The part (2) of Dynamical Dichotomy (Theorem 3.3) now follows. For the part (1), we proceed as below.

Definition 4.2. An action $H \to \mathrm{Homeo}(X)$ is *CO–transitive* (or, *compact-open–transitive*) if for every compact $K \subseteq X$ and a nonempty open $U \subseteq X$, there exists $h \in H$ such that $hK \subseteq U$.

Lemma 4.3 (Higman's Theorem)**.** *Let* H *be a group acting faithfully on a set* X*, and suppose that for all triples* $r, s, t \in H\backslash\{1\}$ *there is an element* $u \in H$ *such that*

$$(\operatorname{supp} r \cup \operatorname{supp} s) \cap (t^u(\operatorname{supp} r \cup \operatorname{supp} s)) = \varnothing.$$

Then the commutator subgroup $H' = [H, H]$ *is simple.*

We will use the following variation of Higman's Theorem. We include a complete proof, without resorting to Higman's Theorem.

Lemma 4.4 (Higman's Theorem, variation)**.** *If* $H \leqslant \mathrm{Homeo}_c(X)$ *for some noncompact Hausdorff space* X *and if* H *is CO-transitive, then* H' *is simple.*

Proof. For a group G and a set $S \subseteq G$, we denote by $\langle\!\langle S \rangle\!\rangle_G$ the normal closure of S in G. We use the notation

$$t^{-u} = (t^{-1})^u = u^{-1}t^{-1}u.$$

Claim 1. *For all* $r, s \in H$ *and* $t \in H\backslash\{1\}$*, there exists* $w \in H'$ *such that*

$$[t^{-w}rt^w, s] = 1.$$

Since X is Hausdorff, we can choose a nonempty open set $B \subseteq X$ such that $B \cap tB = \varnothing$. Choose a compact set $A \subseteq X$ containing $\operatorname{supp} r \cup \operatorname{supp} s$. By the hypothesis, there is $u \in H$ such that $uA \subseteq B$. So we have

$$\operatorname{supp} r \cap t^u \operatorname{supp} s = \varnothing.$$

So we have $[t^{-u}rt^u, s] = 1$. Note that we allow $s = 1$. Let us apply the same argument to (t, u, t) instead of (r, s, t). Then we can find $v \in H$ such that

$$[t^{-v} \cdot t \cdot t^v, u] = 1.$$

We put $w = [t^v, u^{-1}]$ so that $[t, wu^{-1}] = 1$ and $t^w = t^u$. We have

$$[t^{-w}rt^w, s] = [t^{-u}rt^u, s] = 1,$$

and the claim is proved.

Claim 2. *No proper nontrivial subgroup of H' is normal in H.*

Suppose $1 \neq N \leqslant H'$ satisfies $N \trianglelefteq H$. Pick $r, s \in H\backslash 1$ and $t \in N\backslash 1$. By the first claim, we can find $w \in H'$ such that

$$[t^{-w}rt^w, s] = 1.$$

It follows that

$$[[r, t^{-w}], s] = [r \cdot t^{-w}r^{-1}t^w, s] = [r, s].$$

On the other hand, we have

$$[[r, t^{-w}], s] \in \langle\langle t\rangle\rangle_H \leqslant \langle\langle N\rangle\rangle_H = N.$$

Since this is true for all $r, s \in H\backslash 1$, we see $H' \leqslant N$.

Claim 3. $H'' = H'$.

This follows from the second claim and from that $H'' \trianglelefteq H$.

Claim 4. *H' is simple.*

For this claim, suppose we have $1 \neq N \trianglelefteq H'$. Pick $1 \neq t \in N$ and $r, s \in H'\backslash\{1\}$. We have $w \in H'$ such that

$$[t^{-w}rt^w, s] = 1.$$

by the first claim. As in the Claim 2, we see

$$[[r, t^{-w}], s] = [r \cdot t^{-w}r^{-1}t^w, s] = [r, s] \in H''.$$

Since $r, s, w \in H'$, we have

$$[[r, t^{-w}], s] \in \langle\langle t\rangle\rangle_{H'} \leqslant N_{H'} = N.$$

It follows $H'' \leqslant N$. Claim 3 implies that $N = H'$. □

Lemma 4.5. *If G is a minimal chain group, then $G'' = G'$ is compactly supported and CO–transitive.*

Proof. Recall $F'' = F'$ [1]. Since $\langle f_i, f_{i+1}\rangle \cong F$ for each i, we see

$$\begin{aligned} &[f_i, f_{i+1}] \in \langle f_i, f_{i+1}\rangle' = \langle f_i, f_{i+1}\rangle'' \leqslant G'' \trianglelefteq G, \\ &G' = \langle\langle\{[f_i, f_{i+1}] \colon 1 \leqslant i < n\}\rangle\rangle_G \leqslant G''. \end{aligned}$$

This proves $G'' = G'$.

Note our convention that $\operatorname{supp} G = \mathbb{R}$. The germs of G at the infinities form infinite cyclic groups generated by germs of f_1 and f_n, respectively. So the germ of $g \in G'$ is trivial at the infinities. In particular, we have

$$G'' = G' \leqslant \mathrm{Homeo}_c(\mathbb{R}).$$

A typical trick shows that G' acts minimally. The fact that every orbit is accumulated at $\partial^+ I_1$ implies that G' is actually CO-transitive. □

Proof of Theorem 3.3. By Lemma 4.5, we can apply Higman's Theorem to G'. It follows that $G'' = G'$ is simple. This completes the proof of Theorem 3.3. □

5. Diversity

Let us begin with a construction of uncountably many isomorphism types of finitely generated groups due to de la Harpe.

Lemma 5.1 (cf. de la Harpe [2] for (1) and (2))**.** *There exists an orderable group $\Gamma = \langle s, t\rangle$ and a collection of subgroups $\{N_i\}_{i\in I}$ of Γ with the following properties:*

(1) The collection $\{N_i\}_{i\in I}$ is uncountable;
(2) For each i, the group $N_i < \Gamma$ is central;
(3) For each $i \in I$, the quotient $\Gamma_i = \Gamma/N_i$ embeds into $\mathrm{Homeo}^+(\mathbb{R})$ *such that the image of t has no fixed point.*

The construction of Γ and N_i are as follows. Let $S = \{s_i\}_{i\in\mathbb{Z}}$, and let

$$R = \{[[s_i, s_j], s_k] = 1\}_{i,j,k\in\mathbb{Z}} \cup \{[s_i, s_j] = [s_{i+k}, s_{j+k}]\}_{i,j,k\in\mathbb{Z}}.$$

Define $\Gamma_0 = \langle S \mid R\rangle$ and let $\Gamma = \Gamma_0 \rtimes \mathbb{Z}$, where the conjugation action of $\mathbb{Z} = \langle t\rangle$ is given by $t^{-1} s_i t = s_{i+1}$. For each i, we set $u_i = [s_0, s_i]$. For each subset $X \subset \mathbb{Z}\setminus\{0\}$, we can consider the group $N_X = \langle u_i \mid i \in X\rangle$.

We deduce the following, the proof of which is omitted.

Lemma 5.2. *There exists a uncountable collection $\{H_i\}$ of pairwise–non-isomorphic two-generator groups*

$$H_i = \langle x_i, y_i\rangle \leqslant \mathrm{Homeo}^+(\mathbb{R})$$

such that $H_i/H_i' \cong \mathbb{Z}^2$ and such that $\operatorname{Fix} y_i = \varnothing$.

These groups $\{H_i\}$ are images of $\Gamma = \langle s, t\rangle$ in Γ/N_i. Then the image of t on H_i acts freely.

So, what does this construction have anything to do with simple groups? We show that the groups of the form in Lemma 5.2 embed into the (simple) commutator subgroups of 3–chain groups.

Lemma 5.3. *Suppose we have a group*

$$H = \langle x, y \rangle \leqslant \mathrm{Homeo}^+(\mathbb{R})$$

such that $H/H' \cong \mathbb{Z}^2$ *and such that* $\mathrm{Fix}\, y = \varnothing$. *Then there exists a minimal* 3*–chain group* G *and an embedding*

$$H \hookrightarrow G'.$$

Let us postpone the proof of Lemma 5.3, and prove Theorem 3.4 first.

Proof of Theorem 3.4. Let $\mathscr{C}$ be the class of isomorphism types of G' for all minimal 3–chain groups G. Then each group in $\mathscr{C}$ is countable, simple and orderable.

If $\mathscr{C}$ were countable, then there would be only countably many isomorphism types of two generator groups H satisfying the hypothesis of Lemma 5.3. This contradicts Lemma 5.2. □

For the rest of this section, we sketch the idea of the Lemma 5.3. For illustration purpose, we will only show a weaker version:

Lemma 5.4. *Every two generator orderable subgroup embeds into a minimal* 4*–chain group.*

Here is the construction. Recall Thurston's realization of F as piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ actions with rational breakpoints such that each consecutive breakpoints are of the form

$$\frac{p}{q}, \quad \frac{r}{s}$$

for some

$$p, q, r, s \in \mathbb{Z} \cup \{-\infty, \infty\}, \quad qr - ps = 1.$$

Then $a, b \in F$ are realized as the following piecewise-$\mathrm{PSL}_2(\mathbb{Z})$ maps on $\mathbb{R}$:

$$a(x) = x + 1, \qquad b(x) = \begin{cases} x & \text{if } x \leqslant 0 \\ \frac{x}{1-x} & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 3 - \frac{1}{x} & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \\ x + 1 & \text{if } 1 \leqslant x \end{cases}$$

Now suppose we have an arbitrary two generator orderable group

$$H = \langle s, t \rangle \leqslant \mathrm{Homeo}[1/4, 1/2]$$

possibly after a conjugation. Put

$$y = bs, \quad z = bt.$$

Then surprisingly, y and z behave very similarly to b! That is,

$$\mathrm{supp}\, y = \mathrm{supp}\, z = [0, 1]$$

and $x < y(x), z(x) < x + 1$ for each $x \in (0, 1)$. We define

$$\begin{aligned} f_0 &= b^{-1}a \\ f_1 &= ay^{-1}a^{-1} \cdot b \\ f_2 &= a^2z^{-1}a^{-2} \cdot aya^{-1} \\ f_3 &= a^2za^{-2} \end{aligned}$$

It is routine to check:

$$\begin{aligned} \operatorname{supp} f_0 &= (-\infty, 1), \\ \operatorname{supp} f_1 &= (0, 2), \\ \operatorname{supp} f_2 &= (1, 3), \\ \operatorname{supp} f_3 &= (2, \infty). \end{aligned}$$

and $G = \langle f_0, f_1, f_2, f_3 \rangle = \langle a, b, x, y \rangle$ is a chain group containing H. Since $F \cong \langle a, b \rangle$ acts on $\mathbb{R}$ minimally, so does G. This completes the proof of Lemma 5.4.

We also note a corollary:

Corollary 5.5. *(1) Every n-generator subgroup H of $\mathrm{Diff}^r_+(\mathbb{R})$ embeds into a minimal $(n + 2)$-chain group G in $\mathrm{Diff}^r_+(\mathbb{R})$.*
(2) If $H/H' \cong \mathbb{Z}^n$, then this embedding maps H into G'.

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