Symplectic topology and b-symplectic structures

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1. Introduction and statement of main results

A symplectic structure on a manifold M is given by ω a closed 2-form maximally non-degenerate. The first fundamental result for symplectic structures is the existence around any point of Darboux coordinates $x_1, y_1, \ldots, x_n, y_n$ for ω:

$$\omega = \sum_{i_1}^{n} dx_i \wedge dy_i$$

Therefore symplectic structures on a fixed (even) dimension have no local invariants; this is also reflected in having an infinite dimensional group of symmetries: infinitesimal symmetries are vector fields $X \in \mathfrak{X}(M)$ satisfying: $\mathcal{L}_X \omega = 0$. In particular, any function $f \in C^\infty(M)$ produces any such infinitesimal symmetry via its Hamiltonian vector field X_f characterized by: $df = i_{X_f}\omega$.

Not unexpectedly, topology plays a major role in the study of symplectic structures, as illustrated by the following fundamental results:

- Surgery is central to the construction of closed symplectic manifolds, the key fact being the existence of normal forms in plenty of situations, a consequence of the so-called Moser's method [11, 7].
- Closed symplectic manifolds have symplectic submanifolds, which are very carefully chosen Poincaré duals of multiples of the (rational) class of the symplectic form [3].
- The existence of symplectic structures on open manifolds boils down to a homotopical obstruction [8].

Symplectic structures are a particular instance of **Poisson structures**: these are given by a bracket operation on smooth functions such that $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra over $\mathbb{R}$, and the Lie bracket is linked to the geometry of M by requiring $\{f, \cdot\}$ to be a derivation, the so-called Hamiltonian vector field of f.

Poisson structures abound: any Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ has an associated Poisson structure on its dual $\mathfrak{g}^*$; any symplectic manifold (M, ω) is a Poisson manifold with bracket $\{f, g\} := \omega(X_f, X_g)$; more generally, any foliated manifold with a leafwise symplectic form $(M, \mathcal{F}, \omega_{\mathcal{F}})$ is a Poisson manifold. In fact, a Poisson structure formalizes the notion of a possibly singular foliation by symplectic leaves; this foliation is the one integrating the distribution spanned by the Hamiltonian vector fields. For example, for the dual of a Lie algebra the symplectic foliation has as leaves the coadjoint orbits. Symplectic manifolds are exactly those Poisson manifolds whose (symplectic) foliation has just one leaf (the whole manifold).

The speaker would like to acknowledge the finantial support from Chuo University and the organizers of the Topology Symposium.
2010 Mathematics Subject Classification: 53D15.
Keywords: Symplectic, b-symplectic, cobordism, submanifold, h-principle.
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It is natural to investigate which techniques from symplectic geometry go through to Poisson geometry. Unfortunately, the answer is almost none. The reason is that Poisson structures are far too general (for example, having singular foliation forces the appearance of local invariants). Still, one would like to describe families of Poisson structures which behave as much as possible as symplectic structures. The purpose of this presentation is to discuss joint work with P. Frejlich and E. Miranda [6] in which we describe one such family.

Another way to recast the definition of a Poisson structure on M is a as a bivector $\pi \in \mathfrak{X}^2(M)$ which obeys the P.D.E. $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields. If ω is a symplectic form, its associated Poisson structure is the one with bundle map $\pi^\sharp := {\omega^\sharp}^{-1}$ (the Poisson structure in the 'inverse' of the symplectic form). Conversely, Poisson structure π on M^{2n} is symplectic exactly when the section $\wedge^n \pi$ of the line bundle $\bigwedge^{2n} TM$ does not meet the zero section. Hence, it is natural to relax the symplectic condition as follows:

Definition 1 *[10] A Poisson manifold (M^{2n}, π) is of b-symplectic type if $\bigwedge^n \pi$ is transverse to the zero section $M \subset \bigwedge^{2n} TM$.*

Such structures were first defined, in the case of dimension two, by Radko [18], who called them *topologically stable* Poisson structures. Poisson structures of b-symplectic type have also appeared under the names *log symplectic* [9], [2], [13].

Poisson structures of b-symplectic type –also referred to as **b-symplectic structures**– do not stay too far from being symplectic. One can think of them as symplectic structures which blow up to infinity along hypersurfaces in a controlled way. Indeed, The transversality condition $\bigwedge^n \pi \pitchfork M$ ensures that the **singular locus** $Z = Z(\pi) = \bigwedge^n \pi^{-1} 0$ is a codimension-one submanifold of M, which by the Poisson condition is itself foliated in codimension one by symplectic leaves of π.

1.1. Statement of the main results

We shall start by describing a link between b-symplectic manifolds and cobordisms in the symplectic category with appropriate boundary behavior.

Definition 2 *A* **cosymplectic structure** *on a manifold Z^{2n-1} consists of a pair of closed forms $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$, such that $\theta \wedge \eta^{n-1}$ is a volume form.*

The prototype of a cosymplectic structure is a **symplectic mapping torus**, i.e. the suspension of a symplectomorphism of a symplectic manifold. It turns out that the singular locus a b-symplectic manifold is not a just Poisson submanifold. An additional choice of data makes a cosymplectic structure appear [10]. Also, cosymplectic structures appear naturally on boundaries (or hypersurfaces) of symplectic manifolds endowed with a symplectic vector field transverse to the boundary (what one may call a 'flat end' in symplectic geometry).

Definition 3 *A* **cosymplectic cobordism** *(M, ω, θ) is a compact symplectic manifold (M, ω) together with $\theta \in \Omega^1(\partial M)$ making $(\partial M, \theta, \omega_{|\partial M})$ a cosymplectic manifold.*

Our first result makes a direct link between b-symplectic manifolds and cosymplectic cobordisms:

Proposition 1 *[6] A b-symplectic manifold (M, π) can be canonically factored into a composition of (connected) cosymplectic cobordisms. The cobordisms are obtained as the result of cutting M open along its singular locus.*

A main concern for us is the construction of 'enough interesting examples' of closed b-symplectic manifolds. More precisely, it is natural to ask if a cosymplectic structure (Z, θ, η) may appear of singular locus of a closed b-symplectic manifold. Proposition 1 implies that this problem is equivalent to finding a cosymplectic cobordism from (Z, θ, η) to the empty set, i.e, to finding a **symplectic filling** for it:

Proposition 2 *[6] A compact cosymplectic manifold (Z, η, θ) is the singular locus of a compact, orientable b-symplectic manifold (M, π) without boundary, if and only if (Z, η, θ) is symplectically fillable.*

Symplectic fillings of contact manifolds –and more generally symplectic cobordisms with concave/convex boundaries– are central to Symplectic Topology, whereas the case of cosymplectic (or flat) boundaries has received comparatively little attention. In this respect Eliashberg has shown that when Z is a 3-dimensional symplectic mapping torus then it is symplectically fillable [4].

Our second result follows from observing that symplectic fillability of all cosymplectic 3-manifolds would be a consequence of symplectic fillability of all symplectic mapping tori, hence solving the cosymplectic existence problem in dimension 3:

Theorem 1 *[6] Any compact cosymplectic manifold of dimension 3 is the singular locus of a compact b-symplectic 4-manifold without boundary.*

Our third result describes a class of symplectomorphisms φ which yield symplectically fillable symplectic mapping tori in arbitrary dimensions; namely, Dehn twists τ_l around parametrized Lagrangian spheres $l \subset (F, \sigma)$ and their inverses τ_l^{-1}:

Theorem 2 *[6] Let Z be a compact symplectic mapping torus. Assume that φ is Hamiltonian isotopic to*

$$\tau_{l_1} \cdots \tau_{l_m} \tau_{l_{m+1}}^{-1} \cdots \tau_{l_{m'}}^{-1},$$

where $l_i : S^{n-1} \hookrightarrow (F, \sigma)$, $i = 1, \ldots, m'$ are parametrized Lagrangian spheres.

Then there exists a compact b-symplectic manifold without boundary, whose cosymplectic singular locus is Z.

Another important question we address in the construction b-symplectic submanifolds. These are, roughly speaking, submanifolds of M transverse to the singular locus $Z(\pi)$ and such that π induces on them a b-symplectic structure. Any such submanifold, upon factoring (M, π) into cosymplectic cobordisms, would give rise to a symplectic submanifold on each connected cobordism with appropriate boundary behavior. Hence, it is natural to try to construct b-symplectic submanifolds by reversing the previous procedure. Our fourth result shows that this is possible under a mild cohomological assumption.

Theorem 3 *[6] Every (M, π) rational compact b-symplectic manifold without boundary has closed b-symplectic submanifolds $W \hookrightarrow (M, \pi)$ of any dimension intersecting every connected component of $Z(\pi)$. If M has dimension four, then the rationality assumption can be dropped.*

The final issue we address is the existence of symplectic structures on open manifolds:

Theorem 4 *[6] Let M be an orientable, open manifold. Then M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

In fact, the story here is completely analogous to the symplectic case: supporting a b-symplectic structure imposes restrictions on the de Rham cohomology of a compact manifold without boundary [2, 12], but these do not apply to open manifolds. There, the existence of b-symplectic structures becomes a purely homotopical question, and we show that they abide by a version of the h-principle of Gromov [8].

2. The b-tangent bundle and Moser's method

In this section we briefly describe how the reformulation of the b-symplectic condition as a closed non-degenerate section of a suitable bundle, allows for an analog of Moser's method for b-symplectic manifolds. In particular, it produces (semilocal) normal forms around the singular locus.

The category of **b-manifolds** has as objects pairs (M, Z), where $Z \subset M$ is a closed submanifold of codimension one with empty boundary, and as morphisms $f : (M, Z) \to (M', Z')$ those maps $f : M \to M'$ transverse to Z', and pulling back Z' to Z.

The Lie subalgebra $\mathfrak{X}(M, Z) \subset \mathfrak{X}(M)$ consisting of those vector fields v which are tangent to Z can be identified with the space of smooth sections of the **b-tangent bundle** $T(M, Z)^b \to M$. There is a bundle map $T(M, Z)^b \to TM$ which is the identity outside Z. Its restriction to Z defines an epimorphism $T(M, Z)^b|_Z \to TZ$, whose kernel $N(M, Z)^b$ has a canonical trivialization ν: if one expresses Z locally as $x_1 = 0$ in a coordinate chart $(x_1, ..., x_n)$, then $x_1 \frac{\partial}{\partial x_1}$ is independent of choices along Z.

One defines b-forms as sections of $\bigwedge^p \left(T^*(M, Z)^b\right)$, and denotes them by $\Omega^*(M, Z)^b$. They form a complex with a differential d^b given by a Koszul type formula, and which matches the de Rham differential outside Z. In fact, there is a short exact sequence of chain complexes:

$$0 \to (\Omega^\bullet(M), d) \to (\Omega^\bullet(M, Z)^b, d^b) \xrightarrow{\flat} (\Omega^{\bullet-1}(Z), d) \to 0, \tag{1}$$

where $\flat$ maps a b-form ω to its contraction with the canonical ν.

A b-map $f : (M, Z) \to (M', Z')$ induces a maps of b-complexes by pulling back sections in the usual fashion.

A b-form $\omega \in \Omega^2(M, Z)^b$ will be called non-degenerate if ω^n is nowhere vanishing, and **symplectic** if it is non-degenerate and closed, $d^b\omega = 0$.

Example 1 *Let $x_1, y_1, \ldots, x_n, y_n$ be coordinates in $\mathbb{R}^{2n}$, and consider the b-manifold $(\mathbb{R}^{2n}, x_1 = 0)$. Then*

$$\omega = \frac{dx_1}{x_1} \wedge y_1 + \sum_{j=2}^{n} dx_j \wedge dy_j$$

is a b-symplectic form on $(\mathbb{R}^{2n}, x_1 = 0)$

Example 2 *In the unit sphere $S^2 \subset \mathbb{R}^3$, consider h the height function and θ the polar coordinates associated to rotations around the z-axis. Then*

$$\omega = \frac{dh}{h} \wedge d\theta$$

is a b-symplectic form on $(S^2, h = 0)$.

This b-symplectic form is invariant under the antipodal map, hence descending to a b-symplectic form on $(\mathbb{RP}^2, \mathbb{RP}^1)$.

The first important result is:

Proposition 3 *[10] There is a bijective correspondence between symplectic forms on (M, Z), and Poisson structures of b-symplectic type with singular locus Z.*

The Poisson structures of b-symplectic type corresponding to examples 1 and 2 are:

$$x_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{x_1} + \sum_{j=2}^{n} \frac{\partial}{\partial y_j} \wedge \frac{\partial}{x_j}, \quad h \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial h}$$

It is not surprising then that Moser's method carries through to b-symplectic manifolds. At the local level, there is a Darboux theorem [17] that says that around a point in the singular locus Z there exists coordinates so that a b-symplectic form ω writes as in example 1. At a global level, on a compact b-manifold a path of deformations of b-symplectic forms which are cohomologous (for the cohomology of the b-complex) corresponds to an isotopy [13].

Let us discuss the seminormal form around Z and the appearance of a cosymplectic structure on Z. Firstly, the contraction with the canonical section $\theta := i_\nu \omega_{|Z} = \flat\omega_{|Z}$ is a closed 1-form, and it is nowhere vanishing since both ω and ν are nowhere vanishing. Of course, the foliation defined by θ is the symplectic foliation induced on Z by the Poisson structure that corresponds to ω. Assuming for simplicity that M is orientable, we can always find $t \in C^\infty(M)$ a function vanishing linearly exactly at $Z(\omega)$. Then its Hamiltonian vector field X_t is tangent to the singular locus, so it is a section of $T(M, Z)^b$ which again does not vanish near Z. Mimicking the construction in the symplectic category, is not difficult to see that w becomes independent of the local coordinate t. In fact, the difference

$$\eta := \omega - \frac{dt}{t} \wedge \theta$$

becomes an honest 2-form independent of the local 'time' coordinate $t \in [-\epsilon, \epsilon]$, so one can write around Z [10]:

$$\omega = \frac{dt}{t} \wedge \theta + \eta, \quad \eta \in \Omega^2_{\mathrm{cl}}(Z)$$

The closed 2-form η is symplectic on the codimension 1-leaves of the foliation defined for θ, so (Z, θ, η) becomes a cosymplectic manifold.

3. Cobordisms and b-symplectic manifolds

Let (M, ω) be an oriented closed b-symplectic manifold. Upon the choice of a local time coordinate t around the singular locus Z, we have the normal form in $Z \times [-\epsilon, \epsilon]$:

$$\omega = \frac{dt}{t} \wedge \theta + \eta, \quad \eta \in \Omega^2_{\mathrm{cl}}(Z)$$

If we denote $M^c = M \backslash [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, then we obtain:

- An oriented compact symplectic manifold with boundary.
- A cosymplectic structure in the boundary $(\partial M^c, \theta, \eta)$.
- A cosymplectic involution $\iota \colon (\partial M^c, \theta, \eta) \to (\partial M^c, \theta, \eta)$ exchanging boundary components.

(M, ω) can be recovered from $(M^c, \theta, \eta, \iota)$ by gluing the product neighborhood of Z using ι. Conversely, if (X, θ, η, ι) is an oriented compact cosymplectic cobordism with an involution in the boundary exchanging components, one may ask whether there is a structure naturally obtained in M, the result of gluing ∂X using ι. Each boundary component of ∂X can be labeled as an inward or an outward one: The component is outward if the orientation in the component inherited by the orientation of M^{2n} (outward normal first) is the one corresponding to the volume form $\theta \wedge \omega^{n-1}_{|\partial X}$; otherwise it is inward. It follows from standard results in symplectic geometry, that if ι identifies and inward and an outward boundary component, then the symplectic form is compatible with the gluing. If both components $\partial X_j, \iota\partial X_j$, are inward or outward, then the symplectic form will never extend to the gluing, but rather, we can insert a tube:

$$(\partial X_j \times [-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}], \pm \tfrac{dt}{t} \wedge \theta_j + \omega_{|\partial X_j}),$$

the sign corresponding to the inward or outward case. The previous discussion summarizes as follows:

Proposition 4 *[6]*

1. *Any closed oriented b-symplectic manifold (M, ω) can be factored as a composition of b-symplectic cobordisms.*

2. *If (X, θ, η, ι) is an oriented compact cosymplectic cobordism with an involution in the boundary exchanging components, then the result of gluing the boundaries using ι is a closed manifold with a canonical b-symplectic structure; the singular locus corresponds to pairs of identified boundary components which are both inward or outward.*

The previous proposition offers an interesting justification for b-symplectic structures: they appear naturally when composing cosymplectic cobordisms.

Corollary 1 *Let (X, ω, θ) be a cosymplectic cobordism. Then its double has a canonical b-symplectic structure.*

Example 2 is the double of a disk with any symplectic form.

4. Realizing cosymplectic structures

It is natural to ask which cosymplectic structures appear as the singular locus of a closed b-symplectic manifold. The relation with cobordisms readily implies that (Z, θ, η) can be realized as one such symplectic locus if it is cosymplectically cobordant to the empty set (by taking its double!), i.e., if it is what we call symplectically fillable (the flat boundary of a closed symplectic manifold).

Among cosymplectic structures the simplest ones are those for which θ has rational periods, for these are exactly the symplectic mapping tori. It is not difficult to see that using cosymplectic cobordisms which are topologically trivial (products) one can:

1. Reduce the symplectic fillability question for general cosymplectic structures to symplectic mapping tori.

2. Check that the symplectic fillability question for symplectic mapping tori only depends on the Hamiltonian isotopy class of the monodromy.

For the first point simply observe that for (Z, θ, η) a cosymplectic structure on Z closed,

$$(Z \times [0,1], dt \wedge \theta + \eta + dt \wedge \theta')$$

is a cosymplectic cobordism from (Z, θ, η) (inward side) to $(Z, \theta + \theta', \eta)$ (outward side), as long as θ' has small enough C^0-norm. But we can always use such deformations to ensure that $\theta + \theta'$ has rational periods.

In dimension three one can use cosymplectic cobordisms which are topologically non-trivial to modify the monodromy of any surface mapping torus. This is based on the following well-known facts:

1. The mapping class group is generated by (positive) Dehn twists around (oriented) curves.

2. Attaching a 2-handle to one such curve with framing -1 produces a cobordism whose new end has the monodromy of the latter surface bundle composed with the corresponding positive Dehn twist.

3. The construction is symplectic, in the sense that the elementary cobordism admit a symplectic form and a cosymplectic structure in the boundary corresponding to the prescribed surface bundles.

This reduces the filling question to filling surface bundles with monodromy isotopic to the identity. Here arises a subtle point. Dehn twists are defined up to symplectic isotopy and not to Hamiltonian isotopy. So our problem is not quite reduced to the trivial one of filling the cosymplectic manifold $(\Sigma \times S^1, \theta, \eta)$ (filled by $\Sigma \times D^2$ with the obvious product symplectic structure), but filling the 'twisted' $(\Sigma \times S^1, \theta, \eta + \alpha \wedge \theta)$, $\alpha \in \Omega^1_{\mathrm{cl}}(\Sigma)$. Fortunately, there is a rather non-trivial result by Eliashberg [4] that grants the existence of such a symplectic filling, this completing the proof of Theorem 1

In higher dimensions, the problem of filling arbitrary symplectic tori is much harder, because little is known of the structure of the group of Hamiltonian isotopy classes of symplectomorphisms beyond the case of surfaces.

Still, one has:

1. A notion of generalized Dehn twist around a Lagrangian sphere (defined up to Hamiltonian isotopy).

2. A result saying that attaching a middle handle to such a sphere with an appropriate framing produces a cobordism whose new end has the monodromy of the latter surface bundle composed with the corresponding generalized Dehn twist [15], and so that the construction is symplectic, in the sense that the elementary cobordism admits a symplectic form and a cosymplectic structure in the boundary with the corresponding to the prescribed symplectic mapping tori.

This allows to fill symplectic mapping tori whose monodromy are certain words in such generalized Dehn twists, as asserted in Theorem 2.

5. b-symplectic submanifolds

A fundamental result in symplectic topology says that any compact symplectic manifold has symplectic submanifolds [3]. It is natural to ask the same question in the b-symplectic setting. For a b-symplectic (M,ω), a b-symplectic submanifold W is a submanifold intersecting the singular locus Z transversely, and so that ω pull backs to a b-symplectic form on $(W, Z\cap W)$.

If W is such a submanifold, upon factoring M we will see W^c as a submanifold of $(M^c,\omega,\theta,\iota)$ with the following properties:

- W^c is a symplectic submanifold.
- W^c intersects ∂M^c transversely and θ pullbacks to a no-where vanishing 1-form in the boundary of W^c.
- ∂W^c is stable under the involution ι.

Donaldson's results has been refined for cosymplectic cobordisms with involution [14]: if (X,ω,θ,ι) is one such cobordism so that $[\omega]$ is a rational class, then there exists $Y\subset X$ submanifolds with the aforementioned properties.

To try to construct a b-symplectic submanifold, one starts with Y a submanifold of $(M^c,\omega,\theta,\iota)$ as above, and the difficulty is that upon gluing back $(M^c,\omega,\theta,\iota)$ into (M,ω), we will generically get a non-smooth submanifold. This is because if we start with $W\subset(M,\omega)$ a b-symplectic submanifold, additionally, we may choose the local transverse coordinate t so that W also inherits a product structure (i.e. the Hamiltonian vector field X_t is tangent to W).

So one either needs to build submanifolds Y of $(M^c,\omega,\theta,\iota)$ compatible with a given product structure in the boundary, or to analyze when for such a given Y one can find a product structure near the boundary compatible with Y. If there is such a structure, then one easily checks that at points $x\in\partial Y$, the symplectic orthogonal T_xY^ω must be tangent to ∂X at x. In fact, it is not difficult to prove that that infinitesimal tangency condition suffices to construct a product structure compatible with Y.

In any case, one can analyze if for a given Y as above, one can isotope it near the boundary (and fixing the boundary) through symplectic submanifolds so that the tangency condition is achieved. Note that this is by no means straightforward, because this is a 'large deformation', and one cannot use the openness of the symplectic condition.

At the linear level –and in the lowest possible dimension $2n=4$– we have two points in $\mathrm{SympGr}^+(2,4)$ the Grassmannian of oriented two planes in $\mathbb{R}^4$, corresponding to the tangent plane T_xY and the tangency we have to achieve. Any non-linear solution should be based upon choosing 'geodesics' in this linear setting. Fortunately, the submanifolds Y coming from Donaldson theory have the additional property of being 'almost' J-complex w.r.t. any fixed compatible almost complex structure. At the linear level, this fixes a core of the Grassmannian associated to the fixed Cartan decomposition of $\mathrm{Sp}(4)=SU(2)P$, and T_xY belongs to this core. The Cartan decomposition itself provides a 'geodesic' joining our two points.

Based on the above ideas, it is possible to perturb Y into Y^t a symplectic submanifold with the appropriate tangency condition.

To summarize the proof of Theorem 3, we start with (M,ω) a closed rational b-symplectic manifold. The rationality assumption means that $(M^c,\omega,\theta,\iota)$ is a rational symplectic manifold. Then we construct Y the fist symplectic submanifold, and finally we perturb it into Y^t another symplectic submanifold which upon gluing $(M^c, \omega, \theta, \iota)$ into (M, ω) becomes a (smooth) b-symplectic submanifold.

The rationality assumption does not appear in the symplectic setting. This is due to the fact that any symplectic class can be approximated by rational ones, and the symplectic submanifolds constructed by Donaldson's methods are well-behaved w.r.t. this approximation. Unfortunately, our deformation/smoothing process is not well-behaved in this respect. The exception is dimension 4, the reason being that symplectic classes can be approximated by rational ones so that no perturbation occurs near the boundary (this because degree 2 cohomology in dimension 4 is isomorphic to compactly supported degree 2 cohomology).

6. An h-principle for b-symplectic structures

A necessary condition for a manifold M^{2n} to be symplectic is that it carry a non-degenerate two-form, or, equivalently, an almost-complex structure. If M is compact, we have a further necessary condition, namely, that there be a degree-two cohomology class $\tau \in H^2(M)$ with $\tau^n \neq 0$.

For *open* manifolds M – that is, those manifolds, none of whose connected components is compact without boundary – a classical theorem of Gromov [8] states that the sole obstruction to the existence of a symplectic structure is that M be almost-complex. More precisely, given any non-degenerate two-form $\omega_0 \in \Omega^2(M)$ and any degree-two cohomology class $\tau \in H^2(M)$, there is a path $\omega : [0, 1] \to \Omega^2(M)$ of non-degenerate two-forms connecting ω_0 to ω_1, $d\omega_1 = 0$, $[\omega_1] = \tau$.

We consider now the case of b-symplectic structures. Recall that b-symplectic manifolds need *not* be oriented as usual manifolds, so in particular they may fail to be almost-complex. However:

Lemma 1 *If an orientable M admits a b-symplectic structure ω, then $M \times \mathbb{C}$ is almost-complex.*

The proof follows an essentially an argument in linear algebra that can be traced back to [1]: the difficulty is around the singular locus. Using the Darboux normal form

$$\omega = \frac{dx_1}{x_1} \wedge y_1 + \sum_{j=2}^{n} dx_j \wedge dy_j,$$

there is an obvious choice of almost complex structure for $t \neq 0$, the difference being that for $t < 0$ it is i in the x_1, y_1 plane, and for $t > 0$ it is $-i$ on that plane. By adding an extra complex dimension, it is easy to write down a path of almost complex structures J_s in $\mathbb{R}^4$ such that $\mathbb{R}^2 \times \{0\}$ is complex for J_0, J_1, the first one being i and the second one $-i$.

In order to prove that if M open is orientable and $M \times \mathbb{C}$ is almost complex, there exists a b-symplectic structure, we need to introduce the analogs of non-degenerate two-forms:

Definition 4 *A bivector $\pi \in \mathfrak{X}^2(M^{2n})$ is almost b-symplectic if its top exterior power $\bigwedge^n \pi$ is transverse to the zero section, and along the zero locus $Z = Z(\pi)$ we have $\pi^\sharp(T^*M_{|Z}) \subset TZ$*

Theorem 5 *On an* open *manifold M, an almost b-symplectic bivector π_0 is homotopic through almost b-symplectic bivectors to a Poisson bivector π_1. Moreover, one can arrange that $Z(\pi_1)$ be non-empty if $Z(\pi_0)$ is non-empty.*

This statement is a result of checking that 1-jets of Poisson bivectors of b-symplectic type forms a microflexible differential relation, invariant under the pseudogroup of local diffeomorphisms of M, cf. [8]. Alternatively, one may follow the more visual scheme of proof of [5].

Let π_0 be an almost b-symplectic bivector, so it can be interpreted as a non-degenerate b-form ω_0 in $(M, Z(\pi_0))$.

- The b-differential $d^b : \Omega^p(M, Z(\pi_0))^b \to \Omega^{p+1}(M, Z(\pi_0))^b$ can be factored as a composition $\tilde{d}^b \circ j_1$, where j_1 denotes the 1-jet map

$$j_1 : \Gamma(M, \bigwedge^p T^*(M, Z(\pi_0))^b) \to \Gamma(M, J_1 \bigwedge^p T^*(M, Z(\pi_0)^b)$$

and

$$\tilde{d}^b : \Gamma(M, J_1 \bigwedge^p T^*(M, Z(\pi_0)^b) \to \Gamma(M, \bigwedge^{p+1} T^*(M, Z(\pi_0)^b)$$

is induced by a bundle map

$$\check{d}^b : J_1 \bigwedge^p T^*(M, Z(\pi_0))^b \to \bigwedge^{p+1} T^*(M, Z(\pi_0))^b$$

- One checks that $\check{d}^b$ is an epimorphism with contractible fibres; in particular, we can lift ω_0 to $\widetilde{\omega}_0 \in \Gamma(M, J_1 \bigwedge^p T^*(M, Z(\pi_0))^b)$.

- Since M is an open manifold, there exists a a subcomplex K of a smooth triangulation of M, of positive codimension, with the property that, for an arbitrarily small open $U \subset M$ around K, there exists an isotopy of open embeddings $g_t : M \hookrightarrow M$, $h_0 = \mathrm{id}_M$, with $g_1(M) \subset U$ and $g_t|_K = \mathrm{id}_K$. We will refer to K as a core of M, and say that g_t compresses M into U. Note in passing that one can always find a core K of M meeting $Z(\pi_0)$.

- Fix then a core K of M, and a compression of M into an open U around K. The Holonomic Approximation theorem of [5] then says that we can find

  – an isotopy h_t of M mapping K into U;
  – an open $V \subset U$ around $h_1(K)$;
  – a section $\alpha \in \Gamma(V, T^*(M, Z(\pi_0))^b)$

  such that $j_1\alpha$ is so C^0-close to $\widetilde{\omega}_0$ that we can find a homotopy

$$\widetilde{\omega}(t) \in \Gamma(V, J_1 T^*(M, Z(\pi_0))^b),$$

  connecting $\widetilde{\omega}_0|_V$ to $j_1\alpha$, and with $\tilde{d}^b\widetilde{\omega}_t$ non-degenerate b-forms on V.

- Now regard the compression g_t as a smooth family of b-maps

$$g_t : (M, Z_t)) \to (M, Z(\pi_0)), \quad Z_t := g_t^{-1} Z(\pi_0),$$

  and set $\omega_1 := d^b(g_1^*\alpha) \in \Omega^2(M, Z_1)^b$.

- Observe that $\widehat{\omega}_t^1 := g_t^*\widetilde{\omega}_0$ connects $\widetilde{\omega}_0$ to $g_1^*(\widetilde{\omega}_0|_V)$, and $\widehat{\omega}_t^2 := g_1^*\widetilde{\omega}(t)$ connects $g_1^*(\widetilde{\omega}_0|_V)$ to a lift of ω_1. Let $\widehat{\omega}_t$ denote the concatenation of $\widehat{\omega}_t^1$ and $\widehat{\omega}_t^2$:

$$\widehat{\omega}_t := \begin{cases} \widehat{\omega}_{2t}^1 & 0 \leqslant t \leqslant 1/2, \\ \widehat{\omega}_{2t-1}^2 & 1/2 \leqslant t \leqslant 1. \end{cases}$$

  Then $t \mapsto \pi_t := \widehat{\omega}_t^{-1} \in \mathfrak{X}^2(M, Z_t)^b$ defines a homotopy of almost b-symplectic bivectors between π_0 and a Poisson π_1.

The proof of theorem 4 results from observing the the presence of an almost complex structure on $M \times \mathbb{C}$, grants the existence of almost b-symplectic bivectors on M orientable (constructed out of the almost symplectic structure on $M \times \mathbb{C}$).

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