BERKOVICH GEOMETRY AND METRIC SYZ CONJECTURE

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Abstract. This paper is written for the proceedings of the 72nd Geometry Symposium. In recent years, Berkovich geometry has gained recognition through its various applications. One notable example is its contribution to mirror symmetry in the sense of Strominger–Yau–Zaslow. In this paper, we discuss the so-called Metric SYZ Conjecture, a weaker variant of their mirror symmetry proposal, and explain how Berkovich geometry offers an effective perspective on it.

1. Introduction

Let (X, L) be a projective Calabi–Yau manifold X of $\dim_{\mathbb{C}} X = n$ polarized by an ample line bundle L on X. Here, our Calabi–Yau condition merely requires the existence of a nowhere vanishing holomorphic volume form $\Omega \in H^0(X, K_X)$ on X, which yields a measure μ on X induced by $\Omega \wedge \overline{\Omega}$. By Yau's celebrated theorem [Yau78], one has a unique metric $\omega \in c_1(L)$, which is called the *Calabi–Yau metric*, satisfying the *Monge–Ampère equation*

$$\omega^n = (L^n)\frac{\mu}{\mu(X)}. \tag{1.0.1}$$

For these Calabi–Yau manifolds, one can consider the following fibration:

Definition 1.1 (SYZ fibration)**.** Keep the same notation as above. Let U be an open subset of X. A map $f : U \to B$, where B is a manifold of $\dim_{\mathbb{R}} B = n$, is called an *SYZ fibration* on U for L if f is a smooth torus fibration such that $\mathrm{Im}\Omega|_{f^{-1}(b)} = 0$ and $\omega|_{f^{-1}(b)} = 0$ hold for any $b \in B$.

SYZ fibrations play essential roles in mirror symmetry in the sense of Strominger–Yau–Zaslow [SYZ96]. Motivated by their program, we focus on the following problem: Let

$$(\mathcal{X}, \mathcal{L}) \to \mathbb{D}^* := \{t \in \mathbb{C} \mid 0 < |t| < 1\} \tag{1.0.2}$$

be a holomorphic family of polarized Calabi–Yau manifolds $(\mathcal{X}_t, \mathcal{L}_t)$, where $\mathcal{X}$ is defined over the field $\mathbb{C}((t))^{\mathrm{mero}}$ of convergent Laurent series, and $\mathcal{L}$ is a relatively ample line bundle on $\mathcal{X}$.

Conjecture 1.2 (Metric SYZ conjecture; see [Li23, Conjecture 1.1])**.** Assume that the above $\mathcal{X}$ is maximally degenerate (for instance, type III degenerations for K3 surfaces), which is defined in § 2.3. Then, for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t \in \mathbb{D}^*$ with $|t| < \delta$, one has an SYZ fibration $f_t : U_t \to B_t$ from some open subset $U_t \subset \mathcal{X}_t$ with $\int_{U_t} \mu_t \geq 1 - \varepsilon$ to some B_t.

Since this concerns one-parameter families, it is helpful to have a geometry over $\mathbb{C}((t))$ that resembles the geometry over $\mathbb{C}$. Berkovich geometry provides such a natural framework of analytic spaces over a valued field such as $(\mathbb{C}, |\cdot|)$ and $K := \left(\mathbb{C}((t)), e^{-\mathrm{ord}_t(\cdot)}\right)$. Furthermore, we may consider analytic spaces over a Banach ring. Here, we recall the *hybrid norm* $|\cdot|_{\mathrm{hyb}}$ on $\mathbb{C}$. This is a map $|\cdot|_{\mathrm{hyb}} : \mathbb{C} \to \mathbb{R}_{\geq 0}$ defined by $|z|_{\mathrm{hyb}} := \max\{|z|, 1\}$ for $z \neq 0$, and $|0|_{\mathrm{hyb}} := 0$. As a Banach ring over $(\mathbb{C}, |\cdot|_{\mathrm{hyb}})$, we can consider the *hybrid ring* $\mathscr{A}$ defined as follows:

$$(\mathscr{A}, ||\cdot||_{\mathrm{hyb}}) := \left\{ f = \sum_{\alpha \in \mathbb{Z}} c_\alpha t^\alpha \in \mathbb{C}((t)) \;\middle|\; c_\alpha \in \mathbb{C}, ||f||_{\mathrm{hyb}} := \sum \left(|c_\alpha|_{\mathrm{hyb}} \cdot e^{-\alpha}\right) < \infty \right\}. \tag{1.0.3}$$

We may assume the above $\mathcal{X}$ is also defined over $\mathscr{A}$. Since any algebraic scheme X over a Banach ring $\mathscr{B}$ admits an associated analytic space X^{an} over $\mathscr{B}$, we obtain three kinds of analytic spaces associated with $\mathcal{X}$: the one-parameter family $\mathcal{X}$, the $\mathscr{A}$-analytic space $\mathcal{X}^{\mathrm{hyb}} := \mathcal{X}^{\mathrm{an}}_{\mathscr{A}}$, and the K-analytic space $\mathcal{X}^{\mathrm{an}}_K$. In particular, there exists a fibration

$$\pi^{\mathrm{hyb}} : \mathcal{X}^{\mathrm{hyb}} \to \Delta := \{t \in \mathbb{C} \mid |t| \leq e^{-1}\} \tag{1.0.4}$$

such that $\mathcal{X}^{\mathrm{hyb}}|_{\Delta\setminus\{0\}} \simeq \mathcal{X}|_{\Delta\setminus\{0\}}$ and $\mathcal{X}^{\mathrm{hyb}}|_{t=0} \simeq \mathcal{X}^{\mathrm{an}}_K$. This structure suggests a deep relationship between complex analytic spaces and K-analytic spaces. In fact, by making use of such analytic spaces, one can derive the following result:

Theorem 1.3 ([GO24, Theorem 1.3])**.** *Set $n \in \mathbb{Z}_{>0}$. If $(\mathcal{X}, \mathcal{L})$ is a maximally degenerating family of n-dimensional polarized abelian varieties, then Conjecture 1.2 holds. Furthermore, there exists a continuous fibration $f : \mathcal{X}^{\mathrm{hyb}} \to B$ over Δ such that $f_t : \mathcal{X}_t \to B_t$ is an SYZ fibration for $\mathcal{L}_t$ over $t \neq 0$, and $f_0 : \mathcal{X}^{\mathrm{an}}_K \to B_0$ is a* NASYZ fibration *for $\mathcal{L}^{\mathrm{an}}_K$ over $t = 0$, which is a non-Archimedean analog of the SYZ fibration defined in Definition 2.3.*

Besides, let $(\mathcal{X}, \mathcal{L})$ be a maximally degenerating family of n-dimensional *Calabi–Yau complete intersections* $\mathcal{X}_t$ (abbreviated as *CYCI*) in a toric Fano variety Y given by the Batyrev–Borisov construction (see [Gro05, §3]). In particular, one has a closed immersion $\mathcal{X} \hookrightarrow Y \times \mathbb{D}^*$ over $\mathbb{D}^*$. Denote the character group of the maximal torus of Y by M, and its cocharacter group by N. For a pair $(X, L) := (\mathcal{X}_K, \mathcal{L}_K)$, the Batyrev–Borisov construction gives two polytopal complexes: the *essential skeleton* $\mathrm{Sk}(X) \subset N_{\mathbb{R}}(:= N \otimes_{\mathbb{Z}} \mathbb{R})$ of X and the *tropical sphere* $A_L \subset M_{\mathbb{R}}(:= M \otimes_{\mathbb{Z}} \mathbb{R})$ associated to L, endowed with natural Lebesgue measures μ and ν. Furthermore, this construction also gives the support function $\phi_L : N_{\mathbb{R}} \to \mathbb{R}$ of the Newton polytope of L (as an ample line bundle on $Y_K := Y \times_{\mathbb{C}} \operatorname{Spec} K$). After taking a suitable finite base change, we can obtain the following sufficient condition of Conjecture 1.2 for the family $(\mathcal{X}, \mathcal{L})$ of CYCIs:

Theorem 1.4 ([GY24, Theorem 1.2])**.** *Keep the same notation as above. If there is a convex function $\phi : N_{\mathbb{R}} \to \mathbb{R}$ with $\sup_{x \in N_{\mathbb{R}}} |\phi(x) - \phi_L(x)| < \infty$ such that the* real Monge–Ampère equation

$$\mathrm{MA}_{\mathbb{R}}(\phi|_{\sigma}) = \nu(A_L) \cdot \frac{\mu|_{\sigma}}{\mu(\mathrm{Sk}(X))} \tag{1.0.5}$$

holds for any open face $\sigma \subset \mathrm{Sk}(X)$, where $\mathrm{MA}_{\mathbb{R}}(\phi|_{\sigma})$ is the real Monge–Ampère operator for the convex function $\phi|_{\sigma}$ defined over σ as an open set of the affine hull $\mathrm{aff}(\sigma) \simeq \mathbb{R}^n$ of σ (see, e.g., [Vil21, Definition 4.1]*), then Conjecture 1.2 holds for the family $(\mathcal{X}, \mathcal{L})$ of CYCIs.*

2. Berkovich Geometry

2.1. **How to extend the framework of complex analytic spaces.** Consider an n-dimensional complex affine space $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$, and its coordinate ring $A_n := \mathbb{C}[z_1, \ldots, z_n]$. Any point $x \in \mathbb{C}^n$ yields the evaluation map $|\cdot(x)| : A_n \to \mathbb{R}_{\geq 0}$ defined by $|f(x)|$ for any $f \in A_n$. Obviously, this map satisfies the following: for any $f, g \in A_n$,

(i) $|fg(x)| = |f(x)| \cdot |g(x)|$ (*multiplicativity*),
(ii) $|(f + g)(x)| \leq |f(x)| + |g(x)|$ (*triangle inequality*),
(iii) $|0(x)| = 0$ and $|1(x)| = 1$ (which guarantees *the emptiness for the zero ring*).

If a map $\chi : A_n \to \mathbb{R}_{\geq 0}$ satisfies the above three conditions, χ is called a *semi-valuation* on A_n. Further, the map $|\cdot(x)|$ also satisfies the following *bounded condition*: for any $a \in \mathbb{C} \subset A_n$,

(iv) $|a(x)| \leq |a|$, that is, the restriction of $|\cdot(x)|$ to $\mathbb{C}$ is bounded by the norm $|\cdot|$ on $\mathbb{C}$.

Denote by $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ *the set of all semi-valuations χ on A_n that satisfies the above bounded condition* (iv). By Ostrowski's theorem [Ost16], any $\chi \in \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ is given by the evaluation map $|\cdot(x)|$ associated to some unique $x \in \mathbb{C}^n$. Namely, there exists a natural bijection $\mathbb{C}^n \simeq \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$. Moreover, this can be seen as a homeomorphism by equipping $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ with *the weakest topology such that, for any $f \in A_n$, the associated map $|f(\cdot)| : \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}} \to \mathbb{R}$, which is defined by $|f(\chi)| := \chi(f)$, is continuous.* Under this identification, one can also rephrase analytic functions on $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ based on semi-valuations, by considering the *uniform seminorm* $\|\cdot\|_V := \sup_{x\in V} |\cdot(x)|$ on A_n for each compact subset $V \Subset \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$. More precisely, first denote by $\mathscr{B}(V)$ *the completion of the ring of rational functions without poles on a compact subset* $V \Subset \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$*, with respect to the induced norm by* $\|\cdot\|_V$. Then, by the Oka–Weil approximation theorem (see [Oka36] and [Wei35]), for any open set $U \subset \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ and $x \in U$, there exists a compact neighborhood $V \Subset U$ of x such that, for any $f \in \mathcal{O}^{\mathrm{an}}(U)$, one has $f|_V \in \mathscr{B}(V)$. In particular, these $\mathscr{B}(V)$ in turn recover $\mathcal{O}^{\mathrm{an}}(U)$ by gluing.

The above rephrased construction of $(\mathbb{C}^n, \mathcal{O}^{\mathrm{an}}_{\mathbb{C}^n})$ remains valid even when $(\mathbb{C}, |\cdot|)$ is replaced by a unital commutative Banach ring $(\mathscr{B}, \|\cdot\|)$. This is how we obtain an *n-dimensional Berkovich affine space* $\left(\mathbb{A}^{n,\mathrm{an}}_{\mathscr{B}}, \mathcal{O}^{\mathrm{an}}_{\mathbb{A}^n_{\mathscr{B}}}\right)$ as a locally ringed space. In particular, we may consider the case when $n = 0$ by setting $A_0 := \mathscr{B}$. Then $\mathbb{A}^{0,\mathrm{an}}_{\mathscr{B}}$ is often written as $\mathscr{M}(\mathscr{B})$, which is called the *Berkovich spectrum of* $\mathscr{B}$. It is a notable fact that $\mathscr{M}(k)$ is a singleton for any complete valuation field k. In the same way as complex analytic spaces, general $\mathscr{B}$-analytic spaces are given by gluing locally closed subspaces of $A^{n,\mathrm{an}}_{\mathscr{B}}$ together. For instance, any scheme X locally of finite type over $\mathscr{B}$ induces its associated $\mathscr{B}$-analytic space X^{an}. Furthermore, each morphism $f : X \to Y$ between such schemes induces its associated morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. In particular, the structure morphism $\pi : X \to \mathscr{B}$ induces $\pi^{\mathrm{an}} : X^{\mathrm{an}} \to \mathscr{M}(\mathscr{B})$. Note that, for the hybrid ring $\mathscr{A}$ defined by (1.0.3), one has $\mathscr{M}(\mathscr{A}) \simeq \Delta$ (see [Poi10, Proposition 2.1.1]), and the fibration $\pi^{\mathrm{hyb}} : \mathcal{X}^{\mathrm{hyb}} \to \Delta$ in (1.0.4) is nothing but the associated morphism to the structure morphism $\pi : \mathcal{X} \to \mathscr{A}$. For further details; see [LP24].

In what follows, we consider k-analytic spaces, where k is a complete valuation field. For simplicity, let X be a projective variety over k, and L an ample line bundle on X. If $k = (\mathbb{C}, |\cdot|)$, since $X^{\mathrm{an}} = X(\mathbb{C})$ is Kähler, the dd^c-lemma implies that any closed $(1,1)$-form on $X(\mathbb{C})$ is locally given as a potential. This observation leads us to the pluripotential theory over an arbitrary k-analytic space.

Definition 2.1. A *singular metric* (resp. *continuous metric*) ϕ on L^{an} is a family of upper semi-continuous functions (resp. continuous functions) $\phi_s : U^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ (resp. $\mathbb{R}$) given for any open subscheme $U \subset X$ and any nowhere-vanishing section $s \in H^0(U, L)$, such that the family ϕ is compatible with the restriction of sections, and $\phi_{fs} = -\log|f(\cdot)| + \phi_s$ holds for any $f \in H^0(U, \mathcal{O}_U^\times)$.

Furthermore, one can define the class $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ of *semi-positive metrics* on L^{an}, which are possibly singular metrics on L^{an}; see [BE21, § 7.1] for details. In particular, if $k = (\mathbb{C}, |\cdot|)$, then the above $\mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ coincides with the space of semi-positive metrics on $L(\mathbb{C})$ in the usual sense (see [PS23, Theorem 2.24]). As in usual pluripotential theory, each $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ yields a positive Radon measure $\mathrm{MA}(\phi)$ on X^{an} of total mass (L^n), where $n := \dim X$. This $\mathrm{MA}(\cdot)$ is called the *Monge–Ampère operator*, and it serves as an analogue of taking the measure associated to the volume form ω^n for any closed $(1,1)$-form ω on a projective complex manifold. More precisely, we obtain the following map:

$$\mathrm{MA}(\cdot) : \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})/\mathbb{R} \to \{\text{positive Radon measures on } X^{\mathrm{an}} \text{ of total mass } (L^n)\}\,. \tag{2.1.1}$$

See [CLD12, § 3.7] for the non-Archimedean case, and [BE21, § 8] for the general case and details.

2.2. **Non-Archimedean analytic spaces.** In this section, we focus on Berkovich analytic spaces over $K := \left(\mathbb{C}((t)), e^{-\mathrm{ord}_t(\cdot)}\right)$. Let X be a smooth projective K-variety. An *snc model* $\mathscr{X}$ of X (over $R := \mathbb{C}[[t]]$) means a projective flat regular R-scheme with $X \simeq \mathscr{X}_K := \mathscr{X} \times_R \operatorname{Spec} K$ such that the special fiber $\mathscr{X}_0 := \mathscr{X} \times_R \operatorname{Spec}(R/(t))$ has a simple normal crossing support, where we do not need its reduced-ness. The stratification of $\mathscr{X}_0$ gives rise to the *dual intersection complex* $\mathrm{Sk}(\mathscr{X})$, which is a simplicial complex and can be embedded into X^{an} respecting each multiplicity of the irreducible component of $\mathscr{X}_0$. Furthermore, $\mathscr{X}$ also yields a strong deformation retract $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$, which is called a *Berkovich retraction* associated to $\mathscr{X}$. These retractions are compatible with projective birational morphisms between snc models. This actually yields a homeomorphism

$$X^{\mathrm{an}} \simeq \varprojlim_{\mathscr{X}\,:\,\mathrm{snc}} \mathrm{Sk}(\mathscr{X}). \tag{2.2.1}$$

In addition, we have the following analog of the result in [Yau78], which was proved under a mild assumption in [BFJ15, Theorem A], and later in greater generality in [BGGJ+20, Theorem D]:

Theorem 2.2. *Let L be an ample line bundle on X, and $n := \dim X$. For any positive Radon measure μ on X^{an} of total mass (L^n) that is supported on a dual complex $\mathrm{Sk}(\mathscr{X})$ associated to some snc model $\mathscr{X}$ of X, there exists a unique metric $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})/\mathbb{R}$ such that $\mathrm{MA}(\phi) = \mu$.*

2.3. **Calabi–Yau case.** In the sequel, we further assume that the K-variety X discussed in § 2.2 is Calabi–Yau, meaning that $K_X \simeq \mathcal{O}_X$. In addition, by the semistable reduction [KKMSD73], we may assume the existence of a *semistable* snc model $\mathscr{X}$, meaning that $\mathscr{X}_0$ is reduced. Then one can consider the *essential skeleton* $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$ of X, which is introduced in [KS06], and studied in [MN15], [NX16], and [BJ17]. In particular, it is known that $\mathrm{Sk}(X)$ is a *pseudo-manifold*, and $\mathrm{Sk}(X) \subset \mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ holds for any snc model $\mathscr{X}$ of X. Also, one has $\dim_{\mathbb{R}} \mathrm{Sk}(X) \leq \dim X$. We say X is *maximally degenerate* if $\dim_{\mathbb{R}} \mathrm{Sk}(X) = \dim X$. From now on, we further assume that our X is maximally degenerate. Then $\mathrm{Sk}(X)$ has a (non-canonical) $\mathbb{Z}$-affine structure away from some singular locus Γ of codim ≥ 2, which yields a Lebesgue type measure μ on $\mathrm{Sk}(X)$ (see [NXY19, Theorem 6.1]). By Theorem 2.2, there exists a unique $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})/\mathbb{R}$ such that

$$\mathrm{MA}(\phi) = (L^n)\frac{\mu}{\mu(\mathrm{X}^{\mathrm{an}})}, \tag{2.3.1}$$

where $n := \dim X = \dim_{\mathbb{R}} \mathrm{Sk}(X)$ (cf. (1.0.1)). We call this solution $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})/\mathbb{R}$ the *non-Archimedean Calabi–Yau metric* (abbreviated as *NACY metric*), denoted by ϕ^{NACY}.

Definition 2.3 (NASYZ fibration)**.** We call $f : X^{\mathrm{an}} \to \mathrm{Sk}(X)$ a *non-Archimedean SYZ fibration* (abbreviated as *NASYZ fibration*) for L^{an} if there exists a pseudo-manifold structure on $\mathrm{Sk}(X)$ such that, for some union Γ of codim ≥ 2 faces of $\mathrm{Sk}(X)$, f is an n-dimensional *affinoid torus fibration* (see, e.g., [NXY19, (3.3)]) over $\mathrm{Sk}(X) \setminus \Gamma$ and, for any $b \in \mathrm{Sk}(X) \setminus \Gamma$, the NACY metric ϕ^{NACY} is constant on $f^{-1}(b)$, where we identify ϕ^{NACY} with some global function on X^{an} by taking its difference from a reference *model metric* on L^{an}; see [BE21, §5.3] for model metrics.

Remark 2.4. The condition that $\phi^{\mathrm{NACY}}|_{f^{-1}(b)}$ is constant on $f^{-1}(b)$ does not depend on the choice of a reference model metric on L^{an}. Our Definition 2.3 is still under development and may be refined further. In particular, we do not assume anything about what happens on Γ for the moment.

Let us recall the original setting. Let $(\mathcal{X}, \mathcal{L}) \to \mathbb{D}^* := \{t \in \mathbb{C} \mid 0 < |t| < 1\}$ be a holomorphic family of polarized Calabi–Yau manifolds $(\mathcal{X}_t, \mathcal{L}_t)$ such that $\mathcal{X}$ is defined over the hybrid ring $\mathscr{A}$ and $(X, L) := (\mathcal{X}_K, \mathcal{L}_K)$ is maximally degenerate, where each polarization on $\mathcal{X}_t$ is induced by a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$. Then, the following strong theorem is known to hold:

Theorem 2.5 ([Li23, Theorem 1.3])**.** *If there exists a semistable snc model $\mathscr{X}$ such that, for any open face σ of $\mathrm{Sk}(\mathscr{X})$ and $b \in \sigma$, ϕ^{NACY} is constant on $\rho_{\mathscr{X}}^{-1}(b)$, then Conjecture 1.2 holds for $(\mathcal{X}, \mathcal{L})$.*

Remark 2.6. This condition is referred to as the *comparison property.* In this setting, the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ is an affinoid torus fibration over each open face of $\mathrm{Sk}(\mathscr{X})$. Moreover, if $\mathscr{X}$ is minimal, that is, if $K_{\mathscr{X}/R} \equiv 0$, then we have $\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(X)$. In this sense, one may say that the comparison property requires that the Berkovich retraction $\rho_{\mathscr{X}}$ is almost a NASYZ fibration. The essential difference is that the singular locus $\Gamma \subset \mathrm{Sk}(\mathscr{X})$ of $\rho_{\mathscr{X}}$ may have codimension one.

3. Key ideas of the proofs

Thanks to Theorem 2.5, we have a sufficient condition of Conjecture 1.2 expressed in terms of Berkovich geometry. In particular, to verify this comparison property, we need to study the following:

(i) the behavior of the NACY metric ϕ^{NACY},

(ii) the existence of the desired semistable snc model $\mathscr{X}$ such that $\rho_{\mathscr{X}}$ is compatible with ϕ^{NACY}.

In the sequel, we see the key points in Theorem 1.3 and Theorem 1.4 along these lines.

3.1. **Abelian varieties.** In this case, the NACY metrics ϕ^{NACY} have been implicitly studied as the *Néron–Tate height* (or *canonical height*) in Diophantine geometry; see [BG06, §9] for instance. On the other hand, the desired semistable snc model $\mathscr{X}$ is obtained as a so-called *Mumford model*, which is given in [Mum72]. The key observation here is that we can obtain an isomorphism $X^{\mathrm{an}} \simeq T^{\mathrm{an}}/P\mathbb{Z}^n$, where $T := \operatorname{Spec} K[M]$ is a split algebraic torus with character group $M(\simeq \mathbb{Z}^n)$ and $P = (p_{ij})_{i,j} \in \mathrm{Mat}_n(K)$. Then, a *tropicalization map* $\mathrm{trop} : T^{\mathrm{an}} \to N_{\mathbb{R}} := \mathrm{Hom}(M, \mathbb{R})$, defined by $\mathrm{trop}(x) = (m \mapsto -\log |z^m(x)|)$, is compatible with the Berkovich retraction

$$\rho_{\mathscr{X}} : X^{\mathrm{an}}(\simeq T^{\mathrm{an}}/P\mathbb{Z}^n) \twoheadrightarrow \mathrm{Sk}(X)(\simeq N_{\mathbb{R}}/\left(-\log |P|\right)\mathbb{Z}^n), \tag{3.1.1}$$

where $-\log |P| := (\mathrm{ord}_t(p_{ij}))_{i,j} \in \mathrm{Mat}_n(\mathbb{Z})$. In addition, ϕ^{NACY} can be understood as a theta-type function on $N_{\mathbb{R}}$ with respect to $(-\log |P|)\,\mathbb{Z}^n$. In particular, we can prove the retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \twoheadrightarrow \mathrm{Sk}(X)$ is a NASYZ fibration for L^{an} with no singular locus, meaning that $\mathrm{Sk}(X) \supset \Gamma = \emptyset$.

Also, by considering a uniformization-type argument [GO24, Lemma 2.6] for the family $\mathcal{X}$ that is compatible with the isomorphism $X^{\mathrm{an}} \simeq T^{\mathrm{an}}/P\mathbb{Z}^n$, we can prove the latter half of Theorem 1.3.

3.2. **Calabi–Yau complete intersections in a toric Fano variety given by Batyrev–Borisov constructions.** In this case, we can obtain the potentially desired semistable snc model $\mathscr{X}$ by Yamamoto's construction [Yam24, §5.2]. Roughly speaking, our model $\mathscr{X}$ is obtained by successive blow-ups of the *toric degeneration* constructed in [Gro05, §3]. The key observation is that the associated Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \twoheadrightarrow \mathrm{Sk}(X)$ is compatible with a tropicalization map $\mathrm{trop} : T_K^{\mathrm{an}} \to N_{\mathbb{R}}$ for the maximal torus T_K of the ambient toric Fano variety Y_K of X over each open face of $\mathrm{Sk}(X)$; see [GY24, Lemma 3.2]. Making use of this feature, we can consider a class of metrics on L^{an} induced by metrics on $\widetilde{L}^{\mathrm{an}}$ that are compatible with $\mathrm{trop} : T_K^{\mathrm{an}} \to N_{\mathbb{R}}$, which is called *toric metrics* (see [BGPS14, Definition 4.3.2]), where $\widetilde{L}$ is an ample line bundle on Y_K such that $\widetilde{L}|_X = L$. Within this class, the condition that $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ is the NACY metric can be rephrased in terms of the existence of a simultaneous solution to the real Monge–Ampère equations (1.0.5) on each open face of $\mathrm{Sk}(X)$, as described in Theorem 1.4.

For Calabi–Yau hypersurfaces, prior to the work [GY24], the condition of Theorem 1.4 had already been established and studied. In particular, several concrete examples were known to satisfy our condition, implying that Conjecture 1.2 holds for such pairs $(\mathcal{X}, \mathcal{L})$; see [PS22], [HJMM24], and [Li24]. Furthermore, in this case, our condition can be completely reformulated in terms of a condition involving *optimal transportation plans*, as shown in [AH23]. In particular, the authors of [AH23] also found examples that do not satisfy our condition by studying such optimal transportation plans. Here, note that this does not mean Conjecture 1.2 is false. However, this suggests that our condition is restrictive. Nevertheless, all currently known examples for which Conjecture 1.2 holds satisfy our condition; see [DH25] for additional examples.

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