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Weighted Composition Operators and Their Differences

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.

Dedicated to the Memories of

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Short Histry, Background

$\mathbb{D} = \{|z| < 1\}$
$\mathcal{H}(\mathbb{D})$: the space of all analytic functions on $\mathbb{D}$
φ : an analytic self-map of $\mathbb{D}$

Schröder Equation (1874)

$$f \circ \varphi = \lambda f \quad (f \in \mathcal{H}(\mathbb{D}))$$

- Königs (1884): eigen functions, eigenvalues

Subordination

$f, g \in \mathcal{H}(\mathbb{D})$

f is subordinate to $g \iff f = g \circ \varphi \ \exists \varphi$

[Littlewood Subordination Principle(1925)]

g : subharmonic on $\mathbb{D}$

φ : an analytic self-map of $\mathbb{D}$, $\varphi(0) = 0$

$$\int_0^{2\pi} g(\varphi(re^{i\theta}))\ d\theta \leq \int_0^{2\pi} g(re^{i\theta})\ d\theta, 0 < r < 1$$

$g = |f|^p, \ f \in \mathcal{H}(\mathbb{D})$

Hardy Spaces

- Nordgren(1968)
- H. Schwartz, Thesis, Univ. Toledo (1969)

$1 \leq p < \infty$

H^p:the Hardy space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\theta})|^p d\theta < \infty$$

where $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on $\{|z| = 1\}$.
Usually, we will write f^* by f also and put
$dm = d\theta/2\pi$.

φ : an analytic self-map of $\mathbb{D}$

$$C_\varphi f(z) = f \circ \varphi(z)$$

C_φ:bounded on H^p

- compactness ?

Shapiro's work

[Littlewood-Paley Identy]

$f \in \mathcal{H}(\mathbb{D}), dA(z) = dxdy/\pi$

$$\|f\|_{H^2}^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z)$$

[Change-of-Variable Formula]

φ: an analytic self-map of $\mathbb{D}$

$$\|C_\varphi f\|_{H^2}^2 = |f(\varphi(0))|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(z)$$

where

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|} \quad w \in \varphi(\mathbb{D}) \setminus \{0\}$$

: Nevanlinna Counting Function of φ

$$N_\varphi(w) \leq \log \left| \frac{1 - \overline{\varphi(0)} w}{\varphi(0) - w} \right| \; w \in \varphi(\mathbb{D}) \setminus \{0\}$$

Compactness Theorem[Shapiro(1987)]

C_φ: compact on H^2 if and only if

$$\lim_{|w|\to 1} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0$$

If φ is univalent, the compactness is euiqvalent to

$$\lim_{|w|\to 1} \frac{1-|w|^2}{|1-|\varphi(w)|^2} = 0$$

that is, φ has no finite angular derivative at any point of $\partial\mathbb{D}$

This means

$$\|C_\varphi f\|_{H^2}^2 = |f(\varphi(0))|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w)\, dA(z)$$

$$\leq C\|f\|_{H^2}^2$$

$$d\mu(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}} dA(z):$$

"compact Carleson measure"

Aleksandrov-Clark measures

1 Introduction

$\mathbb{D} = \{|z| < 1\}$
$\mathcal{H}(\mathbb{D})$: the space of all analytic functions on $\mathbb{D}$
$u \in \mathcal{H}(\mathbb{D})$
φ : an analytic self-map of $\mathbb{D}$
The weighted composition operator $M_u C_\varphi$:

$$(M_u C_\varphi f)(z) = u(z) f \circ \varphi(z)$$

for $f \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$.

- isometries on function spaces
- dynamical system
- De Branges's original proof of Bieberbach conjecture
- Bennan's conjecture (in univalent function theory)

(Weighted) composition operators have been extensively investigated on various analytic function spaces during some decades.

The main theme is

to characterize the operator-theoretic behavior

of weighted composition operators

in terms of the function-theoretic properties

of the weight u and the symbol φ.

Targets

- boundedness and compactness

- compact differences

One of the recent main subjects:
the topological structure of the set of composition operators on Banach spaces of analytic functions on $\mathbb{D}$.

[Shapiro and Sundberg's conjecture (1990)]:
two composition operators would lie in the same component

if and only if

they have compact difference, that is,
the difference of the two composition operators is compact

Carleson measures

μ: a positive Borel measure on $\mathbb{D}$
X: a Banach space in $\mathcal{H}(\mathbb{D})$ $q > 0$
μ: (X, q)-Carleson measure if

$$\int_{\mathbb{D}} |f(z)|^q \; d\mu(z) \le C \|f\|_X^q$$

that is, the identy map: $X \to L^q(\mu)$ is bounded.

the identity map is compact
$\Longrightarrow$ μ: compact (X, q)-Carleson measure

To characterize the compactness, we will need the so-called “weak convergence theorem”.

Proposition 1.1.[Tjani(1996)]
Let X and Y be Banach spaces in $\mathcal{H}(\mathbb{D})$.
Suppose that

(i) The point evalution functionals on X are continuous.

(ii) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets of $\mathbb{D}$.

(iii) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets of $\mathbb{D}$.

Then T is compact
if and only if
whenever $\{f_n\}$ is a bounded sequence in X and f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$,

$$\|Tf_n\|_Y \to 0.$$

We here consider problems on weighted composition operators between function spaces.

- the Bergman space
- the Hardy space
- Furthermore

2 The Bergman space A^p_α

For $0 < p < \infty$ and $-1 < \alpha < \infty$,
A^p_α: the weighted Bergman space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\alpha} = (1+\alpha)\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha\, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ denotes the Lebesgue area measure on $\mathbb{D}$.
Put $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$.

Then the functions

$$K_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^{2+\alpha}}$$

reproduce the point-evaluations for $\lambda \in \mathbb{D}$.

For any $f \in A^1_\alpha$,

$$f(\lambda) = \int_{\mathbb{D}} f(z) \frac{1}{(1-\lambda\bar{z})^{2+\alpha}} \, dA_\alpha(z).$$

2.1 Weighted composition operators

$$M_u C_\varphi : A^p_\alpha \to A^q_\beta$$

We divide into three cases.

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$$M_u C_\varphi : A^p_\alpha \to A^q_\beta$$

We divide into three cases.

(1) The case $p = q$

At first Moorhouse obtained the following in the case $p = q = 2$ and $\alpha = \beta$.

$$K_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^{2+\alpha}}$$

$$\Rightarrow (M_u C_\varphi)^* K_\lambda(z) = \overline{u(\lambda)} K_{\varphi(\lambda)}(z)$$

$$\Rightarrow \|(M_u C_\varphi)^* \frac{K_\lambda}{\|K_\lambda\|_{A^2_\alpha}}\|^2_{A^2_\alpha}$$

$$= |u(\lambda)|^2 \left(\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}\right)^{2+\alpha}$$

Theorem 2.1.[Moorhouse (2005)]
Suppose that $u \in \mathcal{H}(\mathbb{D})$ is bounded.
Then the following are equivalent.

(i) $M_u C_\varphi$ is compact on A^2_α.

(ii) $\lim_{|z| \to 1} |u(z)|^2 \dfrac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$

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(i) $M_u C_\varphi$ is compact on A^2_α.

(ii) $$\lim_{|z|\to 1} |u(z)|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

(iii) $$\lim_{|z|\to 1} |u(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

And this was generalized.

Theorem 2.2. [Lindström and Saukko (2015)]
Let $1 < p < \infty$.
Suppose that $u \in \mathcal{H}(\mathbb{D})$ is bounded.
Then $M_u C_\varphi : A^p_\alpha \to A^p_\alpha$ is compact
if and only if

$$\lim_{|z|\to 1} |u(z)|^{\frac{p}{\alpha+1}} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

How about the case $p = 1$?

How about the case $p = 1$?
We could obtain the following summarization.

Theorem 2.3.
Suppose that $u \in \mathcal{H}(\mathbb{D})$ is bounded.
Then the following are equivalent.

(i) $M_u C_\varphi$ is compact on A^2_α.

(ii) $\displaystyle \lim_{|z|\to 1} |u(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$

(iii) $M_u C_\varphi$ is compact on A^p_α for $1 < p < \infty$.

(iv) $M_u C_\varphi$ is compact on A^1_α.

(2) The case $p \le q$

- μ: (A^p_α, q)-Carleson measure on $\mathbb{D}$

$$\exists C_1 > 0 \ s.t. \int_{\mathbb{D}} |f(z)|^q \ d\mu(z) \le C_1 \|f\|^q_{A^p_\alpha}$$

$$\Longleftrightarrow \exists C_2 > 0 \ s.t.$$

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left| \frac{1-|a|^2}{(1-\overline{a}z)^2} \right|^{(2+\alpha)q/p} d\mu(z) \le C_2$$

(Hastings(1975), Luecking(1985), Aulaskari, Stegenga and Xiao(1996))

Theorem 2.4. [Čučković and Zhao (2007)]
For $u \in \mathcal{H}(\mathbb{D})$, the following hold.
(1) For $0 < p \le q < \infty$,
$M_u C_\varphi : A^p_\alpha \to A^q_\beta$ is bounded
if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \overline{a}\varphi(z)|^2} \right)^{(2+\alpha)q/p} |u(z)|^q \, dA_\beta(z) < \infty$$

(2) For $1 < p \le q < \infty$, assume that $M_u C_\varphi : A^p_\alpha \to A^q_\beta$ is bounded.
Then $M_u C_\varphi : A^p_\alpha \to A^q_\beta$ is compact if and only if

$$\limsup_{|a|\to 1} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\varphi(z)|^2} \right)^{(2+\alpha)q/p} |u(z)|^q \, dA_\beta(z) = 0.$$

[compactness] The cases $p = 1$ or $q = \infty$ has remained.

$A^\infty_\alpha = H^\infty$:
the Banach algebra of bounded analytic functions on $\mathbb{D}$ with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} = f(z)|$$

[compactness] The cases $p = 1$ or $q = \infty$ has remained.
the case $p = 1$: True

Theorem 2.5.
For $1 \leq p < \infty$, the following hold.
(1) $M_u C_\varphi : A^p_\alpha \to H^\infty$ is bounded
if and only if

$$\sup_{z \in \mathbb{D}} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} < \infty.$$

(2) $M_u C_\varphi : A^p_\alpha \to H^\infty$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{|u(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} = 0.$$

(3) The case $q < p$

Luecking(1986,1993): the characterzation of the (A^p_α, q)-Carleson measure for $1 \le q < p < \infty$

Moreover,

Čučković and Zhao used the weighted φ-Berezin transform of a measurable function h defined as follows.

$$B_{\varphi,\alpha,\beta}(h)(z) = \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\overline{z}\varphi(w)|^2} \right)^{2+\alpha} h(w)\ dA_\beta(w).$$

Theorem 2.6. [Čučković and Zhao (2007)]
Let $u \in \mathcal{H}(\mathbb{D})$.
Let $1 \leq q < p < \infty$ and $\alpha, \beta > -1$.
Then the following are equivalent.

(i) $M_u C_\varphi : A^p_\alpha \to A^q_\beta$ is bounded.
(ii) $M_u C_\varphi : A^p_\alpha \to A^q_\beta$ is compact.
(iii) $B_{\varphi,\alpha,\beta}(|u|^q) \in L^{p/(p-q)}(dA_\alpha(z))$.

How about the case $p = \infty$?

How about the case $p = \infty$?

We have the following.

Theorem 2.7.

Let $u \in \mathcal{H}(\mathbb{D})$.

Let $1 \leq q < \infty$ and $\beta > -1$.

Then the following are equivalent.

(i) $M_u C_\varphi : H^\infty \to A^q_\beta$ is bounded.

(ii) $M_u C_\varphi : H^\infty \to A^q_\beta$ is compact.

(iii) $u \in A^q_\beta$.

Problem 1.

For any unbounded analytic function u on $\mathbb{D}$, function-theoretically characterize the boundedness and comapctness of $M_u C_\varphi : A^p_\alpha \to A^q_\beta$.

In particular, how about the operator

$$M_u C_\varphi : A^2_0 \to A^2_0 \quad \text{or} \quad M_{\varphi'} C_\varphi : A^2_1 \to A^2_1$$

for unbounded u?

Choe,Choi,Koo and Yang present the next example.

Example 2.8.[Choe,Choi,Koo and Yang(2020)]

Let $\alpha > -1, 0 < p < \infty$ and $0 < \varepsilon < \dfrac{\alpha+2}{p}$.

Put

$$u_\varepsilon(z) = \frac{1}{(1-z)^\varepsilon} \quad \text{and} \quad \varphi(z) = 1 - (1-z)^{1/2}.$$

Then

(1) $M_u C_\varphi$ is bounded on A^p_α

$$\iff \quad \varepsilon \le \frac{\alpha + 2}{2p}.$$

(2) $M_u C_\varphi$ is compact on A^p_α

$$\iff \quad \varepsilon < \frac{\alpha + 2}{2p}.$$

2.2 The differences

Let $\sigma(z) = (\varphi(z) - \psi(z))/(1 - \overline{\varphi(z)}\psi(z))$..

Theorem 2.9.[Moorhouse (2005)] For $\alpha > -1$, $C_\varphi - C_\psi$ is compact on A^2_α if and only if

$$\lim_{|z|\to 1} |\sigma(z)| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0$$

$$\lim_{|z|\to 1} |\sigma(z)| \frac{1 - |z|^2}{1 - |\psi(z)|^2} = 0.$$

And this was generalized.

Theorem 2.10.

[Lindström and Saukko (2015)]

For $1 < p < \infty$ and $\alpha > -1$,

$C_\varphi - C_\psi$ is compact on A^p_α if and only if

$$\lim_{|z|\to 1} \left(|\sigma(z)|^{\frac{p}{1+\alpha}} \text{ Max} \left\{ \frac{1-|z|^2}{1-|\varphi(z)|^2}, \frac{1-|z|^2}{1-|\psi(z)|^2} \right\} \right)$$

$$= 0.$$

Motivated by these results,

Saukko (2011, 2012) studied the case
$0 < p, q < \infty$, but errors.

Some were corrected in his Thesis (2015),
but still.

Lindström, Saukko and Shi (2021) correct errors.

Fortunately theorems hold!

Theorem 2.11.
[Saukko (2011), Lindström, Saukko and Shi(2021)] Suppose that $0 < p \leq q < \infty$.

(1) Then $C_\varphi - C_\psi : A^p_\alpha \to A^q_\beta$ is bounded

if and only if

$\sigma C_\varphi, \sigma C_\psi : A^p_\alpha \to L^q(dA_\beta)$ are bounded.

(2) Assume that $C_\varphi - C_\psi : A^p_\alpha \to A^q_\beta$ is bounded.

Then $C_\varphi - C_\psi : A^p_\alpha \to A^q_\beta$ is compact

if and only if

$\sigma C_\varphi, \sigma C_\psi : A^p_\alpha \to L^q(dA_\beta)$ are compact.

Problem 2. How about the cases $p = 1$ or $q = \infty$?

Theorem 2.12.
[Saukko (2012), Lindström, Saukko and Shi(2021)]
Let $0 < q < p < \infty$ and μ a positive Borel measure on $\mathbb{D}$. Then the following are equivalent.

(i) $C_\varphi - C_\psi$ maps A^p_α into $L^q(\mu)$.
(ii) σC_φ and σC_ψ map A^p_α into $L^q(\mu)$.

(iii) The function

$$K_{\varphi,\psi}(z) = \int_{\mathbb{D}} \left| \left(\frac{1-|z|^2}{(1-\overline{z}\varphi(w))^2} \right)^{(2+\alpha)/q} - \left(\frac{1-|z|^2}{(1-\overline{z}\psi(w))^2} \right)^{(2+\alpha)/q} \right|^q d\mu(w)$$

belongs to $L^{p/(p-q)}(dA_\alpha)$.

Problem 3. How about the case $p = \infty$?

Moreover about the compactness?

For differences of weighted composition operators,
Choe, Choi, Koo and Yang (2020, 2021):
a complete characterization in terms of Carleson measures for the boundedness and compactness of $uC_\varphi - vC_\psi : A^p_\alpha \to L^q(dA_\alpha)$ with $u, v \in L^q(dA_\alpha)(0 < p, q < \infty)$.

3 The Hardy space H^p

3.1 Weighted composition operators

φ: a finite Blaschke product

$M_u C_\varphi$: bounded on $H^2 \Longrightarrow u$:bounded

How about inner function φ?

(A. Richards, Thesis Univ. Wisconsin 1974)

3.1 Weighted composition operators

φ: a finite Blaschke product

$M_u C_\varphi$: bounded on $H^2 \Longrightarrow u$:bounded

How about inner function φ?

Negative

We divide two cases.

(1) The case $p \le q$

Contreras and Hernández-Díaz :

the notion of Carleson measures

For $u \in H^q$, define the measure $\mu_{u,\varphi,p}$ on $\overline{\mathbb{D}}$ by

$$\mu_{u,\varphi,p}(E) = \int_{\varphi^{-1}(E)\cap\partial\mathbb{D}} |u|^p \, dm,$$

where E is a measurable subset of $\overline{\mathbb{D}}$.

So it holds that

$$\int_{\partial\mathbb{D}} |u|^p (g \circ \varphi)\ dm = \int_{\overline{\mathbb{D}}} g\ d\mu_{u,\varphi,p},$$

where g is an arbitrary measurable positive function on $\overline{\mathbb{D}}$.

Theorem 3.1.

[Contreras and Hernández-D$\acute{\imath}$az(2003)]

(1) For $1 \le p \le q < \infty$ and $u \in H^q$, $M_u C_\varphi : H^p \to H^q$ is bounded if and only if $\mu_{u,\varphi,q}$ is a (H^p, q)-Carleson measure on $\overline{\mathbb{D}}$.

(2) For $1 \le p \le q < \infty$ and $u \in H^q$, $M_u C_\varphi : H^p \to H^q$ is compact if and only if $\mu_{u,\varphi,q}$ is a (H^p, q)-Carleson compact measure on $\overline{\mathbb{D}}$.

(3) For $1 \leq p < q = \infty$,
$M_u C_\varphi : H^p \to H^\infty$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} < \infty.$$

Moreover, $M_u C_\varphi : H^p \to H^\infty$ is compact if and only if either $\|\varphi\|_\infty < 1$ or

$$\lim_{|\varphi(z)| \to 1} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} = 0.$$

Čučković and Zhao adopted the Cauchy kernel as the test function and characterized.

Theorem 3.2. [Čučković and Zhao (2007)]
Let u be an analytic function on $\mathbb{D}$.
(1) For $0 < p \leq q < \infty$, $M_u C_\varphi : H^p \to H^q$ is bounded if and only if

$$\sup_{a \in \mathbb{D}} \int_{\partial \mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \overline{a}\varphi(\zeta)|^2} \right)^{q/p} |u(\zeta)|^q \, dm(\zeta) < \infty.$$

(2) For $1 < p \leq q < \infty$,
$M_u C_\varphi : H^p \to H^q$ is compact if and only if

$$\limsup_{|a| \to 1} \int_{\partial \mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \overline{a}\varphi(\zeta)|^2} \right)^{q/p} |u(\zeta)|^q \, dm(\zeta) = 0.$$

The ccase $p = 1$ also holds.

Problem 4.

Give the function-theoretic characterizzation for the boundedness and comapctness of

$$M_u C_\varphi : H^p \to H^p.$$

(2) The case $q < p$

For example,

$u \in H^2$

$\Longrightarrow \quad M_u C_\varphi : H^2 \to H^1$ is bounded

Contreras and Hernández-D$\acute{\imath}$az left this case and Čučković and Zhao answered the boundedeness using a result due to Luecking(1991).

Theorem 3.3. [Čučković and Zhao (2007)]
(1) For $1 \le q < p < \infty$, $M_u C_\varphi : H^p \to H^q$ is bounded if and only if

$$\int_0^{2\pi} \left(\int_{\Gamma(\theta)} \frac{d\mu_u(w)}{1 - |w|^2} \right)^{p/(p-q)} dm < \infty,$$

$\mu_u = (|u|^q dm) \circ \varphi^{-1}$ and $\Gamma(\theta)$ is the Stoltz angle at θ which is defined for real θ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{1/2}\}$.

(2) For $1 \leq q < p < \infty$,
let $M_u C_\varphi : H^p \to H^q$ is bounded.

Then $M_u C_\varphi : H^p \to H^q$ is compact

if and only if

$|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.

In the case $p = \infty$ we could have the following.

Theorem 3.4.

[Contreras and Hernández-D$\acute{\imath}$az (2003)]

(1) For $1 \leq q < \infty$,
$M_u C_\varphi : H^\infty \to H^q$ is bounded if and only if $u \in H^q$.

(2) For $1 \leq q < \infty$,
$M_u C_\varphi : H^\infty \to H^q$ is compact if and only if $u \in H^q$ and $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.

3.2 The differences

In the case $1 \leq q < p < \infty$, the topological strucure of the set of weighted composition operators is completely characterized.

Theorem 3.5.[Izuchi and Ohno (2014)]

$1 \leq q < p < \infty$,

$u, v \in \mathcal{H}(\mathbb{D})$: $M_u C_\varphi - M_v C_\psi : H^p \to H^q$ is bounded.

Then $M_u C_\varphi - M_v C_\psi : H^p \to H^q$ is compact if and only if $|\varphi| < 1$ and $|\psi| < 1$ a.e. on $\partial\mathbb{D}$.

the case $p = \infty$: For $1 \leq q < \infty$,
$M_u C_\varphi - M_v C_\psi : H^\infty \to H^q$ is compact
if and only if
$|\varphi| < 1$ and $|\psi| < 1$ a.e. on $\partial\mathbb{D}$.

φ, ψ: analytic self-maps of $\mathbb{D}$
let $\sigma(z) = (\varphi(z) - \psi(z))/(1 - \overline{\varphi(z)}\psi(z))$.

Theorem 3.6.[Shi and Li (2018)]
$1 < p < q < \infty$,
assume that $\sigma C_\varphi, \sigma C_\psi : H^p \to H^q$ is bounded.

Then the following are equivalent.

(i) $C_\varphi - C_\psi : H^p \to H^q$ is compact.

(ii)

$$\lim_{|a|\to 1} \int_{\partial\mathbb{D}} \Bigg| \left(\frac{1-|a|^2}{|1-\overline{a}\varphi(\zeta)|^2} \right)^{1/p} - \left(\frac{1-|a|^2}{|1-\overline{a}\psi(\zeta)|^2} \right)^{1/p} \Bigg|^q \, dm(\zeta) = 0$$

and $\lim_{n\to\infty} \|\varphi^n - \psi^n\|_{H^q} = 0.$

(iii)

$$\lim_{|a|\to 1} \int_{\partial\mathbb{D}} |\sigma(\zeta)|^q \Big(\frac{1-|a|^2}{|1-\overline{a}\varphi(\zeta)|^2} + \frac{1-|a|^2}{|1-\overline{a}\psi(\zeta)|^2} \Big)^{q/p} \, dm(\zeta) = 0.$$

Problem 5. How about the cases $p = 1$ or $q = \infty$?

Problem 6. How about the cases $p = q$?

Choe, Choi, Koo and Yang showed the following as an application of a characterization in the case of Bergman space setting.

Theorem 3.7.[Choe, Choi, Koo and Yang(2020)]
φ, ψ: univalent analytic self-maps of $\mathbb{D}$.
Then

$C_\varphi - C_\psi : H^2 \to H^2$ is compact

if and only if

$$\lim_{|z|\to 1} |\sigma(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right) = 0.$$

Nieminen and Saksman (2004):
the compactness of $C_\varphi - C_\psi$ on H^p is independent of the parameter p.
So the theorem above also holds with H^p for $1 \le p < \infty$

Saukko considered the differences of composition operators between Bergman spaces and posed a question.

Problem 7. For $1 < p, q < \infty$,
the boundedness and compactness of
$C_\varphi - C_\psi : H^p \to H^q$

$\Longleftrightarrow$

the ones of $\sigma C_\varphi, \sigma C_\psi : H^p \to L^q$?

Next we consider the Hilbert-Schmidtness of differences.

Theorem 3.8.

φ, ψ: $|\varphi| < 1, |\psi| < 1$ a.e. on $\partial\mathbb{D}$.

Then the following conditions are equivalent.

(i) $C_\varphi - C_\psi : H^2 \to H^2$ is Hilbert-Schmidt.

(ii) $\sigma C_\varphi, \sigma C_\psi : H^2 \to L^2$ is Hilbert-Schmidt.

4 Products of Toeplitz and composition operators

We could consider the products of Toeplitz and composition operators as a generalization.
So we have a big project.

Problem 8.
Characteterize the compactness of the products of Toeplitz and composition operators on the Hardy and Bergman spaces.

We here add comments on the Hardy space.

We here add comments on the Hardy space.

T: a bounded linear operator on H^2.
T: a Toeplitz operator $\iff S^*TS = T$
where$Sf(z) = zf(z)$ for $z \in \partial\mathbb{D}$ and $f \in H^2$
and S^* is the backward shift on H^2.

[Barr$\acute{\imath}$a and Halmos (1982)]:
T: asymptotically Toeplitz
$\iff$
$\{{S^*}^n T S^n\}$ converges (strongly) on H^2.

[Duna, Gagne, Gu and Shapiro(2014)] "Toeplitzness" of composition operators and their adjoints.

$C_\varphi^* C_\varphi$ is uniformly asymptotically Toeplitz

if and only if

$T_{1-\chi_{\Gamma(\varphi)}} C_\varphi$ is compact,

where $\Gamma(\varphi) = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$.

Problem 9. Characteterize the compactness of $T_{1-\chi_\Gamma(\varphi)}C_\varphi$ on H^2.

Problem 9. Characteterize the compactness of $T_{1-\chi_{\Gamma(\varphi)}}C_\varphi$ on H^2.

If $m(\Gamma(\varphi)) = 0$, then $T_{\chi_{\Gamma^c(\varphi)}}$ is identity, so $T_{\chi_{\Gamma^c(\varphi)}}C_\varphi$ is compact if and only if C_φ is compact.

If $m(\Gamma(\varphi)) = 1$, then $T_{\chi_{\Gamma^c(\varphi)}} = 0$, so trivially $T_{\chi_{\Gamma^c(\varphi)}}C_\varphi$ is compact.

We shall study the case $0 < m(\Gamma(\varphi)) < 1$ and here discuss the Hilbert-Schmidtness.

We shall study the case $0 < m(\Gamma(\varphi)) < 1$ and here discuss the Hilbert-Schmidtness.

Proposition 4.1.
If

$$\int_{\Gamma^c(\varphi)} \frac{1}{1-|\varphi|^2}\, dm < \infty,$$

$T_{\chi_{\Gamma^c(\varphi)}} C_\varphi$ is Hilbert-Schmidt.

Corollary 4.2.

Suppose that $0 < m(\Gamma(\varphi)) < 1$.
If $m(\Gamma(\varphi)) = m(\{|\varphi| > \sigma\})$ for some
$0 < \sigma < 1$,
$T_{\chi_{\Gamma^c(\varphi)}} C_\varphi$ is Hilbert-Schmidt.

The converse.

Proposition 4.3.
If $T_{\chi_{\Gamma^c(\varphi)}} C_\varphi$ is Hilbert-Schmidt,
then
$$\int_{\Gamma^c(\varphi)} \log \frac{1}{1-|\varphi|^2}\, dm < \infty.$$

After all

Thank you for your time!

Weighted Composition Operators and Their Differences

Shûichi Ohno

MSJ Spring Meeting 2022
at Saitama University, March 30

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