WKB analysis of Hartree equations

Satoshi Masaki (Kyoto university)

We consider the semiclassical Hartree equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon, \quad u(0,x) = a_0(x)e^{i\frac{\phi_0(x)}{\varepsilon}}, \tag{1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\lambda \in \mathbb{R}\setminus\{0\}$, $\gamma > 0$, and ε is a positive constant corresponding to the Planck's constant. We would like to seek the WKB-type approximate solution of the form

$$u^\varepsilon(t,x) \sim e^{i\frac{\phi(t,x)}{\varepsilon}}(b_0(t,x) + \varepsilon b_1(t,x) + \varepsilon^2 b_2(t,x) + \cdots) \tag{2}$$

as $\varepsilon \to 0$. One way to obtain this approximation is to employ a modified Madelung transform $u^\varepsilon = a^\varepsilon e^{\phi^\varepsilon/\varepsilon}$ and solve the system for $(a^\varepsilon, \phi^\varepsilon)$:

$$\begin{cases} \partial_t a^\varepsilon + (\nabla\phi^\varepsilon \cdot \nabla)a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^h = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon(0,x) = a_0 \\ \partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, & \phi^\varepsilon(0,x) = \phi_0. \end{cases} \tag{3}$$

The expansion $a^\varepsilon = a + o(1)$ and $\phi^\varepsilon = \psi + \varepsilon\psi_1 + o(\varepsilon)$ yield the desired WKB type estimate $u^\varepsilon = e^{i\psi/\varepsilon}(ae^{i\psi_1} + o(1))$. The system for $(a^\varepsilon, \nabla\phi^\varepsilon)$ can be reduced to a symmetric hyperbolic system with semilinear perturbation. It is shown in [1, 2] that, if $n \geqslant 3$ and $\gamma \in (n/2-2, n-2]$, then we can solve this system for an initial data in a Sobolev-type space, and obtain the expansion of $(a^\varepsilon, \phi^\varepsilon)$ in powers of ε. We consider the principal part $a = \lim_{\varepsilon\to 0} a^\varepsilon$ and $\psi = \lim_{\varepsilon\to 0}\phi^\varepsilon$. If (a,ψ) becomes singular then the WKB estimate (2) ceases to be valid. We put $(\rho, v) := (|a|^2, \nabla\psi)$. Then, it solves hydrodynamical equations

$$\rho_t + \operatorname{div}(\rho v) = 0, \qquad v_t + v\cdot\nabla v + \lambda\nabla(|x|^{-\gamma} * \rho) = 0. \tag{4}$$

If the solution of (4) breaks down in finite time then (a,ϕ) becomes singular at the same time. We restrict our attention to the case $\gamma = n-2$. Then, the Hartree equation corresponds to the Schrödinger-Poisson system, and (4) to the following compressible Euler-Poisson equations:

$$\rho_t + \operatorname{div}(\rho v) = 0, \qquad v_t + v\cdot\nabla v = -\lambda\nabla\Phi, \qquad \Delta\Phi = \rho. \tag{5}$$

In this talk, we discuss global existence/finite-time breakdown of the classical solution of (5) under the radial symmetry. We concentrate on the multi-dimensional isotropic model:

$$r^{n-1}\rho_t + \partial_r(r^{n-1}\rho v) = 0; \qquad \rho(0,r) = \rho_0(r), \tag{6}$$

$$v_t + v\partial_r v + \lambda\partial_r\Phi = 0; \qquad v(0,r) = v_0(r), \tag{7}$$

$$\partial_r(r^{n-1}\partial_r\Phi) = r^{n-1}\rho. \tag{8}$$

Here, $r \geqslant 0$ denotes the distance from the origin. We define a function space $D^m := C([0,\infty)) \cap C^m((0,\infty))$ for $m \geqslant 1$ and $D^m_\rho := D^m \cap L^1((0,\infty), r^{n-1}dr)$.

Theorem 1 (Corollary 1.17 in [4])**.** *Suppose $\lambda > 0$ or $n \geqslant 3$. Suppose $\rho_0 \in D^1_\rho$ is not identically zero and $v_0 \in D^2$ satisfies $v_0(0) = 0$ and $v_0 \to 0$ as $r \to \infty$. Then, the solution of* (6)*–*(8) *is global if and only if $\lambda > 0$ and $n \geqslant 3$, and the initial data is of particular form*

$$v_0(r) = \sqrt{\frac{2\lambda}{(n-2)r^{n-2}} \int_0^r \rho_0(s) s^{n-1} ds}.$$

Moreover, if $\rho_0 \in D^m_\rho$ then the above v_0 belongs to D^{m+1} with $v_0(r) = O(r)$ as $r \to 0$ and $v_0(r) = O(r^{1-n/2})$ as $r \to \infty$, and the corresponding global solution $\rho \in C^2([0,\infty), D^m_\rho) \cap C^\infty((0,\infty), D^m_\rho)$ and $v \in C^1([0,\infty), D^{m+1}) \cap C^\infty((0,\infty), D^{m+1})$ are given explicitly by

$$\rho(t, X(t,R)) = \rho_0(R) \left(1 + \frac{n v_0(R)}{2R} t\right)^{-1} \left(1 + \frac{\lambda R \rho_0(R)}{(n-2) v_0(R)} t\right)^{-1},$$
$$v(t, X(t,R)) = v_0(R) \left(1 + \frac{n v_0(R)}{2R} t\right)^{1-\frac{2}{n}},$$

where $X(t,R) = R(1 + \frac{n v_0(R)}{2R} t)^{2/n}$. Furthermore, the solution is unique and also solves (5) *in the distribution sense.*

The key for the proof of Theorem 1 is a reduction of the radial Euler-Poisson equation in [3, 4]. We set $m(t,r) = \int_0^r \rho(t,s) s^{n-1} ds$ and define characteristic curves X by $\frac{d}{dt} X(t,R) = v(t, X(t,R))$ with $X(0,R) = R$. Then, (6)–(8) is reduced to an ODE for X

$$X''(t,R) = -\frac{\lambda m_0(R)}{X(t,R)^{n-1}}, \quad X'(0,R) = v_0(R), \quad X(0,R) = R, \tag{9}$$

where $m_0(r) = m(0,r)$. Another point is that one necessary and sufficient condition for global existence is that $\partial_R X(t,R) > 0$ holds for all $R \geqslant 0$ and $t \geqslant 0$. We check this condition by an analysis of (9).

References

[1] T. Alazard and R. Carles, *Semi-classical limit of Schrödinger–Poisson equations in space dimension $n \geqslant 3$*, J. Differential Equations **233** (2007), no. 1, 241–275.

[2] R. Carles and S. Masaki, *Semiclassical analysis for Hartree equations*, Asymptotic Analysis **58** (2008), no. 4, 211–227.

[3] S. Engelberg, H. Liu, and E. Tadmor, *Critical thresholds in Euler-Poisson equations*, Indiana Univ. Math. J. **50** (2001), no. Special Issue, 109–157, Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[4] S. Masaki, *Remarks on global existence of classical solution to multidimensional compressible Euler-Poisson equations with geometrical symmetry*, RIMS Kokyuroku bessatsu, to appear.