STABILITY OF THE BURGERS VORTEX

Yasunori Maekawa (Kyushu University)
yasunori@math.kyushu-u.ac.jp

We consider the three dimensional Navier-Stokes equations for viscous incompressible flows

$$
(NS)\quad \begin{cases} \partial_t U - \Delta U + (U, \nabla)U + \nabla P &= 0 \qquad t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot U &= 0 \qquad t >,\ x \in \mathbb{R}^3, \\ U(0,x) &= U_0(x) \qquad x \in \mathbb{R}^3. \end{cases}
$$

Here $U = (U_1, U_2, U_3)^\top$ is an unknown velocity field, P is an unknown pressure field, $\Delta = \sum_{i=1}^3 \partial_i^2$, and $\nabla = (\partial_1, \partial_2, \partial_3)^\top$. In this talk we consider the velocity U of the form

$$
U = \bar{u} + u \tag{0.1}
$$

where

$$
\begin{aligned} \bar{u} &= (-\frac{1}{2}x_1, -\frac{1}{2}x_2, x_3)^\top \\ u(x,t) &= (u_1(x,t), u_2(x,t), u_3(x,t))^\top. \end{aligned}
$$

The velocity $\bar{u}$ represents the (axisymmetric) background straining flow. In the dynamics of fluid flows its vorticity field plays important roles. Since $\bar{u}$ is rotation free, the vorticity field Ω of U is given by

$$
\Omega := \nabla \times U = \nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^\top. \tag{0.2}
$$

The equation for Ω is obtained by taking the curl of (NS) and by expressing the velocity u by the Biot-Savart law:

$$
(V)\quad \begin{cases} \partial_t \Omega - \Delta \Omega + (\bar{u} + u, \nabla)\Omega - (\Omega, \nabla)(\bar{u} + u) &= 0 \qquad t > 0,\ x \in \mathbb{R}^3, \\ u &= \nabla \times (-\Delta)^{-1}\Omega \qquad t > 0,\ x \in \mathbb{R}^3, \\ \Omega(0,x) &= \Omega_0(x) \qquad x \in \mathbb{R}^3. \end{cases}
$$

For (V) there is a family of exact stationary solutions. Let G be two dimensional Gaussian, i.e.,

$$
G(x_h) = \frac{1}{4\pi} \exp(-\frac{|x_h|^2}{4}), \qquad x_h = (x_1, x_2). \tag{0.3}
$$

Set $\mathbf{e}_3 = (0,0,1)^\top$. The velocity field associated with $G\mathbf{e}_3$ is given by

$$
\begin{aligned} u^G(x) &= \nabla \times (-\Delta)^{-1} G\mathbf{e}_3 = (-\frac{x_2}{|x_h|} v^G(|x_h|), \frac{x_1}{|x_h|} v^G(|x_h|), 0)^\top, \\ v^G(r) &= \frac{1}{2\pi r}(1 - e^{-\frac{r^2}{4}}), \qquad r > 0. \end{aligned}
$$

Then direct calculations show that $\{\alpha G\}_{\alpha \in \mathbb{R}}$ is a family of stationary solutions for (V), and $\alpha G\mathbf{e}_3$ is called the axisymmetric Burgers vortex which was discovered by Burgers [1]. The number α is called the circulation number or the vortex Reynolds number which represents the magnitude of the vorticity of the Burgers vortex. The Burgers vortices have been used as a simple model of the vortex tubes observed in numerical simulations of the three dimensional turbulent flows, and their stability has been investigated by many researchers. As for the mathematical researches, the stability with respect to purely two dimensional perturbation flows (in this case $\omega = (0, 0, \omega_3(x_1, x_2, t))^\top$ in the above equation) was firstly studied by Giga-Kambe [5] and their results were extended by Carpio [2], Giga-Giga [6], Gallay-Wayne [3], and [7]. Especially, in [3] it is proved that the Burgers vortex is globally stable with respect to two dimensional perturbations for any circulation numbers. In this talk we discuss the three dimensional stability of the Burgers vortex. The most important step is to analyze the linearized equation around $\alpha G\mathbf{e}_3$ as follows.

$$\partial_t \omega - (L - \alpha T)\omega = 0 \qquad t > 0,\ x \in \mathbb{R}^3. \tag{0.4}$$

Here $\omega = (\omega_1, \omega_2, \omega_3)^\top,\ v = \nabla \times (-\Delta)^{-1}\omega$, and

$$L\omega = \Delta\omega - (Mx, \nabla)\omega + M\omega, \qquad Mx = (-\frac{1}{2}x_h, x_3)^\top, \tag{0.5}$$

$$T\omega = (u^G, \nabla)\omega - (\omega, \nabla)u^G + (v, \nabla)G\mathbf{e}_3 - G\partial_3 v. \tag{0.6}$$

Let us introduce the function spaces.

$$L^2_G(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \mid G^{-\frac{1}{2}} f \in L^2(\mathbb{R}^2)\},$$

$$L^2_{G,0}(\mathbb{R}^2) = \{f \in L^2_G(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} f(x_h)dx_h = 0\},$$

$$X_2 = L^2(\mathbb{R}; L^2_G(\mathbb{R}^2)), \qquad X_{2,0} = L^2(\mathbb{R}; L^2_{G,0}(\mathbb{R}^2)), \qquad \mathbb{X}_2 = X_2 \times X_2 \times X_{2,0}.$$

Then we have

Theorem 1. *Let $\alpha \in \mathbb{R}$. Then for any $\omega_0 \in \mathbb{X}_2$ with $\nabla \cdot \omega_0 = 0$, there is a unique solution $\omega(t) \in C([0, \infty); \mathbb{X}_2)$ to (0.4) satisfying the estimate*

$$||\omega(t)||_{\mathbb{X}_2} \le C||\omega_0||_{\mathbb{X}_2}, \qquad t > 0. \tag{0.7}$$

Here the constant C depends only on $|\alpha|$.

The more detailed estimates will be shown in the talk. From these results we can prove the existence of time-global solutions to the full vorticity equations (V) near the Burgers vortices. Especially, these solutions remain large for all time if the circulation number is large.

References

[1] Burgers, J. M., A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech., 1948, 171-199.

[2] Carpio, A., Asymptotic behavior for the vorticity equations in dimensions two and three. Comm. P. D. E., 19, 1994, 827-872.

[3] Gallay, Th. and Wayne, C. E., Global Stability of vortex solutions of the two dimensional Navier-Stokes equation. Comm. Math. Phys., 255(1), 2005, 97-129.

[4] Gallay, Th. and Wayne, C. E., Three-dimensional stability of Burgers vortices : the low Reynolds number case. Phys. D., 213, 2006, no. 2, 164-180.

[5] Giga, Y. and Kambe, T., Large time behavior of the vorticity of two dimensional viscous flow and its application to vortex formation. Comm. Math. Phys., 117, 1988, 549-568.

[6] Giga, Y. and Giga, M.-H., *Nonlinear Partial Differential Equation, Self-similar solutions and asymptotic behavior*; Kyoritsu: 1999 (in Japanese), English version to be published by Birkhäuser.

[7] Maekawa, Y., Spectral properties of the linearization at the Burgers vortex in the high rotation limit. to appear in J. Math. Fluid Mech.