Spectral Projection for Barrier Top Resonances and Applications

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This report is concerned with a joint work with Jean-François Bony (Bordeaux I), Thierry Ramond (Paris XI) and Maher Zerzeri (Paris XIII).

We consider the semiclassical Schrödinger equation in $\mathbb{R}^n$

$$Pu = zu, \quad P := -h^2\Delta + V(x),$$

where $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $h > 0$ is a semiclassical parameter, $z \in \mathbb{C}$ is a spectral parameter and $V(x)$ is a short-range real-valued analytic potential with a positive non-degenerate maximum E_0 at the origin $x = 0$. More precisely, we assume the following conditions (A1)-(A3) on the potential V.

(A1) $V(x)$ is real on $\mathbb{R}^n$, and analytic in a domain $\mathcal{D} := \{x \in \mathbb{C}^n; |\operatorname{Im} x| \le \delta\langle \operatorname{Re} x\rangle\}$ for $\delta > 0$, and verifies $V(x) = \mathcal{O}(|x|^{-\rho})$ for $\rho > 1$ as $|x| \to \infty$ in $\mathcal{D}$.

To the self-adjoint operator P on $L^2(\mathbb{R}^n)$ with $\sigma_{ess}(P) = \mathbb{R}_+$, we associate a *distorted* operator $P_\mu = U_\mu P U_{-\mu}$, where $(U_\mu f)(x) := (\det(\mathrm{Id} + \mu dF))^{-1/2} f(x + \mu F(x))$ for small $|\mu|$, and $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $F = 0$ on $|x| < R$, $F = x$ on $|x| > R + 1$ for large R. The complex eigenvalues of $P_{i\theta}$ for $\theta > 0$ are independent of θ, and called *resonances* of P (Hunziker '86).

(A2) $V(0) = E_0 > 0$, $V'(0) = 0$, $V''(0) < 0$, i.e. for suitable coordinates, $V(x) = E_0 - \sum_{j=1}^n \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(|x|^3)$ as $x \to 0$, for some $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Let us also consider the corresponding classical dynamics in the phase space $\mathbb{R}^{2n}_{(x,\xi)}$, i.e. the vector field $H_p = \nabla_\xi p \cdot \nabla_x - \nabla_\xi p \cdot \nabla_x$ of the Hamitonian $p(x,\xi) = \xi^2 + V(x)$. The assumption (A2) implies that the origin $(x,\xi) = (0,0)$ is a hyperbolic fixed point of H_p, and there exist outgoing and incoming stable manifolds

$$\Lambda_\pm = \{(x,\xi) \in \mathbb{R}^{2n}; \exp tH_p(x,\xi) \to (0,0) \quad \text{as } t \to \mp\infty\}$$

which are tangent to the eigenspaces associated respectively with the positive and negative eigenvalues $\{\lambda_j\}_{j=1}^n$ and $\{-\lambda_j\}_{j=1}^n$ of the fundamental matrix of H_p. They are Lagrangian manifolds, and, near the origin, they have generating functions $\phi_\pm(x) = \pm\sum \frac{\lambda_j}{4} x_j^2 + O(|x|^3)$; $\Lambda_\pm = \{(x,\xi) \in \mathbb{R}^{2n}; \xi = \nabla_x \phi_\pm(x)\}$.

(A3) The origin $(x,\xi) = (0,0)$ is the unique trapped set in $p^{-1}(E_0)$, i.e. if $(x,\xi) \in p^{-1}(E_0)\backslash(0,0)$, then $|\exp tH_p(x,\xi)| \to \infty$ as $t \to +\infty$ or as $t \to -\infty$.

The assumption (A3) implies that V reaches its global maximum E_0 only at $x = 0$.

Under the assumptions (A1)-(A3), the semiclassical distribution of resonances are known near the barrier top energy E_0 (Sjöstrand '87, Briet-Combes-Duclos '87): In a complex disc centered at E_0 of radius Ch, there exists a bijection $b(h)$ between the set $\{z_\alpha^0\}_{\alpha\in\mathbb{N}^n}$ and the set of resonances such that $b(h)z - z = o(h)$, where $z_\alpha^0 = E_0 - ih\sum_{j=1}^n \lambda_j(\alpha_j + \frac{1}{2})$.

Let us fix $\alpha \in \mathbb{N}^n$ satisfying the condition

(A4) If $z_\alpha^0 = z_{\alpha'}^0$, then $\alpha = \alpha'$.

We denote by $z_\alpha = b(h)z_\alpha^0$ the corresponding resonance, and by Π_α the corresponding spectral projection; $\Pi_\alpha = \frac{1}{2i\pi}\int_{|z-z_\alpha|=\epsilon}(P_\theta - z)^{-1}dz$.

Theorem 1 (Spectral projection)**.** *Assume (A1)-(A4). Then, for any $\chi \in C_0^\infty(\mathbb{R}^n)$,*

$$\chi\Pi_\alpha\chi = c(h)(\cdot, \overline{\chi f})\chi f, \quad c(h) = \left(\frac{h}{i}\right)^{n/2-|\alpha|}\prod_{j=1}^{n}\lambda_j^{\alpha_j+\frac{1}{2}},$$

where $f = f(x,h)$ is a solution to $Pf = z_\alpha f$, locally L^2 uniformly in h, 0 in the incoming region (in the microlocal sense) and has an asymptotic expansion

$$f = d(x,h)e^{i\phi_+(x)/h}, \quad d(x,h) \sim \sum d_j(x)h^j, \quad d_0(x) = x^\alpha + \mathcal{O}(|x|^{|\alpha|+1})$$

near the origin.

The point consists in the calculus of $(P_\theta - z)^{-1}u$ for non-resonant z and for a certain u that we choose to have its microsupport near a point on the incoming stable manifold Λ_-. Our recent work (BFRZ '07) enables to represent $(P_\theta - z)^{-1}u$ near the fixed point $(0,0)$ as superposition of WKB solutions to the time-dependent Schrödinger equation. The microlocal argument is based on the resolvent estimate:

Theorem 2 (Resolvent estimate)**.** *Assume (A1)-(A3). For any $C > 0$ and $z \in D(E_0, Ch) = \{z \in \mathbb{C}; |z - E_0| < Ch\}$, there exists $K > 0$ such that*

$$||(P_\theta - z)^{-1}|| \le h^{-K}\prod_{z_\beta\in D(E_0,2Ch)}(z - z_\beta)^{-1}.$$

As applications, we obtain the following Theorem 3 and Theorem 4:

Let Λ_ω, $\Lambda_{\omega'}$ be the set of Hamiltonian curves with asymptotic direction $\omega, \omega' \in S^{n-1}$ as $t \to +\infty$, $t \to -\infty$ respectively. We assume

(A5) Λ_+ and Λ_ω [resp. Λ_- and $\Lambda_{\omega'}$] intersect transversally along a Hamiltonian curve γ_+ [resp. γ_-].

Let $x_\pm(t)$ be x-projections of $\gamma_\pm$. Then $x_\pm(t)^\alpha = \sum_{\lambda\cdot\beta<N} g_\beta^\pm(t)e^{-\lambda\cdot\beta t} + \mathcal{O}(e^{-Nt})$ as $t \to \mp\infty$ for every $N \in \mathbb{N}$. $g_\beta^\pm(t)$ are at most polynomials and $g_\alpha^\pm$ are constants.

Theorem 3 (Residu of the scattering amplitude)**.** *Assume (A1)-(A5). Let $\mathcal{A}(z;\omega,\omega')$ be the scattering amplitude with outgoing and incoming directions ω, ω' respectively. Then its residu at $z = z_\alpha$ is given by*

$$Res_{z_\alpha}\mathcal{A}(z;\omega,\omega') = a_\alpha(\omega,\omega')g_\alpha^- g_\alpha^+ h^{\frac{3n}{2}+1-|\alpha|} + \mathcal{O}(h^{\frac{3n}{2}+2-|\alpha|}),$$

where $a_\alpha(\omega,\omega')$ is a non-zero constant independent of h.

Theorem 4 (Long time asymptotic of the Schrödinger group)**.** *Assume (A1)-(A3) and that $\{\lambda_j\}_{j=1}^n$ is $\mathbb{Z}$-independent (i.e. every resonance is simple). Then, for any $C > 0$, $\chi \in C_0^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty(\mathbb{R})$ supported near E_0, there exists $\epsilon > 0$ and $K > 0$ such that*

$$\chi e^{-itP/h}\chi\psi(P) = \sum_{z_\beta\in D(E_0,Ch)} e^{-itz_\beta/h}\,\chi\Pi_\beta\chi\,\psi(P) + \mathcal{O}(h^\infty) + \mathcal{O}(e^{-(C+\epsilon)t}h^{-K}).$$

This is a consequence of Theorem 2. Thanks to Theorem 1, the error is smaller than the first term in RHS when t is larger than $A|\log h|$ for large enough A.