On the Navier–Stokes Equations with Coriolis Force Term

Shin'ya Matsui
(Hokkaido Information University)

I will talk on the Navier–Stokes equations in $D \subset \mathbb{R}^3$:

$$\text{(NS)} \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \Omega e_3 \times u + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in } D \times (0,T), \\ u|_{\partial D} = 0, \quad u|_{t=0} = u_0, \end{cases}$$

Here $u = u(x,t) = (u_1, u_2, u_3)$ is an unknown velocity field and $p = p(x,t)$ is an unknown pressure. In the equation (NS) e_3 is the vertical unit vector $(= (0,0,1))$ and the term $\Omega e_3 \times u$ is called the Coriolis force term with the Coriolis parameter $\Omega \in \mathbb{R}$.

The Coriolis force term is obtained in the following way. Consider the standard Navier–Stokes equation in $\mathbb{R}^3$:

$$v_t - \Delta v + (v \cdot \nabla)v + \nabla q = 0, \quad \nabla \cdot v = 0$$

with initial velocity

$$v|_{t=0}(y) = v_0(y) + (\Omega/2)e_3 \times y = v_0(y) + (\Omega/2)Jy \quad \text{with } J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that $\mathrm{rot} v|_{t=0} = \mathrm{rot} v_0 + \Omega e_3$. In this equation we introduce the transform

$$\begin{cases} u(x,t) = \exp\{-\Omega J t/2\}\, v(y,t) - (\Omega/2)Jx \\ p(x,t) = \exp\{-\Omega J t/2\}\, q(y,t) + \Omega^2 |y'|^2/8. \end{cases}$$

Here $y = (y', y_3) = \exp\{\Omega J t/2\}\, x$. Then $(u, \nabla p)$ satisfies (NS).

From a meteorological point of view, if Coriolis parameter Ω is sufficiently large, flow will be independent of vertical direction x_3 asymptotically, that is 3 dim. flow will close to 2 dim. flow as $|\Omega| \to \infty$. This phenomena is called the Taylor–Proudman theorem. Our main purpose is to study this singular perturbation problem.

In my talk I will discuss the following results.

For a fixed Ω, we obtained several unique existence theorems of the Cauchy problem and boundary value problem ($\mathbb{R}^3_+$) in some function spaces which include periodic functions, almost periodic functions and some L^∞ functions.

In the case of the stationary problem in the Poincaré domain we noted some asymptotic behavior of velocity field in $L^2(D)$ as $\Omega \to \infty$.

My talk is based on joint works with Prof. Y. Giga, Prof. K. Inui, Prof. A. Mahalov and Prof. J. Saal:

1. Y. Giga, K. Inui, A. Mahalov and S. Matsui; Uniform local solvability for the Navier-Stokes equations with the Coriolis force, Methods and Applications of Analysis, Vol. 12 (2005) 381–384
2. Y. Giga, K. Inui, A. Mahalov and S. Matsui; Navier-Stokes Equations in a Rotating Frame in $\mathbb{R}^3$ with Initial Data Nondecreasing at Infinity, Hokkaido Math. J., Vol. 35 (2006) 321–364.
3. Y. Giga, K. Inui, A. Mahalov, S. Matsui, and J. Saal; Rotating Navier-Stokes Equations in $\mathbb{R}^3_+$ with Initial Data Nondecreasing at Infinity: The Ekman Boundary Layer Problem, to appear in Archive for Rational Mechanics and Analysis.