Propagation of the homogeneous wavefront set for Schrödinger equations

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• Some wavefront sets

Definition (S. Nakamura [2]) Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(z_0, \zeta_0) \in \mathbb{R}^{2n} \setminus \{0\}$. Then

$(z_0, \zeta_0) \notin \mathrm{HWF}(u)$

$\overset{\text{def}}{\Longleftrightarrow} \exists \varphi \in C_0^\infty(\mathbb{R}^{2n})$ s.t. $\varphi(z_0,\zeta_0) \neq 0$, and $\|a^w(hz, hD_z)u\|_{L^2} \le C_N h^N$ $(\forall N > 0)$.

Remark If we delete h in front of z in the above definition, this gets to be a characterization of the wavefront set.

Definition (J. Wunsch [3]) For an operator

$$A = a^w(x, D) \in \mathrm{Op}\, S\left(1, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right)$$

we can consider well-defined characteristic set also for the spatial direction:

$$\begin{aligned}\Sigma_{\mathrm{sc}}(A) = &\left\{(z,\zeta) \in S^{n-1} \times \mathbb{R}^n; \liminf_{t\to+\infty} |a(tz,\zeta)| = 0\right\} \\ &\bigsqcup \left\{(z,\zeta) \in \mathbb{R}^n \times S^{n-1}; \liminf_{t\to+\infty} |a(z,t\zeta)| = 0\right\}.\end{aligned} \qquad \text{(disjoint union)}$$

Then we define *the scattering* and *the quadratic scattering wavefront set* by

$$\begin{aligned}\mathrm{WF}_{\mathrm{sc}}(u) &= \bigcap \left\{\Sigma_{\mathrm{sc}}(A); A \in \mathrm{Op}\, S\left(1, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right),\ Au \in \mathcal{S}(\mathbb{R}^n)\right\}, \\ \mathrm{WF}_{\mathrm{qsc}}(u) &= \mathrm{WF}_{\mathrm{sc}}(\tilde{u}), \quad \tilde{u}(q) = u((1+\langle q\rangle)^{-\frac{1}{2}} q).\end{aligned}$$

Remark Since $\mathcal{F} a^w(z, D_z) \mathcal{F}^{-1} = a^w(-D_\zeta, \zeta)$, we have a correspondence

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \mathrm{WF}(u), \quad \mathrm{WF}_{\mathrm{sc}}(u) \cap (S^{n-1} \times \mathbb{R}^n) \longleftrightarrow \mathrm{WF}(\mathcal{F}u).$$

Put $z = (1+\langle q\rangle)^{-\frac{1}{2}} q$, then $\langle z\rangle^{-2} = \langle q\rangle^{-1}$. If we embed $\mathbb{R}^n$ into S^n_+ through the stereographic projection:

$$\mathrm{SP}: \mathbb{R}^n \to S^n_+ = \left\{w \in \mathbb{R}^{n+1} \,|\, |w| = 1, w_n \ge 0\right\},\ z \mapsto \left(\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}}\right),$$

then the qsc-wf set is the sc-wf set of u on $\mathbb{R}^n \subset S^n_+$ that has a new C^∞ structure whose boundary defining function is the square of the original one. The qsc-wf set is nothing but the coordinate-changed sc-wf set, so there is a correspondence

$$\mathrm{WF}_{\mathrm{qsc}}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \mathrm{WF}(u).$$

For $\mathrm{WF}_{\mathrm{qsc}}(u) \cap (S^{n-1} \times \mathbb{R}^n)$ we have another kind of correspondence.

Theorem 1 Let $\Psi : \mathbb{R}^n \setminus \{0\} \to \mathrm{GL}(n; \mathbb{R})$, $z \mapsto \Psi(z) = \left(\delta_{ij} + z^i z^j |z|^{-2}\right)_{ij}$, then

$$\begin{aligned}&\left\{(tz, t\zeta) \in \mathbb{R}^{2n} \middle| (z,\zeta) \in \mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right), t > 0\right\} \\ &= \left\{(z, \Psi(z)\zeta) \in \mathbb{R}^{2n} \middle| (z,\zeta) \in \mathrm{HWF}(u) \setminus (\{0\} \times \mathbb{R}^n)\right\}.\end{aligned}$$

• Propagation of singularities

Let $(\mathbb{R}^n, g)$ be a Riemannian manifold and consider the Schrödinger equation:

$$(i\partial_t + \triangle_g - V(z))u_t(z) = 0, \quad \triangle_g = \frac{1}{2}\partial_i g^{ij}(z)\partial_j.$$

Suppose that g is of the form

$$g = \frac{dx^2}{x^4} + \frac{h(x,y,dx,dy)}{x^2}, \quad x = |z|^{-1}, \quad y\text{: local coordinates of } S^{n-1}$$

for $z \in \mathbb{R}^n$ far from the origin, where h is a 2-cotensor on $\mathbb{R}^n$ that approaches some Riemannian metric on S^{n-1} as $x \to 0$. In other words, g is asymptotically conic.

V is assumed to be in $C^\infty(\mathbb{R}^n;\mathbb{R})$ and satisfies for some $\nu < 2$

$$|\partial_z^\alpha u(z)| \le C_\alpha \langle z\rangle^{\nu-|\alpha|} \quad \forall \alpha \in \mathbb{Z}_+^n.$$

Under these assumptions the potential term can be completely ignored as a small perturbation to the Laplacian.

Theorem 2 Let $u_0 \in L^2$, $\omega_- \in S^{n-1}$, and $t_0 > 0$, and assume that $(-t_0\omega_-, \omega_-) \notin \mathrm{HWF}(u_0)$. Then, if $\gamma(t) = (z(t), \zeta(t))$ is a free backward nontrapped classical trajectory with limiting direction ω_-, i.e., if

$$\dot\gamma(t) = \left(\partial_\zeta p(\gamma(t)), -\partial_z p(\gamma(t))\right), \quad p(z,\zeta) = \frac{1}{2}\sum_{i,j=1}^n g^{ij}(z)\zeta_i\zeta_j,$$

$$\lim_{t\to-\infty}|z(t)| = \infty, \quad \omega_- = \lim_{t\to-\infty}\zeta(t)/|\zeta(t)| = -\lim_{t\to-\infty} z(t)/|z(t)|,$$

then we have

$$\mathrm{WF}(u_{t_0}) \cap \{\gamma(t); t \in \mathbb{R}\} = \varnothing$$

Remark For our metric, if a trajectory γ is backward nontrapped, there always exists a limiting direction $\omega_- \in S^{n-1}$.

Professor S. Nakamura has proved Theorem 2 for asymptotically flat g.

A part of results by Wunsch [3] on the Euclidean space with an optimally weak assumption on potential follows from Theorems 1 and 2, since for any $t > 0$ we have the equivalence

$$(-t\omega_-, \omega_-) \in \mathrm{HWF}(u) \iff (-\omega_-, \omega_-/2t) \in \mathrm{WF}_{\mathrm{qsc}}(u).$$

References

[1] K. Ito, Propagation of singularities for Schrödinger equations on the Euclidean space with a scattering metric, Comm. Partial Differential Equations, **31**, 1735-1777, (2006)

[2] S. Nakamura, Propagation of the Homogeneous Wave Front Set for Schrödinger Equations, Duke Math. J. **126(2)**, 349-367, (2005).

[3] J. Wunsch, Propagation of singularities and growth for Schrödinger operators, Duke Math. J. **98**, 137-186 (1999).