Vorticity directions near the blow-up time for the 3D Navier-Stokes flows with infinite energy

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This talk is based on a joint work with Yoshikazu Giga (University of Tokyo). We consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \partial u/\partial t - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \times (0,T). \end{cases} \tag{NS}$$

There is a large number of work on regularity criterion or non blow up criterion for the 3D Navier-Stokes flow started by Serrin [Se]. Most of them are analytic in the sense that boundedness for some (scale-invariant) quantity is assumed. In 1993, Constantin and Fefferman [CF] gave a geometric condition for the vorticity direction $\zeta = \omega/|\omega|$, where $\omega = \operatorname{curl} u$. It says that if the vorticity direction is Lipschitz continuous in space uniformly in time in the region where vorticity is large, then the Leray-Hopf weak solution is regular. Since then there are several improvement, e.g., [BB, CKL, GZ]. However, all these work discuss a solution having a finite energy. In this talk, we consider a mild solution (i.e., a solution of integral equations) of (NS) in $\mathbb{R}^3 \times (0,T)$, where a solution is just bounded in space and may not have a finite energy. We shall give a condition for the vorticity direction so that the solution can be extended smoothly across $t = T$. Roughly speaking our result reads: the solution does not blow up at $t = T$ if the blow up is type I and the vorticity direction is just uniformly continuous in space variables. For the assumption on the vorticity direction, our result improves existing one. However, we are forced to assume the growth of L^∞-norm of solution is bounded by a self-similar rate which is called type I blow up rate, that is,

$$||u||_{L^\infty(\mathbb{R}^3)}(t) \leq C_0(T-t)^{1/2} \qquad \text{for } \ t \in (0,T), \tag{I}$$

where the time $t = T$ is considered as a possible blow up time. We now state our main result.

Theorem *Let u be a mild solution of (NS) in $\mathbb{R}^3 \times (0,T)$ satisfying (I). For a given $d > 0$ let η be a modulus of continuity such that*

$$|\zeta(x,t) - \zeta(y,t)| \leq \eta(|x-y|) \text{ for all } x, y \in \Omega_d(t),\ t \in (0,T) \tag{CA}$$

where $\Omega_d(t) = \{x \in \mathbb{R}^3 |\ |\omega(x,t)| > d\}$. Then u does not blow up at $t = T$.

Remark In [CF] η is taken $\eta(\sigma) = A\sigma$ in (CA) where A is a positive constant. In [BB] η is taken $\eta(\sigma) = A\sigma^{1/2}$. However, the authors did not assume that the singularity is type I.

Our theorem is proved by a blow up analysis. Assume the contrary so that the solution blows up at $t = T$. We are able to rescale the solution near the blow up time by $M_k := \|u\|_\infty(t_k) \uparrow +\infty$ for some $\{t_k\}_{k=1}^\infty$ with $t_k \uparrow T$. Taking a blow up limit of the sequence of the rescaled solutions we obtain a bounded backward global solution of (NS) $\overline{u}$ defined in $\mathbb{R}^3 \times (-\infty, 0)$. It follows from the assumption (I) that the vorticity of the bounded backward global solution $\overline{\omega} = \operatorname{rot} \overline{u}$ is not identically zero. On the other hand, we also show the following Liouville type theorem, which contradicts with the above conclusion.

Proposition *Assume that u is a mild solution of (NS) in $\mathbb{R}^3 \times (0,T)$ which is bounded in $\mathbb{R}^3 \times (0, T-\delta)$ for all $\delta > 0$. If ζ satisfies (CA), then $\overline{\omega} \equiv 0$.*

The key observation of the proof is that the assumption (CA) implies the blow up limit $\overline{u}$ is a 2D Navier-Stokes flow. Since the vorticity of the 2D Navier-Stokes flow satisfies the strong maximum principle, we are able to show that the vorticity must be zero.

All previous results [CF, BB, GZ, CKL] are proved by integral estimates where some special structure of the nonlinear term plays a crucial role, while our result is by the blow up argument and the Liouville type theorem. The blow up argument is not only simple but also clarifies the nature of our problem. We note that similar blow up argument is effectively applied to the analysis of various kind of equations, e.g., [G, H, PQS].

We also remark that the question whether the type I blow up solutions for (NS) exist or not is still open. Necas, Ruzicka and Sverak [NRS] proved that non-trivial backward self-similar solutions do not exist. Giga, Inui and Matsui [GIM] showed that the solution does not blow up if C_0 is sufficiently small. Recently, Koch, Nadirashvili, Seregin and Sverak [KNSS] proved that type I axisymmetric flows must be regular by using a Liouville type theorem for (NS); see also [CSYT] for a different proof. Our result gives another sufficient condition for nonexistence of the type I blow up solutions.

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