A mathematical analysis of tsunami generation in shallow water due to seabed deformation

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In numerical computations of tsunamis due to submarine earthquakes, it is frequently assumed that the initial displacement of the water surface is equal to the permanent shift of the seabed and that the initial velocity field is equal to zero, and the shallow water equations are often used to simulate the propagation of tsunamis. In this talk we give a mathematically rigorous justification of this tsunami model starting from the full water wave problem by comparing the solution of the full problem and that of the tsunami model. We also show that in some case we have to impose a nonzero initial velocity field, which arises as a nonlinear effect.

We will consider a water wave in $(n+1)$-dimensional space and assume that the water surface $\Gamma(t)$ and the seabed $\Sigma(t)$ at time t are of the forms

$$\Gamma(t) = \bigl\{ X = (x, x_{n+1}) \in \mathbf{R}^{n+1} \, ; \, x_{n+1} = h + \eta(x,t) \bigr\},$$
$$\Sigma(t) = \bigl\{ X = (x, x_{n+1}) \in \mathbf{R}^{n+1} \, ; \, x_{n+1} = b(x,t) \bigr\},$$

where h is the mean depth of the fluid. The functions b and η represent the bottom topography and the surface elevation, respectively. In this problem b is a given function, while η is the unknown. We assume that the seabed deforms only for the time interval $[0, t_0]$, so that the function b can be written in the form $b(x,t) = \beta(x, t/t_0)$, where

$$\beta(x,\tau) = \begin{cases} b_0(x) & \text{for} \quad \tau \le 0, \\ b_1(x) & \text{for} \quad \tau \ge 1. \end{cases}$$

We assume that the water is incompressible and inviscid fluid, and that the flow is irrotational. Then, the fluid motion is described by the velocity potential $\Phi = \Phi(X,t)$. We introduce a new unknown function ϕ by $\phi(x,t) := \Phi(x, h+\eta(x,t), t)$, which is the trace of the velocity potential on the water surface. Rewriting the equations in a nondimensional form, we see that η and ϕ satisfy

$$(1) \qquad \begin{cases} \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\bigl(\Lambda^{\mathrm{DN}}\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi\bigr)^2 = 0, \\ \eta_t - \Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau = 0 \qquad \text{for} \quad t > 0, \end{cases}$$

where δ and ε are nondimensional parameters, $b(x,t) = \beta(x, t/\varepsilon)$, $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ and $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ are linear operators depending on η, b, and δ. More precisely, the operators are defined in the following way.

Definition 1 Under appropriate assumptions on η and b, for any function φ on the water surface in some class there exists a unique solution Φ of the boundary value problem

$$\begin{cases} \triangle\Phi + \delta^{-2}\partial_{n+1}^2\Phi = 0 & \text{in} \quad b(x) < x_{n+1} < 1 + \eta(x), \\ \Phi = \varphi & \text{on} \quad x_{n+1} = 1 + \eta(x), \\ \delta^{-2}\partial_{n+1}\Phi - \nabla b\cdot\nabla\Phi = \gamma & \text{on} \quad x_{n+1} = b(x). \end{cases}$$

Using the solution Φ we define linear operators $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ and $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ by

$$\Lambda^{\mathrm{DN}}(\eta, b, \delta)\varphi + \Lambda^{\mathrm{NN}}(\eta, b, \delta)\gamma = (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta\cdot\nabla\Phi)\big|_{x_{n+1}=1+\eta(x)}.$$

We will consider the initial value problem for (1) with the initial conditions

$$\eta = \eta_0, \quad \phi = \phi_0 \qquad \text{at} \quad t = 0. \tag{2}$$

In this talk, the two parameters

$$\delta = \frac{\text{mean depth}}{\text{wave length}}, \qquad \varepsilon = \frac{\text{time of seabed deformation}}{\text{time period of tsunami}}$$

play an important role. We will consider the limit $\delta, \varepsilon \to 0$ under the restriction $\delta^2/\varepsilon \to \sigma$ with some nonnegative constant σ. It is known that the well-posedness of the initial value problem for water waves may be broken unless a generalized Rayleight–Taylor sign condition $-\partial p/\partial N \geq c_0 > 0$ on $\Gamma(t)$ is satisfied, where p is the pressure and N is the unit outward normal. To ensure this sign condition we impose the following conditions on the seabed.

Assumption 1 There exist constants $C, c > 0$ such that for any $(x, \tau) \in \mathbf{R}^n \times (0,1)$ the following conditions are satisfied.

(i) In the case $\delta/\varepsilon \to 0$: No conditions.
(ii) In the case $\delta/\varepsilon \to \nu$: $1 + \nu^2 \beta_{\tau\tau}(x,\tau) \geq c$.
(iii) In the case $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to 0$: $\beta_{\tau\tau}(x,\tau) \geq 0$.
(iv) In the case $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to \sigma$: $\beta_{\tau\tau}(x,\tau) \geq 0$ and $1 + \sigma\big(a^{(0)} + \sigma C \beta_{\tau\tau}\big)(x,\tau) \geq c$.

From a technical point of view, we also impose the following conditions.

Assumption 2 For any $(x,\tau) \in \mathbf{R}^n \times (0,1)$ the following conditions are satisfied.

(i) In the case $\delta/\varepsilon \to \nu$: No conditions.
(ii) In the case $\delta/\varepsilon \to \infty$: $\beta_{\tau\tau\tau}(x,\tau) \leq 0$.

Our main result in this talk is the following.

Theorem 1 *Let $c_0 > 0$, $\sigma \geq 0$, and $s > (n+7)/2$. In addition to Assumptions* 1 *and* 2*, we assume that*

$$\begin{cases} \eta_0 \in H^{s+5}, \quad \nabla\phi_0 \in H^{s+4}, \quad \beta \in C(\mathbf{R}; H^{s+5+1/2}) \cap B^3((0,1); H^{s+6}), \\ 1 + \eta_0(x) - b_0(x) \geq c_0 > 0 \quad \textit{for} \quad x \in \mathbf{R}^n. \end{cases}$$

Then, there exist a time $T > 0$ and constants $C_0, \delta_0, \varepsilon_0, \gamma_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \varepsilon_0]$ satisfying $|\delta^2/\varepsilon - \sigma| \leq \gamma_0$ the initial value problem (1) *and* (2) *has a unique solution $(\eta, \phi) = (\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ on the time interval $[0,T]$ satisfying*

$$\sup_{\varepsilon \leq t \leq T} \big(\|\eta^{\delta,\varepsilon}(t) - \eta^0(t)\|_s + \|\nabla\phi^{\delta,\varepsilon}(t) - u^0(t)\|_s\big) \leq C_0\big(\varepsilon + |\delta^2/\varepsilon - \sigma|\big),$$

where (η^0, u^0) is a unique solution of the shallow water equations

$$\begin{cases} \eta^0_t + \nabla \cdot \big((1 + \eta^0 - b_1)u^0\big) = 0, \\ u^0_t + (u^0 \cdot \nabla)u^0 + \nabla\eta^0 = 0 \qquad \textit{for} \quad 0 < t < T \end{cases}$$

with the initial conditions

$$\begin{cases} \eta^0 = \eta_0 + (b_1 - b_0), \\ u^0 = \nabla\Big(\phi_0 + \dfrac{\sigma}{2}\displaystyle\int_0^1 \beta_\tau(\cdot,\tau)^2 d\tau\Big) \qquad \textit{at} \quad t = 0. \end{cases}$$