Value distribution of the sixth Painlevé transcendents in sectorial domains

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In order to find new special functions, in the end of the ninteenth century, P. Painlevé studied on the second order algebraic differential equations which are free from movable singularities except for poles. And, he and G. Gambier, one of his colleagues, obtained six *Painlevé equations*. The sixth Painlevé equation

$$\begin{aligned} y'' &= \frac{1}{2}\left\{\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right\}(y')^2-\left\{\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right\}y' \\ &\quad +\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\frac{\beta x}{y^2}+\frac{\gamma(x-1)}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right] \end{aligned} \tag{1}$$

is one of them. Here $' = \mathrm{d}/\mathrm{d}x$ and $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. The solutions to the sixth Painlevé equation are called the sixth Painlevé transcendents.

In this talk, we study the value distribution of the sixth Painlevé transcendents. Kießling uniformized (1) by use of an automorphic function in [1], that yields new difficulties; in order to study the value distribution of solutions to the uniformized sixth Painlevé equation, we must study the value distribution of the automorphic functions. So, in this talk, we avoid the uniformization by any automorphic functions, and analyse (1) directly in sectorial domains.

Fix $\varphi_0 \in \mathbb{R}$ and $\varphi > 0$. We take a sectorial domain

$$S(\varphi_0,\varphi;r) := \{x|\ |\arg x-\varphi_0|<\varphi,\ 2<|x|<r\}.$$

And, $n(f,\varphi_0,\varphi;r)$ denotes the number of zeros in the sectorial domain, i.e.

$$n(f,\varphi_0,\varphi;r) := \sharp\{x \in S(\varphi_0,\varphi;r)|\ f(x)=0\}.$$

Similarly to [2], we can prove

Theorem. *There exists a positive constant C, independent of $(\alpha,\beta,\gamma,\delta)\in\mathbb{C}^4$, such that for any solution $y=y(x)$ to (1), $n(y,\varphi_0,\varphi;r)=O(\mathrm{e}^{Cr})$ as $r\to\infty$.*

References

[1] Kießling, H., *Zur Werteverteilung der Lösungen algebraisher Differentialgleichungen,* Thesis, Berlin, 1996.

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