Blowup Rate of Type II and the Braid Group Theory

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We are concerned with blowup rate of solutions to a Cauchy problem for a semilinear heat equation

$$(1) \qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \mathbf{R}^N. \end{cases}$$

It was shown in [1] that if p is subcritical in the Sobolev sense, then any solution of (1) blowing up at $t = T$ fulfills

$$(2) \qquad |u(t)|_\infty \leq C(T-t)^{-\frac{1}{p-1}} \quad \text{for } t \in [0,T)$$

with some constant $C > 0$. The blowup satisfying (2) is called of type I and of type II otherwise. Let p_{JL} and p_L be the exponent of Joseph-Lundgren and of Lepin, respectively, i.e.,

$$p_{JL} = \begin{cases} \infty & \text{if } N \leq 10, \\ 1 + \dfrac{4}{N-4-2\sqrt{N-1}} & \text{if } N \geq 11; \end{cases}$$

$$p_L = \begin{cases} \infty & \text{if } N \leq 10, \\ 1 + \dfrac{6}{N-10} \quad (> p_{JL}) & \text{if } N \geq 11. \end{cases}$$

If $p > p_{JL}$, then there exist radial solutions of (1) which undergo type II blowup by [2] and [3].

Put

$$w(r,s) = (T-t)^{\frac{1}{p-1}} u(|x|,t), \quad r = (T-t)^{-1/2}|x|, \quad s = -\log(T-t)$$

for a radial solution u of (1). Then w satisfies

$$(3) \qquad \begin{cases} w_s = w_{rr} + \dfrac{N-1}{r} w_r - \dfrac{r}{2} w_r - \dfrac{1}{p-1} w + w^p & \text{in } (0,\infty) \times (s^T, \infty), \\ w(r, s^T) = T^{1/(p-1)} u_0(T^{1/2} r) & \text{in } [0,\infty), \end{cases}$$

where $s^T = -\log T$. Let φ_∞ be the radial singular steady state of (1) defined by

$$\varphi_\infty(\xi) = \left\{\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right\}^{\frac{1}{p-1}} \xi^{-\frac{2}{p-1}} \quad \text{for } \xi = |x|.$$

It is immediate that φ_∞ is also a singular steady state of (3). If $p > p_{JL}$, then the spectrum of the linearized operator of (3) at φ_∞ consists of countable eigenvalues $\{\lambda_j : j = 0, 1, 2, \cdots\}$.

For a function f on $[0,\infty)$ with $f \not\equiv 0$, let $z(f)$ be the supremum over all k such that there exist $0 \le r_1 < r_2 < \cdots < r_{k+1} < \infty$ with $f(r_i)\cdot f(r_{i+1}) < 0$ for $i = 1, 2, \cdots, k$. The following was obtained in [4].

Theorem 1 *Let $p > p_L$. Let u be a radial solution of (1) with $z(u_0 - \varphi_\infty) < \infty$ which undergoes type II blowup at $t = T$. Denote by $\xi_i(t)$ the ith zero of $u(t) - \varphi_\infty$ for positive integer i. Let $\ell = \max\{i : \liminf_{t\nearrow T} (T-t)^{-1/2}\xi_i(t) < \infty\}$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1(T-t)^{-\gamma_\ell - \frac{1}{p-1}} \le |u(t)|_\infty \le C_2(T-t)^{-\gamma_\ell - \frac{1}{p-1}} \quad \textit{for } t \in [0,T),$$

where

$$\gamma_j = \frac{2\lambda_j}{(p-1)|\alpha| - 2} \quad \textit{for } j = 0, 1, 2, \cdots$$

and

$$\alpha = \frac{1}{2}\left\{-(N-2) + \sqrt{\left(N-2-\frac{4}{p-1}\right)^2 - 4(p-1)c_\infty^{p-1}}\right\}$$

Our purpose is to weaken the assumption on p.

Theorem 2 *Let $p > p_{JL}$ and $\lambda_j \neq 0$ for any j. Let u be a radial solution of (1) with $z(u_0 - \varphi_\infty) < \infty$ which undergoes type II blowup at $t = T$. Denote by $\xi_i(t)$ the ith zero of $u(t) - \varphi_\infty$ for positive integer i. Let $m = \max\{i : \liminf_{t\nearrow T} \xi_i(t) = 0\}$. Then there exists a constant $\overline{C} > 0$ such that*

$$|u(t)|_\infty \le \overline{C}(T-t)^{-\gamma_m - \frac{1}{p-1}} \quad \textit{for } t \in [0,T).$$

Theorem 3 *Suppose the same conditions as in Theorem 2. If m is even, then there exists $\underline{C} > 0$ such that*

$$|u(t)|_\infty \ge \underline{C}(T-t)^{-\gamma_m - \frac{1}{p-1}} \quad \textit{for } t \in [0,T), \tag{4}$$

When m is odd, we need a technical assumption to prove (4). The above results are also valid for the Cauchy-Dirichlet problem.

References

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