Global Existence for Systems of Nonlinear Wave Equations in Exterior Domains

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This talk is based on a joint work with Hideo Kubo (Osaka University).

Let $\mathcal{O}(\subset \{x \in \mathbb{R}^3; |x| \le 1\})$ be either a non-trapping obstacle, or a trapping obstacle which was treated by Ikawa ('82, '88), with smooth boundary.

We set $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$, and consider the Dirichlet problem

$$(\partial_t^2 - c_i^2 \Delta_x) u_i = F_i(u, \partial u), \qquad (t,x) \in (0,\infty) \times \Omega, \tag{1}$$

$$u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, \tag{2}$$

$$u(0,x) = \phi(x),\ (\partial_t u)(0,x) = \psi(x), \qquad x \in \Omega, \tag{3}$$

for $i = 1, \dots, N$, where $c_i > 0$, $u = (u_1, \dots, u_N)$ and $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla_x)$. In the following, we always suppose that ϕ and ψ are small in some suitable norm, and (ϕ, ψ, F) satisfies the compatibility condition to infinite order.

First we consider the Cauchy problem with $\Omega = \mathbb{R}^3$. We say that **the null condition** associated with $(c_1, \dots, c_N)$ is satisfied if F_i can be written as

$$\begin{aligned} F_i(u, \partial u) = & \sum_{j,k;\, c_j = c_k = c_i} A_{ijk} Q_0(u_j, u_k; c_i) + \sum_{j,k;\, c_j=c_k=c_i} \sum_{a,b} B_{ijk}^{ab} Q_{ab}(u_j, u_k) \\ & + \sum_{j,k;\, c_j = c_k \ne c_i} \sum_{a,b} C_{ijk}^{ab} (\partial_a u_j)(\partial_b u_k) + \sum_{j,k;\, c_j \ne c_k} \sum_{a,b} D_{ijk}^{ab} (\partial_a u_j)(\partial_b u_k) \\ & + O(|u|^3 + |\partial u|^3) \quad (i = 1, \dots, N), \end{aligned}$$

where the **null forms** are defined by

$$Q_0(v, w\,; c) = (\partial_t v)(\partial_t w) - c^2 (\nabla_x v) \cdot (\nabla_x w), \tag{4}$$

$$Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w) \quad (0 \le a < b \le 3). \tag{5}$$

This condition was first introduced by Klainerman ('86) for the single speed case ($c_1 = \cdots = c_N = 1$) and global existence of small solutions under the null condition is proved (see also Christodoulou '86). Klainerman used the vector fields method with

$$S = t\partial_t + x \cdot \nabla_x,\ L_j = t\partial_j + x_j \partial_t,\ \Omega_{jk} = x_j \partial_k - x_k \partial_j.$$

This global existence result under the null condition is extended to the multiple speeds case by many authors. L_j's are excluded from the arguments in these works.

Now we consider the Dirichlet problem in the exterior domain $\Omega(\subsetneq \mathbb{R}^3)$.

Metcalfe–Nakamura–Sogge ('05) proved

Theorem 1. *Let s be a sufficiently large integer. Suppose that F satisfies the null condition associated with $(c_1, c_2, \dots, c_N)$. If $\|\phi\|_{H^{s+2,s}(\Omega)} + \|\psi\|_{H^{s+1,s}(\Omega)} \ll 1$, then the mixed problem (1)–(3) admits a unique solution $u \in C^\infty([0,\infty) \times \overline{\Omega}; \mathbb{R}^N)$.*

They used S and Ω_{jk} as in the Cauchy problem, but because of the boundary, the usage of S makes the argument complicated.

The aim of this talk is to give an alternative approach where S is not used.

We write $\langle a\rangle = \sqrt{1+|a|^2}$. We define

$$|\varphi(t,x)|_k = \sum_{|\alpha|\le k} |Z^\alpha \varphi(t,x)|, \tag{6}$$

where $Z = (\Omega_{12}, \Omega_{23}, \Omega_{31}, \partial_t, \partial_1, \partial_2, \partial_3)$. Setting $c_0 = 0$, we also define

$$\Phi_\rho(t,x) = \langle t-|x|\rangle^\rho \text{ for } \rho>0,\ \Phi_0(t,x) = \left\{\log\left(2+\frac{\langle t+|x|\rangle}{\langle t-|x|\rangle}\right)\right\}^{-1}, \tag{7}$$

$$W_{\nu,\kappa}(t,x) = \langle t+|x|\rangle^\nu \Big(\min_{0\le j\le N} \langle c_j t - |x|\rangle\Big)^\kappa, \tag{8}$$

$$W^{(c)}_{\nu,\kappa}(t,x) = \langle t+|x|\rangle^\nu \Big(\min_{0\le j\le N; c_j\ne c} \langle c_j t - |x|\rangle\Big)^\kappa, \tag{9}$$

$$N_k[f;\mathcal{W}](t) = \sup_{(s,x)\in[0,t]\times\Omega} \langle x\rangle\, \mathcal{W}(s,x)\, |f(s,x)|_k \text{ for a weight function } \mathcal{W}, \tag{10}$$

$$\mathcal{A}_{\rho,k}[v_0,v_1] = \sup_{y\in\Omega} \langle y\rangle^\rho \big(|v_0(y)|_k + |\nabla_x v_0(y)|_k + |v_1(y)|_k\big). \tag{11}$$

Theorem 2. *Let v be the solution to the Dirichlet problem $(\partial_t^2 - c^2\Delta_x)v = f$ in $(0,\infty)\times\Omega$, $v(t,x)|_{x\in\partial\Omega} = 0$ for $t\in(0,\infty)$, $(v,\partial_t v) = \vec{v}_0 (\equiv (v_0,v_1))$ at $t=0$.*
Set $r=|x|$, and let $\mu>0$.

(i) *If $(\nu=\rho\ge 1,\ \kappa>1)$ or $(\rho\ge1,\ \nu=\rho+\mu,\ \kappa=1-\mu)$, then*

$$\langle t+r\rangle \Phi_{\rho-1}(ct,x)|v(t,x)|_k \le C\mathcal{A}_{\rho+1,k+4}[\vec{v}_0] + C\sum_{|\beta|\le 4} N_k[\partial^\beta f; W_{\nu,\kappa}](t).$$

(ii) *If $(\nu=\rho>1,\ \kappa>1)$, or $(0<\rho\le1,\ \nu=1+\mu,\ \kappa=\rho-\mu)$, then we have*

$$\langle r\rangle\, \langle ct-r\rangle^\rho\, |\partial v(t,x)|_k \le C\mathcal{A}_{\rho+2,k+5}[\vec{v}_0] + CN_{k+5}[f;W_{\nu,\kappa}](t).$$

(iii) *Let $\rho>0$ and $\kappa>1$. Then we have*

$$\langle r\rangle\, \langle ct-r\rangle^\rho\, |\partial v(t,x)|_k \le C\mathcal{A}_{\rho+2,k+5}[\vec{v}_0] + CN_{k+5}[f;W^{(c)}_{\rho,\kappa}](t).$$

(iv) *Let $\rho\le 2$. If $(\nu=\rho\ge1,\ \kappa>1)$ or $(\rho\ge1,\ \nu=\rho+\mu,\ \kappa=1-\mu)$, then*

$$\frac{\langle r\rangle\,\langle t+r\rangle\,\langle ct-r\rangle^{\rho-1}}{\log(2+t+r)} \sum_{|\alpha|\le k} |D_{+,c} Z^\alpha v(t,x)| \le C\mathcal{A}_{\rho+1,k+6}[\vec{v}_0] + CN_{k+6}[f;W_{\nu,\kappa}](t),$$

where $D_{+,c} = \partial_t + c\partial_r$. The above estimates are true for $(t,x)\in[0,\infty)\times\overline{\Omega}$.

Let $\chi=\chi(x)$ be a cut–off function supported on $|x|\le 5$. (i)–(iii) are obtained by combining the corresponding estimates for the Cauchy problem (Asakura '86, Kubota–Yokoyama '01, K.–Yokoyama '05) with decay of local energy

$$\langle t\rangle^\rho \|v(t,\cdot)\|_{H^k(\Omega;|x|<5)} \le C\big(\|\chi\vec{v}_0\|_{H^{k+1}(\Omega)\times H^k(\Omega)} + C\sup_{s\in[0,t]} \langle s\rangle^\rho \sum_{|\alpha|\le k} \|\chi\partial^\alpha f(s)\|_{L^2(\Omega)}\big),$$

via the cut–off method by Shibata ('83) and Shibata–Tsutsumi ('86). For $r>1$, we can obtain (iv) by integrating

$$(\partial_t - c\partial_r)D_{+,c}(rZ^\alpha v) = rZ^\alpha f + \frac{c^2}{r}\sum_{1\le j<k\le3} \Omega_{jk}^2 Z^\alpha v$$

along the certain ray in Ω, using (i) to estimate $\Omega_{jk}^2 Z^\alpha v$. (iv) enables us to treat the null forms without using S.