On the long-time behavior of viscosity solutions to Hamilton-Jacobi equations*

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This talk is concerned with the long-time behavior of the viscosity solution to the Cauchy problem

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

We are particularly interested in the asymptotic behavior of the form

$$u(x,t) - (\phi(x) - at) \longrightarrow 0 \qquad \text{in } C(\mathbb{R}^n) \qquad \text{as } t \to \infty \tag{2}$$

for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$. The function $\phi(x) - at$, called the asymptotic solution of (1), enjoys the time-independent Hamilton-Jacobi equation

$$H(x, D\phi) = a \qquad \text{in } \mathbb{R}^n. \tag{3}$$

Thus, (2) claims that the solution $u(x,t)$ converges to a "steady" state as the time tends to infinity.

The standing assumptions on H are the following:

(A1) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$, where $B(0,R) := \{x \in \mathbb{R}^n \mid |x| \le R\}$,
(A2) $\inf\{H(x,p) \mid x \in \mathbb{R}^n,\ |p| \ge R\} \longrightarrow +\infty$ as $R \to +\infty$,
(A3) $H(x,p)$ is strictly convex with respect to p for every $x \in \mathbb{R}^n$,
(A4) there exist $a \in \mathbb{R}$, $\phi_0 \in \mathcal{S}^-_{H-a}$ and $\psi_0 \in \mathcal{S}^+_{H-a}$ such that $\phi_0 \le \psi_0$ in $\mathbb{R}^n$,

where $\mathcal{S}^-_{H-a}$ (resp. $\mathcal{S}^+_{H-a}$ and $\mathcal{S}_{H-a}$) stands for the set of continuous viscosity subsolutions (resp. supersolutions and solutions) of (3). As a class of initial functions, we set

$$\Phi_0 := \{u_0 \in \mathrm{UC}(\mathbb{R}^n) \mid \phi_0 - C \le u_0 \le \psi_0 + C \text{ in } \mathbb{R}^n \text{ for some } C > 0\}.$$

The study on asymptotic problems of this kind has been developed in the last decade. As a typical case in the development, it has been proved that if H satisfies (A1)-(A3) and

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$$H(x,p) \text{ is } \mathbb{Z}^n\text{-periodic with respect to } x \text{ for every } p \in \mathbb{R}^n,$$

then there exists a unique $a \in \mathbb{R}$ such that (A4) holds for some $\mathbb{Z}^n$-periodic $\phi_0 \in \mathcal{S}^-_{H-a}$ and $\psi_0 \in \mathcal{S}^+_{H-a}$ and the convergence (2) is valid for every $\mathbb{Z}^n$-periodic $u_0 \in \Phi_0$.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A4) are insufficient to obtain the convergence (2) for every $u_0 \in \Phi_0$. The aim of this talk is, therefore, to present some sufficient conditions on H and u_0 so that (2) holds in general situations.

Our approach is based on the following classical variational formula:

$$u(x,t) = \inf\Big\{ \int_{-t}^{0} L(\eta(s), \dot\eta(s))\, ds + u_0(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,0];x) \Big\}, \tag{4}$$

where $L(x,\xi) := \sup_{x\in\mathbb{R}^n}(p\cdot\xi - H(x,p))$ and

$$\mathcal{C}([-t,0];x) := \{\eta \in \mathrm{AC}([-t,0],\mathbb{R}^n) \,|\, \eta(0) = x\}.$$

The rough idea of showing (2) is to construct, for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$, a curve $\mu_t \in \mathcal{C}([-t,0];x)$ which attains the minimum of the right-hand side of (4) and to investigate the asymptotic behavior of μ_t as $t \to \infty$. The motion of $s \mapsto \mu_t(s)$ depends obviously on the value of $\int_{-t}^{0} L(\mu_t(s), \dot\mu_t(s))\, ds$ (running cost) and $u_0(\mu_t(-t))$ (initial cost). In this talk, we show that if μ_t has a "good" behavior when t is large, then one has the required convergence (2). We also give some typical examples of such motions.

References

[1] Ichihara, N., Ishii, H. Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians. To appear in Comm. Partial Differential Equations.

[2] Ichihara, N., Ishii, H. Notes on the dynamical approach to asymptotic solutions of Hamilton-Jacobi equations. In preparation.