RANDOM DATA CAUCHY THEORY FOR SUPERCRITICAL WAVE EQUATIONS

NICOLAS BURQ AND NIKOLAY TZVETKOV

In the study of nonlinear evolutionary PDE's, one often encounters the presence of a critical threshold for the well-posedness theory. A typical situation is to have a method showing well-posedness in Sobolev spaces H^s where s is greater than a critical index s_{cr}. This index is often related to a scale invariance (leading to solutions concentrating at a point of the space-time) of the considered equation. In some cases (but not all), a good local well-posedness theory is valid all the way down to the scaling regularity. On the other hand, at least in the context of nonlinear dispersive equations, no reasonable local well-posedness theory is known for any supercritical equation, i.e. for data having less regularity than the scaling one. In fact, recently, several methods to show ill-posedness, or high frequency instabilities, for $s < s_{cr}$ emerged (see the works by Burq, Gérard and Tzvetkov [4, 3], Lebeau [6] and Christ Colliander and Tao [5]). The goal of this paper is to give a class of equations for which, using probabilistic arguments, one can still obtain a suitable well-posedness theory below the critical threshold. Our model will be the cubic nonlinear wave equation posed on a compact manifold.

Let (M, g) be a three dimensional compact smooth riemannian manifold (without boundary) and let $\mathbf{\Delta}$ be the Laplace-Beltrami operator associated to the smooth metric g. For $s \in \mathbb{R}$, we denote by $H^s(M)$ the classical Sobolev space equipped with the norm $\|u\|_{H^s(M)} = \|(1-\mathbf{\Delta})^{s/2}u\|_{L^2(M)}$. Consider the following cubic wave equation

$$(\partial_t^2 - \mathbf{\Delta})u + u^3 = 0, \quad (u, \partial_t u)|_{t=0} = (f_1, f_2) \tag{0.1}$$

with *real valued* initial data $(f_1, f_2) \equiv f \in H^s(M) \times H^{s-1}(M) \equiv \mathcal{H}^s(M)$.

Using Strichartz estimates for the free evolution (see Section 2) one can show that for $s > 1/2$ the Cauchy problem (0.1) is locally well-posed for data in $\mathcal{H}^s(M)$. This means that for every $f \in \mathcal{H}^s(M)$ there exists $T > 0$ and a unique solution u of (0.1), in a suitable class, such that $(u, u_t) \in C([0, T]; \mathcal{H}^s(M))$, i.e. the solution u represents a continuous curve in $H^s(M)$ (we call such a solution strong solution since the classical construction of weak solutions does not yield the continuity in time). Moreover, we can show that the time existence T may be chosen the same for all f belonging to a fixed bounded set B of $\mathcal{H}^s(M)$ and the map $f \mapsto (u, u_t)$ is continuous (and even Lipschitz continuous) from B to $C([0, T]; \mathcal{H}^s(M))$.

For $s = 1/2$ one can still construct local strong solution for $f \in \mathcal{H}^{1/2}(M)$ but the dependence of T on f is more complicated and the Sobolev space $\mathcal{H}^{1/2}(M)$ is called critical space for (0.1).

1991 *Mathematics Subject Classification.* 35Q55, 35BXX, 37K05, 37L50, 81Q20 .

Key words and phrases. nonlinear wave equation, eigenfunctions, dispersive equations.

For $s < 1/2$, the argument to construct local solutions by Strichartz estimates breaks down. Moreover one may show (see [5], [6] for the case of constant coefficient metrics or the appendix of this paper for the case of non constant coefficient metrics) that if the initial data belong to $\mathcal{H}^s(M)$, $s < 1/2$, the Cauchy problem (0.1) is ill-posed in a strong sense: there exists initial data $(f_1, f_2) \in \mathcal{H}^s(M)$ such that any *reasonable* solution of (0.1), i.e. satisfying the finite speed of propagation ceases *instantaneously* to be in $\mathcal{H}^s$ for positive times (by finite speed of propagation, we mean the fact that the value of the solution at (x_0, t_0) depends only on the values of the initial data on the set of points located at distance smaller that $|t_0|$: $\{x : d_g(x, x_0) \leq |t_0|\}$). However, the functions for which one can prove such a pathological behavior are highly non generic and a natural question is whether despite this result one can still prove that the problem (0.1) possesses local strong solutions for a "large class of functions" in $\mathcal{H}^s(M)$, $s < 1/2$. Our purpose in this paper is precisely to give a positive answer to this question.

Furthermore, in a *very particular case* we can combine the local theory with some invariant measure arguments inspired by Bourgain [1, 2] to obtain *global* solutions. Namely, we consider the nonlinear wave equation posed on the unit ball of $\mathbb{R}^3$ with Dirichlet boundary condition and radial initial data. Replacing the cubic non linearity u^3 by a sub quadratic non linearity, $|u|^{\alpha-1}u$, $\alpha < 4$, we obtain global solutions for data which are essentially in $H^{1/2}(\Theta) \times H^{-1/2}(\Theta)$ but not more regular. Observe that the equation (0.1) is $H^{3/2-2/\alpha}$ critical. As a consequence, for $2 < \alpha < 3$, we obtain global existence of strong solutions for a supercritical model, a result which seems to be completely out of reach of the present deterministic methods, even for the local existence theory.

References

[1] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. 166 (1994) 1-26.

[2] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996) 421-445.

[3] N. Burq, P. Gérard and N. Tzvetkov. *Two singular dynamics of the nonlinear Schrödinger equation on a plane domain*, *Geom. Funct. Anal.*, 13 (2003) 1 1–19.

[4] N. Burq, P. Gérard and N. Tzvetkov. *An instability property of the nonlinear Schrödinger equation on S^d*, *Math. Res. Lett.*, 9:2-3 (2002) 323–335.

[5] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, Preprint 2003.

[6] G. Lebeau, *Perte de régularité pour les equations d'ondes sur-critiques*, Bull. Soc. Math. France 133 (2005) 145-157.

Laboratoire de Mathématiques, Bât. 425, Université Paris Sud, 91405 Orsay Cedex, France et Institut Universitaire de France

E-mail address: nicolas.burq@math.u-psud.fr

Département de Mathématiques, Université Lille I, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: nikolay.tzvetkov@math.univ-lille1.fr