COMPUTATION OF AUTOMORPHISM GROUPS OF $K3$ AND ENRIQUES SURFACES

ICHIRO SHIMADA

Abstract. In virtue of the advances of computer, we can calculate various geometric data of $K3$ surfaces and Enriques surfaces by brute-force method. The lattice theory plays an important role in this method. We give a survey of the lattice theory used in the computer-aided algebraic geometry of $K3$ surfaces and Enriques surfaces. In particular, we explain Borcherds' method for the calculation of automorphism groups of these surfaces. As an example, we compute the automorphism groups of Enriques surfaces covered by a general Jacobian Kummer surface.

1. Introduction

Machine-aided computation has now become a very strong tool in the study of $K3$ surfaces and Enriques surfaces. In this survey, we show how far we can go with computers in the algebraic geometry of these surfaces.

We mainly deal with the automorphism group of a $K3$ surface or an Enriques surface. The automorphism group is calculated from the numerical Néron–Severi lattice and the nef-and-big cone of the surface. Borcherds [2], [3] developed a computational method to determine the shape of the nef-and-big cone by embedding the Néron–Severi lattice into an even unimodular hyperbolic lattice $\mathrm{II}_{1,25}$ of rank 26, which is unique up to isomorphism. In this paper, we write L_{26} for $\mathrm{II}_{1,25}$. The lattice L_{26} has many beautiful combinatorial properties related to the *Leech lattice*, and these properties are used in the study of geometry of $K3$ and Enriques surfaces.

We explain Borcherds' method and its generalization. In particular, we present our recent result (joint work [4] with Simon Brandhorst) on Borcherds' method for Enriques surfaces. As an example, we compute the automorphism groups of complex Enriques surfaces covered by a general Jacobian Kummer surface.

2. Lattices

First we fix notation and terminologies about lattices. A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

Let $e_1, \dots, e_n$ be a basis of a lattice L of rank n. The *Gram matrix* of L with respect to $e_1, \dots, e_n$ is the $n \times n$ matrix whose (i, j)-component is $\langle e_i, e_j\rangle$. The automorphism group of a lattice L is denoted by $\mathrm{O}(L)$. The action of $\mathrm{O}(L)$ on L is from the *right*, and we write the action as $v \mapsto v^g$ for $v \in L$ and $g \in \mathrm{O}(L)$. A lattice L is *unimodular* if the determinant of the Gram matrix is ± 1. A lattice L is *even* (or *of type* II) if $\langle x, x\rangle \in 2\mathbb{Z}$ holds for all $x \in L$. A lattice L of rank n is *hyperbolic*

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(resp. *positive-definite*, resp. *negative-definite*) if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(1, n-1)$ (resp. $(n, 0)$, resp. $(0, n)$).

Let L be an even hyperbolic lattice. A *positive cone* of L is one of the two connected components of

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \,\}.$$

Let $\mathcal{P}$ be a positive cone of L. We put

$$\mathrm{O}(L, \mathcal{P}) := \{\, g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P} \,\}.$$

Then we have $\mathrm{O}(L) = \mathrm{O}(L, \mathcal{P}) \times \{\pm 1\}$. For a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$, we put

$$(v)^{\perp} := \{\, x \in \mathcal{P} \mid \langle v, x \rangle = 0 \,\},$$

which is a real hyperplane of $\mathcal{P}$. Let D be a closed subset of $\mathcal{P}$ defined by countably many inequalities of the form

$$\langle x, v_i \rangle \geq 0 \quad (v_i \in L \otimes \mathbb{Q}).$$

Suppose that D contains a non-empty open subset of $\mathcal{P}$. A closed subset w of D is a *wall* of D if there exists a hyperplane $(v)^{\perp}$ of $\mathcal{P}$ such that w is written as $D \cap (v)^{\perp}$, that $(v)^{\perp}$ is disjoint from the interior of D, and that $D \cap (v)^{\perp}$ contains a non-empty open subset of $(v)^{\perp}$. Let w be a wall of D. We say that a vector $v \in L \otimes \mathbb{Q}$ *defines* the wall w if $w = D \cap (v)^{\perp}$ and $\langle v, x \rangle \geq 0$ holds for all points x of D.

A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r \rangle = -2$. A (-2)-vector $r \in L$ defines the *reflection* $s_r \in \mathrm{O}(L, \mathcal{P})$ into the mirror $(r)^{\perp}$, which is given by

$$s_r \colon x \mapsto x + \langle x, r \rangle\, r.$$

Let $W(L)$ denote the subgroup of $\mathrm{O}(L, \mathcal{P})$ generated by all reflections s_r with respect to (-2)-vectors r. Note that $W(L)$ is a normal subgroup in $\mathrm{O}(L, \mathcal{P})$. A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp},$$

where r runs through the set of all (-2)-vectors. Then $W(L)$ acts on the set of standard fundamental domains simple-transitively. Let N be a standard fundamental domain. We put

$$\mathrm{O}(L, N) := \{\, g \in \mathrm{O}(L) \mid N^g = N \,\}.$$

Then $W(L)$ is generated by the reflections s_r with respect to the (-2)-vectors r defining walls of N, and we have $\mathrm{O}(L, \mathcal{P}) = W(L) \rtimes \mathrm{O}(L, N)$. Therefore, for the study of $\mathrm{O}(L)$, it is important to calculate the walls of a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}$.

The following is well-known. See, for example, [31, Chapter V].

Theorem 2.1. *For a positive integer n with $n \equiv 2 \bmod 8$, there exists an even unimodular hyperbolic lattice L_n of rank n. (A more standard notation is $\mathrm{II}_{1,n-1}$.) For each n, the lattice L_n is unique up to isomorphism.*

We denote by U (instead of L_2) the *hyperbolic plane*, a Gram matrix of which is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When $n = 8m + 2$, the lattice L_n is obtained as the orthogonal direct-sum of U and m copies of the negative-definite root lattice of type E_8.

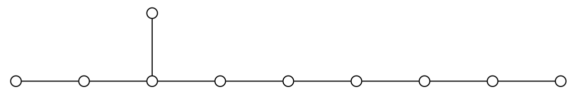

FIGURE 2.1. Coxeter graph of $W(L_{10})$

Example 2.2. Vinberg [41] proved the following. Let $\mathcal{P}_{10}$ be a positive cone of L_{10}. A standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$ has exactly 10 walls defined by (-2)-vectors that form the dual graph given in Figure 2.1. Hence $W(L_{10})$ is the Coxeter group whose Coxeter graph is Figure 2.1. Since this graph has no non-trivial symmetries, we have $\mathrm{O}(L_{10}, \mathcal{P}_{10}) = W(L_{10})$.

3. Geometry of $K3$ surfaces

For simplicity, we work over the complex number field $\mathbb{C}$.

For a non-singular projective surface Z, we denote by S_Z the lattice of numerical equivalence classes of divisors of Z. For a divisor Γ of Z, let $[\Gamma] \in S_Z$ denote the class of Γ. Note that S_Z is hyperbolic by Hodge index theorem. If Z is a $K3$ surface, then S_Z is even. If Z is an Enriques surface, then S_Z is isomorphic to L_{10}. Let $\mathcal{P}_Z$ be the positive cone of S_Z containing an ample class of Z. We put

$$N_Z := \{\, x \in \mathcal{P}_Z \mid \langle x, [C] \rangle \geq 0 \ \text{ for all curves } C \text{ on } Z \,\},$$

and call it the *nef-and-big cone* of Z.

Suppose that X is a complex $K3$ surface. The following is well-known.

Theorem 3.1. *The nef-and-big cone N_X of X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$. The mapping $C \mapsto [C]$ gives rise to a bijection from the set of smooth rational curves C on X to the set of (-2)-vectors $[C]$ defining the walls of N_X.*

The following is a corollary of Torelli theorem for complex algebraic $K3$ surfaces [29].

Theorem 3.2. *The natural homomorphism* $\mathrm{Aut}(X) \to \mathrm{O}(S_X, N_X)$ *is an isomorphism up to finite kernel and finite cokernel.*

Remark 3.3. The kernel and cokernel can be calculated by looking at the period $H^{2,0}(X)$ of X and the action of $\mathrm{Aut}(X)$ on the discriminant form of S_X, which is canonically anti-isomorphic to the discriminant form of the transcendental lattice of X [23].

Let $a \in S_X$ be an ample class. Note that a is an interior point of N_X. Since $\langle a, a \rangle > 0$, the orthogonal complement of $\mathbb{Z}\, a$ in S_X is negative-definite. Therefore, for integers c and d, we can calculate the finite set

$$\{\, v \in S_X \mid \langle a, v \rangle = c, \ \ \langle v, v \rangle = d \,\}.$$

Then we have the following algorithms.

- A vector $v \in \mathcal{P}_X \cap S_X$ is nef (that is, $v \in N_X$) if and only if the finite set

  $$\{\, r \in S_X \mid \langle r, r \rangle = -2, \langle r, a \rangle > 0, \langle r, v \rangle < 0 \,\}$$

  is empty. See [32] for the algorithm to calculate this set.

- Let $r \in S_X$ be a (-2)-vector such that

$$d := \langle r, a \rangle > 0,$$

so that r is the class of an effective divisor Γ. Then Γ is irreducible if and only if $\langle r, [C'] \rangle \geq 0$ for all smooth rational curves C' with $\langle [C'], a \rangle < d$. Hence we can determine whether r is the class of a smooth rational curve or not by induction on d.

Example 3.4. The Fermat quartic surface

$$X_{\mathrm{FQ},p} \quad : \quad x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

in characteristic p is a $K3$ surface if $p \neq 2$. The complex Fermat quartic surface $X_{\mathrm{FQ},0}$ is a singular $K3$ surface whose transcendental lattice is

$$\left[\begin{array}{cc} 8 & 0 \\ 0 & 8 \end{array}\right],$$

whereas the Fermat quartic surface $X_{\mathrm{FQ},3}$ in characteristic 3 is a supersingular $K3$ surface with Artin invariant 1. Let h_4 be the class of a hyperplane section of $X_{\mathrm{FQ},p}$. We denote by R_d the set of smooth rational curves C on $X_{\mathrm{FQ},p}$ such that $\langle h_4, [C] \rangle = d$. Then the sizes $|R_d|$ of the sets R_d are given as follows:

d	1	2	3	4	5	6	7
$p=0$	48	320	1152	15456	136896	743808	3851136
$p=3$	112	0	0	18144	0	0	2177280

.

See [37] and [28].

Example 3.5. Suppose that $h_2 \in S_X$ is a nef vector with $\langle h_2, h_2 \rangle = 2$, and $\mathcal{L}$ a line bundle whose class is h_2. By [25], the complete linear system $|\mathcal{L}|$ is fixed-component free if and only if the finite set

$$\{\, e \in S_X \mid \langle e, e \rangle = 0, \langle e, h_2 \rangle = 1 \,\}$$

is empty. Suppose that $|\mathcal{L}|$ is fixed-component free. Then $|\mathcal{L}|$ is base-point free by [30], and hence $|\mathcal{L}|$ defines a double covering $\Phi\colon X \to \mathbb{P}^2$. The set of classes of smooth rational curves contracted by Φ is the fundamental root system in

$$\{\, r \in S_X \mid \langle r, r \rangle = -2, \langle r, h_2 \rangle = 0 \,\}$$

with respect to the ample class a, and hence we can compute it explicitly. (See, for example, [11].) The matrix representation of the action of the deck transformation $\iota(h_2) \in \mathrm{Aut}(X)$ of $\Phi\colon X \to \mathbb{P}^2$ on S_X is then calculated from this set of classes of contracted curves. See [32] or [34]. Calculating vectors h_2 with $\langle h_2, h_2 \rangle = 2$ and $\langle h_2, a \rangle = d$ for small d, we can obtain many involutions $\iota(h_2) \in \mathrm{Aut}(X)$.

Remark 3.6. In [19], a generating set of the automorphism group of $X_{\mathrm{FQ},3}$ is obtained by the method above. In [34], using randomly generated involutions, we carried out an experiment on the characteristic polynomials of automorphisms of supersingular $K3$ surfaces.

no.	R	$N/\langle R\rangle$
1	$24\,A_1$	$(\mathbb{Z}/2\mathbb{Z})^{12}$
2	$A_{11}+D_7+E_6$	$\mathbb{Z}/12\mathbb{Z}$
3	$2\,A_{12}$	$\mathbb{Z}/13\mathbb{Z}$
4	$A_{15}+D_9$	$\mathbb{Z}/8\mathbb{Z}$
5	$A_{17}+E_7$	$\mathbb{Z}/6\mathbb{Z}$
6	$12\,A_2$	$(\mathbb{Z}/3\mathbb{Z})^6$
7	A_{24}	$\mathbb{Z}/5\mathbb{Z}$
8	$8\,A_3$	$(\mathbb{Z}/4\mathbb{Z})^4$
9	$6\,A_4$	$(\mathbb{Z}/5\mathbb{Z})^3$
10	$4\,A_5+D_4$	$\mathbb{Z}/2\mathbb{Z}\times(\mathbb{Z}/6\mathbb{Z})^2$
11	$4\,A_6$	$(\mathbb{Z}/7\mathbb{Z})^2$
12	$2\,A_7+2\,D_5$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$
13	$3\,A_8$	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$
14	$2\,A_9+D_6$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}$
15	$D_{10}+2\,E_7$	$(\mathbb{Z}/2\mathbb{Z})^2$
16	$2\,D_{12}$	$(\mathbb{Z}/2\mathbb{Z})^2$
17	$D_{16}+E_8$	$\mathbb{Z}/2\mathbb{Z}$
18	D_{24}	$\mathbb{Z}/2\mathbb{Z}$
19	$6\,D_4$	$(\mathbb{Z}/2\mathbb{Z})^6$
20	$4\,D_6$	$(\mathbb{Z}/2\mathbb{Z})^4$
21	$3\,D_8$	$(\mathbb{Z}/2\mathbb{Z})^3$
22	$4\,E_6$	$(\mathbb{Z}/3\mathbb{Z})^2$
23	$3\,E_8$	0
24	none	$\mathbb{Z}^{24}$

TABLE 4.1. Niemeier lattices

4. Conway theory

A positive-definite even unimodular lattice of rank 24 is called a *Niemeier lattice*. Niemeier showed that there exist exactly 24 isomorphism classes of Niemeier lattices, one of which is the famous *Leech lattice* Λ. See [8, Chapter 16] or [40]. The isomorphism classes of Niemeier lattices are described in Table 4.1, where the second column is the ADE-type of the set R of vectors $r\in N$ with $\langle r,r\rangle=2$, and the third column shows the group $N/\langle R\rangle$, where $\langle R\rangle$ is the sublattice of N generated by R. The Leech lattice Λ (no. 24) is characterized as the unique Niemeier lattice that does not contain any vectors of square-norm 2.

Recall that L_{26} is an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. Note that the lattice L_{26} is written as

$$U\oplus N^-,$$

where N is a Niemeier lattice and N^- is the negative-definite lattice obtained from N by multiplying the intersection form by -1. We fix a positive cone $\mathcal{P}_{26}$ of L_{26}, and let $\overline{\mathcal{P}}_{26}$ denote the closure of $\mathcal{P}_{26}$ in $L_{26}\otimes\mathbb{R}$. We put $\partial\,\overline{\mathcal{P}}_{26}:=\overline{\mathcal{P}}_{26}\setminus\mathcal{P}_{26}$.

Definition 4.1. A vector $\mathbf{w}\in L_{26}$ is called a *Weyl vector* if $\mathbf{w}$ is a non-zero primitive vector of L_{26} contained in $\partial\,\overline{\mathcal{P}}_{26}$ (in particular, we have $\langle\mathbf{w},\mathbf{w}\rangle=0$ and hence $\mathbb{Z}\mathbf{w}\subset(\mathbb{Z}\mathbf{w})^\perp$) such that $(\mathbb{Z}\mathbf{w})^\perp/\mathbb{Z}\mathbf{w}$ is isomorphic to the negative-definite Leech lattice Λ^-. A (-2)-vector $r\in L_{26}$ is called a *Leech root* with respect to a Weyl vector $\mathbf{w}$ if $\langle\mathbf{w},r\rangle=1$.

Let $\langle\ ,\ \rangle_\Lambda$ denote the intersection form of the (positive-definite) Leech lattice Λ. Note that every Weyl vector $\mathbf{w}$ is written as $(1,0,\mathbf{0})$ in an orthogonal direct-sum decomposition

$$L_{26}=U\oplus\Lambda^-,$$

and, under this decomposition, Leech roots with respect to $\mathbf{w}$ are written as

$$r_\lambda:=\left(\frac{\langle\lambda,\lambda\rangle_\Lambda}{2}-1,1,\lambda\right),\quad\text{where }\lambda\in\Lambda.$$

For a Weyl vector $\mathbf{w}$, we put

$$N_{26}(\mathbf{w}) := \{\, x \in \mathcal{P}_{26} \mid \langle x, r\rangle \ge 0 \ \text{ for all Leech roots } r \text{ with resepct to } \mathbf{w} \,\}.$$

Definition 4.2. A standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$ is called a *Conway chamber.*

Conway [5] proved the following.

Theorem 4.3. *The mapping $\mathbf{w} \mapsto N_{26}(\mathbf{w})$ gives a bijection from the set of Weyl vectors to the set of Conway chambers.*

We fix a Conway chamber N_{26}. Let $\mathbf{w}_0$ be the corresponding Weyl vector, and let $L_{26} = U \oplus \Lambda^-$ be an orthogonal direct-sum decomposition such that $\mathbf{w}_0 = (1, 0, \mathbf{0})$.

Corollary 4.4. *The group $\mathrm{O}(L_{26}, N_{26}) = \{\, g \in \mathrm{O}(L_{26}) \,|\, N_{26}^g = N_{26} \,\}$ is the group Co_∞ of* affine *isometries of Λ, that is, the group generated by $\mathrm{Co}_0 = \mathrm{O}(\Lambda)$ and the affine translations of Λ.*

Let $\overline{N}_{26}$ be the closure of N_{26} in $L_{26}\otimes\mathbb{R}$. We investigate the rays in $\overline{N}_{26}\cap\partial\,\overline{\mathcal{P}}_{26}$. Suppose that

$$v := (a, b, x) \in L_{26} \otimes \mathbb{R}$$

be a non-zero vector in $\overline{N}_{26}\cap\partial\,\overline{\mathcal{P}}_{26}$, where $(a, b) \in U\otimes\mathbb{R}$ and $x \in \Lambda\otimes\mathbb{R}$. Then we have

$$\langle v, v\rangle = 2ab - \langle x, x\rangle_\Lambda = 0,$$

and, for every $\lambda \in \Lambda$, we have

$$\langle v, r_\lambda\rangle = a + b\left(\frac{\langle \lambda, \lambda\rangle_\Lambda}{2} - 1\right) - \langle x, \lambda\rangle_\Lambda \ge 0. \tag{4.1}$$

Considering the limit of $\langle v, r_\lambda\rangle$ when $\langle \lambda, \lambda\rangle_\Lambda \to \infty$, we see that $b \ge 0$. If $b = 0$, then we have $x = 0$ and $a > 0$. Hence the ray $\mathbb{R}_{\ge 0}v$ is equal to $\mathbb{R}_{\ge 0}\mathbf{w}_0$. Suppose that $b > 0$. We can assume that $b = 1$ without changing $\mathbb{R}_{\ge 0}v$. Then we have $a = \langle x, x\rangle_\Lambda/2$ and hence

$$\langle v, r_\lambda\rangle = \frac{1}{2}\langle x - \lambda, x - \lambda\rangle_\Lambda - 1. \tag{4.2}$$

Therefore we have $\langle x - \lambda, x - \lambda\rangle_\Lambda \ge 2$ for all $\lambda \in \Lambda$, which means that $x \in \Lambda\otimes\mathbb{R}$ is a *deep hole* of the Leech lattice [6]. In particular, we have $x \in \Lambda\otimes\mathbb{Q}$ and $a \in \mathbb{Q}$, and hence there exists a primitive vector $f \in L_{26}$ with $\langle f, f\rangle = 0$ such that $\mathbb{R}_{\ge 0}v = \mathbb{R}_{\ge 0}f$. Thus we obtain the following:

Proposition 4.5. *The intersection $\overline{N}_{26}\cap\partial\,\overline{\mathcal{P}}_{26}$ consists of countably many rational rays. One of them is $\mathbb{R}_{\ge 0}\mathbf{w}_0$, and the other rays are in one-to-one correspondence with the deep holes of Λ.*

Let $f \in L_{26}$ be as above, and let $x_f \in \Lambda\otimes\mathbb{Q}$ be the corresponding deep hole. Since f is primitive and L_{26} is unimodular, we have a vector $z \in L_{26}$ such that $\langle f, z\rangle = 1$ and $\langle z, z\rangle = -2$. Let $U_{f,z} \subset L_{26}$ be the hyperbolic plane generated by f and z, and let $N_{f,z}^-$ be the orthogonal complement of $U_{f,z}$ in L_{26}, which is obtained from a Niemeier lattice $N_{f,z}$ by changing the sign of the intersection form. By (4.2), the mapping $\lambda \mapsto r_\lambda$ gives a bijection from the set of vectors $\lambda \in \Lambda$ with $\langle x_f - \lambda, x_f - \lambda\rangle_\Lambda = 2$ to the set

$$R_f := \{\, r \in L_{26} \mid \langle r, r\rangle = -2, \langle r, \mathbf{w}_0\rangle = 1, \langle r, f\rangle = 0 \,\}$$

of Leech roots r with $\langle r, f\rangle = 0$. By [6] and [7], we see that R_f form a Dynkin diagram whose ADE-type is the ADE-type of the Niemeier lattice $N_{f,z}$ given in Table 4.1, and we obtain one-to-one correspondences between the following three sets:

- the set of deep holes of Λ modulo the action of Co_∞,
- the set of rays in $\overline{N}_{26} \cap \partial\,\overline{\mathcal{P}}_{26}$ other than $\mathbb{R}_{\ge 0}\mathbf{w}_0$ modulo the action of $\mathrm{O}(L_{26}, N_{26}) = \mathrm{Co}_\infty$, and
- the isomorphism classes of Niemeier lattices other than the Leech lattice.

Remark 4.6. Let $\mathbb{X}_{26}$ be a $K3$ surface such that the Néron–Severi lattice of $\mathbb{X}_{26}$ is isomorphic to L_{26}. Of course, such a $K3$ surface $\mathbb{X}_{26}$ does *not* exist. We introduce $\mathbb{X}_{26}$ only for heuristic purpose: Using this *non-existing* "$K3$ surface" $\mathbb{X}_{26}$, we can state results about the lattice L_{26} as results about geometry of $\mathbb{X}_{26}$. Note that the nef-and-big cone of $\mathbb{X}_{26}$ can be identified with the Conway chamber N_{26}.

"Theorem". The smooth rational curves C_λ on $\mathbb{X}_{26}$ are indexed by vectors $\lambda \in \Lambda$ in such a way that $[C_\lambda] = r_\lambda$, and $\mathrm{Aut}(\mathbb{X}_{26})$ is isomorphic to Co_∞ up to finite kernel and finite cokernel in such a way that the action of $\mathrm{Aut}(\mathbb{X}_{26})$ on the set of smooth rational curves on $\mathbb{X}_{26}$ and the action of Co_∞ on Λ are compatible under the correspondence $C_\lambda \leftrightarrow \lambda$.

Using the theory of elliptic $K3$ surfaces (see, for example, [39]), the classical result of Niemeier can also be regarded as a "theorem" on the elliptic fibrations of the "$K3$ surface" $\mathbb{X}_{26}$.

"Theorem". Modulo the action of Co_∞, there exist exactly 24 elliptic fibrations on $\mathbb{X}_{26}$. Each of them has a zero section. The ADE-type of singular fibers and the Mordell–Weil group of each of these elliptic fibrations are given in Table 4.1. In particular, the Leech lattice is realized as the Mordell–Weil lattice of a Jacobian fibration of $\mathbb{X}_{26}$.

5. Borcherds' method

Let X be a $K3$ surface. Suppose that we have a primitive embedding

$$S_X \hookrightarrow L_{26}.$$

By this embedding, we regard the positive cone $\mathcal{P}_X$ of S_X as a subspace of a positive cone $\mathcal{P}_{26}$ of L_{26}. Recall that the positive cone $\mathcal{P}_{26}$ is tessellated by Conway chambers $N_{26}(\mathbf{w})$.

Definition 5.1. An *induced chamber* is a closed subset D of $\mathcal{P}_X$ that contains a non-empty open subset of $\mathcal{P}_X$ and is obtained as the intersection $\mathcal{P}_X \cap N_{26}(\mathbf{w})$ of $\mathcal{P}_X$ with a Conway chamber $N_{26}(\mathbf{w})$. The tessellation of $\mathcal{P}_{26}$ by Conway chambers induces a tessellation of $\mathcal{P}_X$ by the induced chambers, which we call the *induced tessellation* of $\mathcal{P}_X$.

Since the nef-and-big cone N_X is bounded by hyperplanes $(r)^\perp$ defined by (-2)-vectors $r \in S_X$, and a (-2)-vector r of S_X is a (-2)-vector of L_{26}, the cone N_X is also tessellated by induced chambers.

We assume the following mild assumption:

Assumption 5.2. The orthogonal complement of S_X in L_{26} contains at least one (-2)-vector.

Then any induced chamber $D = \mathcal{P}_X \cap N_{26}(\mathbf{w})$ of $\mathcal{P}_X$ has only finite number of walls, and these walls can be calculated explicitly from the Weyl vector $\mathbf{w}$ of the Conway chamber $N_{26}(\mathbf{w})$ inducing D. See [33] for the detail. A linear programming plays an important role in this algorithm.

Definition 5.3. We say that the induced tessellation of $\mathcal{P}_X$ is *simple* if the induced chambers are congruent to each other by the action of $\mathrm{O}(S_X, \mathcal{P}_X)$.

When the induced tessellation of $\mathcal{P}_X$ is simple, we can calculate the shape of N_X by means of this tessellation. This method was contrived by Borcherds [2], [3], and the automorphism groups $\mathrm{Aut}(X)$ of many $K3$ surfaces X have been calculated by this method.

Remark 5.4. Borcherds' method is regarded as a calculation of $\mathrm{Aut}(X)$ by a generalization of "the $K3$ surface" $\mathbb{X}_{26}$ to X, that is, we regard the embedding $S_X \hookrightarrow L_{26}$ as the embedding induced by a "specialization" of X to $\mathbb{X}_{26}$.

5.1. **Jacobian Kummer surface.** Let

$$X := \mathrm{Km}(\mathrm{Jac}(C))$$

be the Kummer surface associated with the Jacobian variety $\mathrm{Jac}(C)$ of a complex general genus 2 curve

$$C \;:\; y^2 = (x-\lambda_1)\cdots(x-\lambda_6).$$

The $K3$ surface X has three famous projective models.

- The $K3$ surface X is embedded into $\mathbb{P}^3$ as a quartic surface $X_4 \subset \mathbb{P}^3$ with 16 ordinary nodes corresponding to the points of the 2-torsion subgroup of $\mathrm{Jac}(C)$. This quartic surface is called the *Kummer quartic surface*.
- The dual $X_4^\vee \subset (\mathbb{P}^3)^\vee$ of the Kummer quartic surface $X_4 \subset \mathbb{P}^3$ is also a quartic surface with 16 ordinary nodes.
- The $K3$ surface X is embedded into $\mathbb{P}^5$ as a smooth $(2,2,2)$-complete intersection $X_{2,2,2}$ defined by

$$\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = \sum_{i=1}^{6} \lambda_i^2 x_i^2 = 0. \tag{5.1}$$

  The surface $X_{2,2,2}$ contains 32 lines, which are the exceptional curves over the ordinary nodes of X_4 and of $X_4^\vee$.

The Néron–Severi lattice S_X of X is of rank 17. Kondo [18] found a primitive embedding $S_X \hookrightarrow L_{26}$ such that $\mathcal{P}_X$ is *simply* tessellated by induced chambers, and using this embedding, he obtained the following:

Theorem 5.5. *Every induced chamber D has $32 + 60 + 32 + 192$ walls.*

There exists a unique induced chamber D_0 that contains the class h_8 of a hyperplane section of $X_{2,2,2} \subset \mathbb{P}^5$. Then the 32 walls of D_0 are defined by the classes of the 32 lines on $X_{2,2,2}$, and the group

$$\mathrm{Aut}(X, D_0) := \{\, g \in \mathrm{Aut}(X) \mid D_0^g = D_0 \,\}$$

is equal to the projective automorphism group

$$\mathrm{Aut}(X_{2,2,2}) = \{\, g \in \mathrm{Aut}(X) \mid h_8^g = h_8 \,\} \;\cong\; (\mathbb{Z}/2\mathbb{Z})^5$$

of $X_{2,2,2}$. For each of the other $60+32+192$ walls w of D_0, there exists an involution $g_w \in \mathrm{Aut}(X)$ that maps D_0 to the induced chamber adjacent to D_0 across the wall

w. These automorphisms g_w are classically known and described geometrically as follows.

(a) 60 *involutions are obtained as Hutchinson–Göpel involutions. See Hutchinson* [13] *and* [14].
(b) $32 = 16 + 16$ *involutions are obtained as the deck-transformation of the double covering $X \to \mathbb{P}^2$ given by the projection with the center being an ordinary node of X_4 or of $X_4^\vee$.*
(c) 192 *involutions are obtained as Hutchinson–Weber involutions. See Hutchinson* [15].

Corollary 5.6. *The group* $\mathrm{Aut}(X)$ *is generated by* $\mathrm{Aut}(X_{2,2,2}) \cong (\mathbb{Z}/2\mathbb{Z})^5$ *and* $60 + 32 + 192$ *involutions described above.*

Remark 5.7. The fact that the 192 involutions above are Hutchinson–Weber involutions was proved by Ohashi [27].

5.2. **Fifteen nodal quartic surface.** As is expected from Remark 5.4, Borcherds' method is especially suitable for the analysis of the change of automorphism group under generalization/specialization of $K3$ surfaces.

The surface $X = \mathrm{Km}(\mathrm{Jac}(C))$ is obtained as the minimal resolution of a quartic surface $X_4 \subset \mathbb{P}^3$ with 16 ordinary nodes (Kummer quartic surface), and it is classically known (see, for example, [12, Chapter 6]) that a Kummer quartic surface is related to the line congruence of type $(2, 2)$ in $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$. We generalize X to a $K3$ surface X' that is the minimal resolution of a general quartic surface X'_4 with 15 ordinary nodes. This surface was investigated by Dolgachev [9] in the relation to the line congruence of type $(2, 3)$ in $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$. Using this result, we determined $\mathrm{Aut}(X')$ in [10].

Compositing Kondo's embedding $S_X \hookrightarrow L_{26}$ with the primitive embedding $S_{X'} \hookrightarrow S_X$ induced by the specialization of X' to X, we obtain a primitive embedding $S_{X'} \hookrightarrow L_{26}$. It turns out that this embedding also induces a simple tessellation of $\mathcal{P}_{X'}$, and we obtain the following:

Theorem 5.8. *The automorphism group of X' is generated by*

$$6 + 45 + 6 + 15 + 120 + 72$$

automorphisms, each of which is described explicitly and geometrically.

Example 5.9. Let $p_1, \dots, p_5$ be distinct ordinary nodes of the 15-nodal quartic surface X'_4, no four of them are coplanar. Then we obtain a birational involution g of X'_4 defined as follows. Let q be a general point of X'_4. There exists a unique twisted cubic curve Γ in $\mathbb{P}^3$ passing through $p_1, \dots, p_5$ and q. Let q' be the point such that $\Gamma \cap X'_4 = \{p_1, \dots, p_5, q, q'\}$. Then the involution g interchanges q and q'. Choosing suitable 5-tuples $p_1, \dots, p_5$ of ordinary nodes of X'_4, we obtain 72 generators of $\mathrm{Aut}(X')$ in Theorem 5.8.

Remark 5.10. Enumerating the faces of an induced chamber with codimension 2, we also obtained a set of defining relations of $\mathrm{Aut}(X')$ with respect to the generators given in Theorem 5.8.

Remark 5.11. In [35], we calculated the automorphism group $\mathrm{Aut}(\mathcal{X}/R)$ of a certain $K3$ surface $\mathcal{X}$ defined over a complete discrete valuation ring R of mixed characteristics by comparing the automorphism groups of the special fiber and of the generic fiber.

5.3. **Non-simple tessellation.** For every complex $K3$ surface X, we can embed S_X into L_{26} primitively. Usually, however, the induced tessellation of $\mathcal{P}_X$ is not simple. In [33], Borcheds' method is generalized for the case where the induced tessellation of $\mathcal{P}_X$ is not simple.

Let X be a singular $K3$ surface whose transcendental lattice is

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

Then we have 1098 distinct types of induced chambers, and obtain a generating set of $\mathrm{Aut}(X)$ consisting of 764 elements.

Recall that $X_{\mathrm{FQ},0}$ is the complex Fermat quartic surface. We have observed that, for $X_{\mathrm{FQ},0}$, there exist more than 10^5 types of induced chambers, and we have not yet obtained a generating set of $\mathrm{Aut}(X_{\mathrm{FQ},0})$.

6. Borcherds method for Enriques surfaces

We work over $\mathbb{C}$. An involution ε of a $K3$ surface X is called an *Enriques involution* if ε is fixed-point free, or equivalently, if the quotient surface $Y := X/\langle\varepsilon\rangle$ is an Enriques surface. Ohashi [26] showed that the set of Enriques involutions of a $K3$ surface X is a union of finitely many conjugacy classes of $\mathrm{Aut}(X)$.

Example 6.1. The Hutchinson–Göpel involutions and the Hutchinson–Weber involutions on $\mathrm{Km}(\mathrm{Jac}(C))$ are Enriques involutions.

Let $\pi\colon X \to Y$ be the universal covering of an Enriques surface $Y = X/\langle\varepsilon\rangle$. Then the pull-back by π gives a primitive embedding

$$\pi^*\colon S_Y(2) \cong L_{10}(2) \hookrightarrow S_X,$$

where $S_Y(2)$ is the lattice with the same underlying $\mathbb{Z}$-module as S_Y and with the intersection form being that of S_Y multiplied by 2. The image of π^* is equal to the invariant part $\{\, v \in S_X \mid v^\varepsilon = v \,\}$ of the action of the Enriques involution ε on S_X. Since π is étale, the orthogonal complement of the image of π^* does not contain any (-2)-vector. The following is due to Keum [16].

Theorem 6.2. *An involution ε of a $K3$ surface X is an Enriques involution if and only if the fixed sublattice $\{\, v \in S_X \mid v^\varepsilon = v \,\}$ of S_X is isomorphic to $L_{10}(2)$ and its orthogonal complement in S_X contains no (-2)-vectors.*

In a joint work with S. Brandhorst [4], we have classified all primitive embeddings of $L_{10}(2)$ into L_{26}.

Theorem 6.3. *Up to the action of* $\mathrm{O}(L_{10})$ *and* $\mathrm{O}(L_{26})$*, there exist exactly* 17 *primitive embeddings*

`12A, 12B, 20A, ..., 20F, 40A, ..., 40E, 96A, 96B, 96C, infty`

of $L_{10}(2)$ into L_{26}.

Recall that the positive cone $\mathcal{P}_{26}$ of L_{26} is tessellated by Conway chambers. A primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ induces a tessellation of the positive cone $\mathcal{P}_{10}$ of L_{10} that is mapped into $\mathcal{P}_{26}$ by $L_{10}(2) \hookrightarrow L_{26}$. The following theorem is very useful in the calculation of the automorphism group of an Enriques surface.

Theorem 6.4. *Except for the embedding of type* `infty`*, the following hold.*

(i) *The induced tessellation on $\mathcal{P}_{10}$ is simple.*

No.	name	rt	m4	og
1	12A	D_8	1376	$2^{29} \cdot 3^7 \cdot 5^3 \cdot 7^2$
2	12B	A_7	1824	$2^{23} \cdot 3^6 \cdot 5^2 \cdot 7^2$
3	20A	$D_4 + D_5$	1760	$2^{25} \cdot 3^7 \cdot 5^2 \cdot 7$
4	20B	$2D_4$	1888	$2^{29} \cdot 3^4 \cdot 5 \cdot 7$
5	20C	$10A_1 + D_6$	1632	$2^{28} \cdot 3^6 \cdot 5^3 \cdot 7$
6	20D	$A_3 + A_4$	2016	$2^{16} \cdot 3^6 \cdot 5^3 \cdot 7$
7	20E	$5A_1 + A_5$	1952	$2^{20} \cdot 3^7 \cdot 5^3$
8	20F	$2A_3$	2080	$2^{23} \cdot 3^4 \cdot 5^2$
9	40A	$4A_1 + 2A_3$	2016	$2^{25} \cdot 3^5 \cdot 5$
10	40B	$8A_1 + 2D_4$	1760	$2^{30} \cdot 3^6 \cdot 5 \cdot 7$
11	40C	$6A_1 + A_3$	2080	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$
12	40D	$12A_1 + D_4$	1888	$2^{28} \cdot 3^5 \cdot 5^2$
13	40E	$2A_1 + 2A_2$	2144	$2^{16} \cdot 3^6 \cdot 5^2$
14	96A	$8A_1$	2144	$2^{28} \cdot 3^3$
15	96B	$16A_1$	2016	$2^{31} \cdot 3^5$
16	96C	$4A_1$	2208	$2^{22} \cdot 3^5$
17	infty		2272	$2^{26} \cdot 3^2 \cdot 5 \cdot 7$

TABLE 6.1. Primitive embeddings of $L_{10}(2)$ into L_{26}

(ii) *Each wall of every induced chamber is defined by a (-2)-vector r of L_{10}.*
(iii) *For each wall $D \cap (r)^{\perp}$ of an induced chamber D with $\langle r, r\rangle = -2$, the reflection s_r maps D to the induced chamber adjacent to D across the wall $D \cap (r)^{\perp}$.*

Table 6.1 shows the 17 primitive embeddings of $L_{10}(2)$ into L_{26}, and Table 6.2 shows properties of induced chambers.

- The `name` of the primitive embedding indicates the number of walls of an induced chamber. For example, each induced chamber of the embedding `96A` has 96 walls.
- The column `rt` in Table 6.1 shows the ADE-type of the set of (-2)-vectors in the orthogonal complement $\iota^{\perp}$ of the image of the primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$. For the embedding `infty`, the orthogonal complement $\iota^{\perp}$ contains no (-2)-vectors (that is, Assumption 5.2 is not satisfied), and the induced chamber has infinitely many walls.
- The column `m4` in Table 6.1 shows the number of vectors v with $\langle v, v\rangle = -4$ in the orthogonal complement $\iota^{\perp}$, and the column `og` gives the order of the orthogonal group $\mathrm{O}(\iota^{\perp})$ of $\iota^{\perp}$.
- Recall that the standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$ is bounded by 10 walls defined by (-2)-vectors that form the dual graph in Figure 2.1. Since every wall of an induced chamber is defined by a (-2)-vector, each induced chamber is a union of standard fundamental domains.

No.	name	volume	|aut|	isom	NK
1	12A	269824	2^2		I
2	12B	12142080	$2^3 \cdot 3$		II
3	20A	64757760	$2^3 \cdot 3$		V
4	20B	145704960	2^6		III
5	20C	777093120	$2^3 \cdot 3 \cdot 5$	20D	VII
6	20D	777093120	$2^3 \cdot 3 \cdot 5$	20C	VII
7	20E	906608640	$2^3 \cdot 3 \cdot 5$		VI
8	20F	2039869440	$2^6 \cdot 5$		IV
9	40A	8159477760	$2^7 \cdot 3$		
10	40B	18650234880	$2^7 \cdot 3^2$	40C	
11	40C	18650234880	$2^7 \cdot 3^2$	40B	
12	40D	32637911040	$2^5 \cdot 3^2 \cdot 5$	40E	
13	40E	32637911040	$2^5 \cdot 3^2 \cdot 5$	40D	
14	96A	163189555200	$2^{13} \cdot 3$		
15	96B	652758220800	$2^{12} \cdot 3^3$	96C	
16	96C	652758220800	$2^{12} \cdot 3^3$	96B	
17	infty	∞			

TABLE 6.2. Induced chambers of $\mathcal{P}_{10}$

The column `volume` in Table 6.2 shows how many standard fundamental domains are contained in an induced chamber.

- The column $|$`aut`$|$ in Table 6.2 shows the order of the automorphism group
$$\mathrm{O}(L_{10}, D) := \{ g \in \mathrm{O}(L_{10}) \mid D^g = D \}$$
of an induced chamber D in $\mathrm{O}(L_{10})$.
- Distinct embeddings can produce congruent induced chambers. The column `isom` in Table 6.2 shows that, for example, two embeddings `20C` and `20D` yield congruent induced chambers.
- Nikulin [24] and Kondo [17] classified Enriques surfaces Y with finite automorphism group. If $\mathrm{Aut}(Y)$ is finite, then Y contains only finite number of smooth rational curves. By the configuration of these smooth rational curves, Enriques surfaces with finite automorphism group are divided into 7 classes $\mathrm{I}, \mathrm{II}, \dots, \mathrm{VII}$. These 7 configurations appear as the configurations of (-2)-vectors defining walls of an induced chamber of $\mathcal{P}_{10}$. The column `NK` in Table 6.2 shows this correspondence.

The induced chambers are much bigger than the standard fundamental domain Δ of the action of $W(L_{10})$ on $\mathcal{P}_{10}$, and hence we need only small number of copies of chambers to describe the nef-and-big cone N_Y of an Enriques surface Y. For example, let Y be a complex *generic* Enriques surface. We have $N_Y = \mathcal{P}_Y$. By Barth–Peters [1], the fundamental domain $\mathcal{F}$ of the action of $\mathrm{Aut}(Y)$ on $N_Y = \mathcal{P}_Y$

number	ε	name
6	Hutchinson–Weber	`20E`
15	Hutchinson–Göpel	`40A`
10	in $\mathrm{Aut}(X_{2,2,2})$	`40C` .

TABLE 7.1. Conjugacy classes of Enriques involutions

is a union of

$$|\mathrm{O}(L_{10} \otimes \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600$$

copies of Δ. If we use induced chambers of the embedding `96C`, we can express $\mathcal{F}$ as a union of

$$\frac{46998591897600}{652758220800} = 72$$

copies of induced chambers.

7. Enriques surfaces covered by a Jacobian Kummer surface

We illustrate Borcherds' method for Enriques surfaces by applying it to Enriques surfaces covered by the Kummer surface

$$X = \mathrm{Km}(\mathrm{Jac}(C))$$

associated with the Jacobian variety $\mathrm{Jac}(C)$ of a general curve C of genus 2. Recall that $\mathrm{Aut}(X)$ was calculated by Kondo [18], as was explained in Section 5.1. Ohashi [27] gave the complete classification of conjugacy classes of Enriques involutions in $\mathrm{Aut}(X)$, which had been conjectured by Mukai [21].

Theorem 7.1. *There exist exactly* $6+15+10$ *conjugacy classes of Enriques involutions in* $\mathrm{Aut}(X)$. *A representative of each conjugacy class is given in Table 7.1.*

Remark 7.2. A representative of the conjugacy class in the third line of Table 7.1 is given as follows. The projective automorphism group $\mathrm{Aut}(X_{2,2,2}) \cong (\mathbb{Z}/2\mathbb{Z})^5$ of the $(2,2,2)$-complete intersection $X_{2,2,2}$ in $\mathbb{P}^5$ defined by (5.1) consists of the involutions

$$(x_1 : x_2 : \cdots : x_6) \mapsto (\pm x_1 : \pm x_2 : \cdots : \pm x_6).$$

This involution is fixed-point free if and only if there exist exactly three minuses in $(\pm x_1 : \pm x_2 : \cdots : \pm x_6)$, and hence there exist exactly ten Enriques involutions in $\mathrm{Aut}(X_{2,2,2})$.

Kondo [18] used a primitive embedding $\iota_X \colon S_X \hookrightarrow L_{26}$ to calculate $\mathrm{Aut}(X)$. Let ε be an Enriques involution of X with the quotient morphism

$$\pi \colon X \to Y = X/\langle\varepsilon\rangle.$$

Then the composite of $\pi^* \colon S_Y(2) \hookrightarrow S_X$ and $\iota_X \colon S_X \hookrightarrow L_{26}$ gives a primitive embedding

$$\iota_Y \colon S_Y(2) \cong L_{10}(2) \hookrightarrow L_{26}.$$

The type of this primitive embedding is given in Table 7.1. We investigate the automorphism groups of these Enriques surfaces $Y = X/\langle\varepsilon\rangle$. We have a canonical isomorphism

$$\mathrm{Aut}(Y) \cong \mathrm{Cen}(\varepsilon)/\langle\varepsilon\rangle,$$

where $\mathrm{Cen}(\varepsilon)$ is the centralizer of ε in $\mathrm{Aut}(X)$. Therefore, to calculate $\mathrm{Aut}(Y)$, it is enough to calculate $\mathrm{Cen}(\varepsilon)$.

Remark 7.3. Mukai and Ohashi [22] investigated automorphisms of an Enriques surface in the conjugacy class of of type `40A` (an Enriques surface of Hutchinson–Göpel type). See also [20].

Recall that $h_8 \in S_X$ is the class of a hyperplane section of $X_{2,2,2} \subset \mathbb{P}^5$. Let D_X be the induced chamber in $\mathcal{P}_X$ containing h_8. (This induced chamber was denoted by D_0 in Section 5.1.) Let ε and $\pi\colon X \to Y = X/\langle\varepsilon\rangle$ be as above. We identify S_Y (resp. $\mathcal{P}_Y$) with the invariant part of the action of ε on S_X (resp. on $\mathcal{P}_X$). Then the nef-and-big cone N_Y of Y is equal to the intersection $\mathcal{P}_Y \cap N_X$. Suppose that $\varepsilon' = g^{-1}\varepsilon g$ is a conjugate of ε, where $g \in \mathrm{Aut}(X)$, and let $\pi'\colon X \to Y' = X/\langle\varepsilon'\rangle$ be the corresponding covering morphism. Then we have $\mathcal{P}_{Y'} = \mathcal{P}_Y^g$. Recall that N_X is tessellated by the induced chambers D_X^g, where g runs through $\mathrm{Aut}(X)$. Therefore, replacing ε with a conjugate of ε, we can and will assume that

$$D_Y := \mathcal{P}_Y \cap D_X$$

contains a non-empty open subset of $\mathcal{P}_Y$, and hence D_Y is an induced chamber of the primitive embedding $\iota_Y\colon S_Y(2) \hookrightarrow L_{26}$. We put

$$\mathrm{Cen}(\varepsilon, D_Y) := \{\, g \in \mathrm{Cen}(\varepsilon) \mid D_Y^g = D_Y \,\},$$

which is a finite subgroup of $\mathrm{Cen}(\varepsilon)$. We then put

$$\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y := \{\, g|\mathcal{P}_Y \mid g \in \mathrm{Cen}(\varepsilon, D_Y) \,\},$$

where $g|\mathcal{P}_Y$ is the restriction of g to $\mathcal{P}_Y$.

Definition 7.4. Let $w = D_Y \cap (r)^\perp$ be a wall of D_Y, and let r be the (-2)-vector defining w. We say that w is an *outer wall* if the following mutually equivalent conditions are satisfied.

- w is contained in a wall of N_Y,
- the induced chamber of $\mathcal{P}_Y$ adjacent to D_Y across the wall w is not contained in N_Y,
- the (-2)-vector r is the class of a smooth rational curve on Y, and
- there exists a smooth rational curve C of X such that $\pi^*(r) = [C] + [\varepsilon(C)]$.

Otherwise we say that w is an *inner wall*.

Definition 7.5. We say that a wall $D_X \cap (v)^\perp$ of D_X is *perpendicular* to D_Y if the vector $v \in S_X \otimes \mathbb{Q}$ defining $D_X \cap (v)^\perp$ belongs to $S_Y \otimes \mathbb{Q}$.

In the following, a configuration of (-2)-vectors in L_{10} is described by a pair (Γ, μ), where Γ is a set of indexes γ of (-2)-vectors r_γ, and $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ gives the intersection pairing $\mu(\gamma, \gamma') = \langle r_\gamma, r_{\gamma'} \rangle$. The configuration of type τ means the configuration of (-2)-vectors defining the walls of an induced chamber obtained by the primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ of type τ.

7.1. **An Enriques surface in the conjugacy class of type `20E`.** First we describe the configuration of type `20E`. This configuration is isomorphic to the configuration of Nikulin–Kondo type VI (Fig. 6.4 of [17]). The description below of this configuration was obtained in [36]. Let A be the set of subsets a of $\{1, \dots, 5\}$ with

$|a| = 3$. Let A_1 and A_2 be two copies of A with the natural bijection to A denoted by $a \mapsto \bar{a}$. We then put

$$\Gamma := A_1 \sqcup A_2,$$

and define a symmetric function $\mu\colon \Gamma \times \Gamma \to \mathbb{Z}$ with $\mu(a, a) = -2$ for all $a \in \Gamma$ as follows.

- Suppose that $a, a' \in A_1$ with $a \neq a'$. Then
$$\mu(a, a') = \begin{cases} 1 & \text{if } |a \cap a'| = 1, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $a, a' \in A_2$ with $a \neq a'$. Then
$$\mu(a, a') = \begin{cases} 1 & \text{if } |a \cap a'| = 2, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $a \in A_1$ and $a' \in A_2$. Then
$$\mu(a, a') = \begin{cases} 2 & \text{if } \bar{a} = \bar{a}', \\ 0 & \text{otherwise.} \end{cases}$$

Then (Γ, μ) defines the configuration of type `20E`.

Remark 7.6. The sub-configuration $(A_1, \mu|A_1)$ of (Γ, μ) is isomorphic to the famous Petersen graph, and the sub-configuration $(A_2, \mu|A_2)$ is isomorphic to the complement of the Petersen graph. The automorphism group of (Γ, μ) is equal to the automorphism group of the Petersen graph, which is isomorphic to $\mathfrak{S}_5$.

Let ε be a Hutchinson–Weber involution of X. Calculating the ADE-type of the set of (-2)-vectors in the orthogonal complement $\iota_Y^{\perp}$ of the image of ι_Y, we see that ι_Y is of type `20E`. We assume that ε is the involution that maps D_X to the induced chamber of $\mathcal{P}_X$ adjacent to D_X, and let $w := D_X \cap (v_\varepsilon)^\perp$ be the wall of D_X between D_X and D_X^ε, that is, w is the wall in Theorem 5.5 (c). Then $D_Y := \mathcal{P}_Y \cap D_X$ is contained in w, and D_Y contains an interior point of w (as a subset of the hyperplane $(v_\varepsilon)^\perp$). The group $\mathrm{Cen}(\varepsilon, D_Y)$ is contained in $\mathrm{Aut}(X_{2,2,2}) \cup \mathrm{Aut}(X_{2,2,2})\varepsilon$. Looking at all $32 + 32$ elements of this set, we see that $\mathrm{Cen}(\varepsilon, D_Y)$ is equal to $\{1, \varepsilon\}$, and hence $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ is trivial.

In 20 walls of D_Y, 10 are outer and 10 are inner. There exists an indexing $\gamma \mapsto r_\gamma$ of the (-2)-vectors defining the walls $D_Y \cap (r_\gamma)^\perp$ of D_Y by the set $\Gamma = A_1 \sqcup A_2$ above with the following properties.

- Suppose that $\gamma \in A_1$. Then the wall $D_Y \cap (r_\gamma)^\perp$ of D_Y is outer. There exists a unique pair of lines $C, \varepsilon(C)$ of $X_{2,2,2}$ such that $D_Y \cap (r_\gamma)^\perp$ is equal to $\mathcal{P}_Y \cap D_X \cap ([C])^\perp$, and we have $\pi^*(r) = [C] + [\varepsilon(C)]$.
- Suppose that $\gamma \in A_2$. Then the wall $D_Y \cap (r_\gamma)^\perp$ of D_Y is inner. There exists a unique wall $D_X \cap (v_\gamma)^\perp$ of D_X such that the wall $D_Y \cap (r_\gamma)^\perp$ of D_Y is equal to $\mathcal{P}_Y \cap D_X \cap (v_\gamma)^\perp$. The wall $D_X \cap (v_\gamma)^\perp$ is perpendicular to D_Y, and corresponds to a Hutchinson–Göpel involution g_γ of X (Theorem 5.5 (a)). The defining vector v_γ is perpendicular to the defining vector v_ε of the wall w. There exists exactly one element h_γ in $\mathrm{Aut}(X_{2,2,2}) = \mathrm{Aut}(X, D_X)$ such that $h_\gamma g_\gamma$ commutes with ε. Then the restriction $h_\gamma g_\gamma|\mathcal{P}_Y$ of $h_\gamma g_\gamma$ to $\mathcal{P}_Y$ maps D_Y to the induced chamber adjacent to D_Y across the wall $D_Y \cap (r_\gamma)^\perp$. The automorphism $h_\gamma g_\gamma$ is of order 2, and the eigenvalues of $h_\gamma g_\gamma|\mathcal{P}_Y$ are $1^6(-1)^4$.

Therefore the group $\mathrm{Cen}(\varepsilon)$ is generated by ε and 10 involutions $h_\gamma g_\gamma$, where γ runs through A_2.

7.2. **An Enriques surface in the conjugacy class of type** `40A`**.** We describe the configuration of type `40A`. Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two copies of the cube $I^3 \subset \mathbb{R}^3$, where $I = [0,1] \subset \mathbb{R}$ is the unit interval. Let σ be $+$ or $-$. A vertex of $\mathcal{C}_\sigma$ is written as $((a_x, a_y, a_z), \sigma)$, where $a_x, a_y, a_z \in \{0,1\}$, and a face of $\mathcal{C}_\sigma$ is written as $(w = a, \sigma)$, where $w \in \{x,y,z\}$ and $a \in \{0,1\}$. Let V be the set of vertices of $\mathcal{C}_\pm$, and let F be the set of faces of $\mathcal{C}_\pm$. Let P be the set of pairs of a face $f_+ = (w = a_+)$ of $\mathcal{C}_+$ and a face $f_- = (w = a_-)$ of $\mathcal{C}_-$ that are parallel. Each element of P is written as $(w = a_+, w = a_-)$, where $w \in \{x,y,z\}$ and $a_\pm \in \{0,1\}$. We have $|V| = 16$, $|F| = 12$, $|P| = 12$. We put

$$\Gamma := V \sqcup F \sqcup P,$$

and define a symmetric function $\mu \colon \Gamma \times \Gamma \to \mathbb{Z}$ with $\mu(a,a) = -2$ for all $a \in \Gamma$ as follows.

- Suppose that $v_1, v_2 \in V$ with $v_1 \neq v_2$. Then
$$\mu(v_1, v_2) = \begin{cases} 0 & \text{if } v_1 v_2 \text{ is an edge of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\ 4 & \text{if } v_1 v_2 \text{ is a diagonal of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\ 2 & \text{otherwise.} \end{cases}$$
- Suppose that $v \in V$ and $f \in F$. Then
$$\mu(v, f) = \begin{cases} 2 & \text{if } v \in f, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $v \in V$ and $p = (f_+, f_-) \in P$. Then
$$\mu(v, p) = \begin{cases} 2 & \text{if } v \in f_+ \cup f_-, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $f_1, f_2 \in F$ with $f_1 \neq f_2$. Let f_i be $(w_i = a_i, \sigma_i)$, where $w_i \in \{x,y,z\}$, $a_i \in \{0,1\}$, and $\sigma_i \in \{+,-\}$. Then
$$\mu(f_1, f_2) = \begin{cases} 1 & \text{if } \sigma_1 \neq \sigma_2 \text{ and } w_1 \neq w_2, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $f = (w = a, \sigma) \in F$ and $p = (f'_+, f'_-) \in P$. Let $\bar{f}$ be the unique face of $\mathcal{C}_\sigma$ that is disjoint from f. Then
$$\mu(f, p) = \begin{cases} 2 & \text{if } \bar{f} = f'_+ \text{ or } \bar{f} = f'_-, \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $p_1, p_2 \in P$ with $p_1 \neq p_2$. Let $\mathrm{faces}(p_i)$ denote the set of 2 faces contained in p_i, and let $\mathrm{verts}(p_i)$ denote the set of 8 vertices contained in the two faces of p_i.
$$\mu(p_1, p_2) = \begin{cases} 2 & \text{if } \mathrm{verts}(p_1) \cap \mathrm{verts}(p_2) = \emptyset, \\ 0 & \text{if } \mathrm{faces}(p_1) \cap \mathrm{faces}(p_2) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Then (Γ, μ) defines the configuration of type `40A`.

Remark 7.7. The automorphism group $\mathrm{Aut}(\Gamma, \mu)$ is of order 768, and V, F, P are the orbits of the action of $\mathrm{Aut}(\Gamma, \mu)$ on Γ. Let V_+ and V_- be the set of vertices of $\mathcal{C}_+$ and of $\mathcal{C}_-$, respectively. We regard each of $V_\pm$ as a graph with edges being the edges of the cube. The automorphism group $\mathrm{Aut}(V_+)$ of the graph V_+ is of order 48. The stabilizer subgroup $\mathrm{Stab}(V_+)$ of V_+ in $\mathrm{Aut}(\Gamma, \mu)$ is of index 2, the natural homomorphism $\mathrm{Stab}(V_+) \to \mathrm{Aut}(V_+)$ is surjective, and its kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ acting on V_- as $((a_x, a_y, a_z), -) \mapsto ((\pm a_x, \pm a_y, \pm a_z), -)$.

Let ε be a Hutchinson–Göpel involution of X. Calculating the ADE-type of the set of (-2)-vectors in $\iota_Y^\perp$, we see that ι_Y is of type `40A`. As in the previous section, we assume that ε maps D_X to the induced chamber of $\mathcal{P}_X$ adjacent to D_X across the wall $w := D_X \cap (v_\varepsilon)^\perp$, that is, w is a wall in Theorem 5.5 (a). Then $D_Y := \mathcal{P}_Y \cap D_X$ is an induced chamber of $\mathcal{P}_Y$, and D_Y is contained in w. Moreover D_Y contains an interior point of w (as a subspace of the hyperplane $(v_\varepsilon)^\perp$).

The group $\mathrm{Cen}(\varepsilon, D_Y)$, which is a subset of $\mathrm{Aut}(X_{2,2,2}) \cup \mathrm{Aut}(X_{2,2,2})\varepsilon$, is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, and the group $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. The eigenvalues of a non-trivial element of $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ are $1^6(-1)^4$.

Remark 7.8. In [22], it was shown that the action of $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y \cong (\mathbb{Z}/2\mathbb{Z})^3$ characterizes the Enriques surfaces of Hutchinson–Göpel type.

In the 40 walls of D_Y, 20 are outer and 20 are inner. There exists an indexing $\gamma \mapsto r_\gamma$ of the (-2)-vectors defining the walls of D_Y by the set $\Gamma = V \sqcup F \sqcup P$ above with the following properties. Let V_0 (resp. V_1) be the subset of V consisting of $((a_x, a_y, a_z), \pm)$ such that $a_x + a_y + a_z$ is even (resp. odd).

- Suppose that $\gamma \in V_0$. Then the wall $D_Y \cap (r_\gamma)^\perp$ is inner. There exists a unique wall $D_X \cap (v_\gamma)^\perp$ of D_X such that $D_Y \cap (r_\gamma)^\perp = \mathcal{P}_Y \cap D_X \cap (v_\gamma)^\perp$. We have $\langle v_\varepsilon, v_\gamma \rangle = 0$. This wall $D_X \cap (v_\gamma)^\perp$ is perpendicular to D_Y, and corresponds to an involution $g_\gamma \in \mathrm{Aut}(X)$ obtained by the projection from an ordinary node of X_4 or of $X_4^\vee$ (Theorem 5.5 (b)). The involution g_γ commutes with ε, and its restriction $g_\gamma|\mathcal{P}_Y$ to $\mathcal{P}_Y$ maps D_Y to the induced chamber adjacent to D_Y across the wall $D_Y \cap (r_\gamma)^\perp$. The eigenvalues of $g_\gamma|\mathcal{P}_Y$ are $1^9(-1)^1$. Hence $g_\gamma \in \mathrm{Cen}(\varepsilon)$ induces a numerically reflective involution [21] on Y.
- Suppose that $\gamma \in V_1$. Then the wall $D_Y \cap (r_\gamma)^\perp$ is outer. There exists a unique line C of $X_{2,2,2}$ such that $D_Y \cap (r_\gamma)^\perp = \mathcal{P}_Y \cap D_X \cap ([C])^\perp$ and $\pi^*(r) = [C] + [\varepsilon(C)]$. The curve $\varepsilon(C)$ is of degree 5 with respect to h_8.
- Suppose that $\gamma \in F$. Then $D_Y \cap (r_\gamma)^\perp$ is outer. There exists a unique pair of lines C, $\varepsilon(C)$ of $X_{2,2,2}$ such that $D_Y \cap (r_\gamma)^\perp$ is equal to $\mathcal{P}_Y \cap D_X \cap ([C])^\perp$, and we have $\pi^*(r) = [C] + [\varepsilon(C)]$.
- Suppose that $\gamma \in P$. Then $D_Y \cap (r_\gamma)^\perp$ is inner. There exists a unique wall $D_X \cap (v_\gamma)^\perp$ of D_X such that $D_Y \cap (r_\gamma)^\perp = \mathcal{P}_Y \cap D_X \cap (v_\gamma)^\perp$. We have $\langle v_\varepsilon, v_\gamma \rangle = 0$. This wall $D_X \cap (v_\gamma)^\perp$ is perpendicular to D_Y, and corresponds to a Hutchinson–Göpel involution g_γ (Theorem 5.5 (a)). The involution g_γ commutes with ε, and $g_\gamma|\mathcal{P}_Y$ maps D_Y to the induced chamber adjacent to D_Y across $D_Y \cap (r_\gamma)^\perp$. The eigenvalues of $g_\gamma|\mathcal{P}_Y$ are $1^6(-1)^4$.

Therefore $\mathrm{Cen}(\varepsilon)$ is generated by a subgroup of $\mathrm{Aut}(X_{2,2,2})$ of order 2^4, eight involutions associated with projections from ordinary nodes of X_4 or of $X_4^\vee$, and 12 Hutchinson–Göpel involutions.

7.3. **An Enriques surface in the conjugacy class of type** `40C`**.** We describe the configuration of type `40C`. We put $F := \{1,2,3,4\}$. Let P be the set $F \times F$ with the projections $\mathrm{pr}_1 \colon P \to F$ and $\mathrm{pr}_2 \colon P \to F$, and let B be the set of bijections $f \colon F \to F$. We put

$$\Gamma := P \sqcup B.$$

We define a symmetric function $\mu \colon \Gamma \times \Gamma \to \mathbb{Z}$ with $\mu(a,a) = -2$ for all $a \in \Gamma$ as follows.

- Suppose that $p, p' \in P$ with $p \neq p'$. Then
$$\mu(p,p') = \begin{cases} 1 & \text{if } \mathrm{pr}_1(p) = \mathrm{pr}_1(p') \text{ or } \mathrm{pr}_2(p) = \mathrm{pr}_2(p'), \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $p \in P$ and $f \in B$. Then
$$\mu(p,f) = \begin{cases} 2 & \text{if } f(\mathrm{pr}_1(p)) = \mathrm{pr}_2(p), \\ 0 & \text{otherwise.} \end{cases}$$
- Suppose that $f, f' \in B$ with $f \neq f'$. Then $\gamma := ff'^{-1}$ is a permutation of F. Let $\tau(\gamma)$ denote the lengths of cycles in the cycle decomposition of $\gamma \in \mathfrak{S}_4$. Then
$$\mu(f,f') = \begin{cases} 2 & \text{if } \tau(\gamma) = (4), \\ 2 & \text{if } \tau(\gamma) = (2,2), \\ 1 & \text{if } \tau(\gamma) = (3,1), \\ 0 & \text{if } \tau(\gamma) = (2,1,1). \end{cases}$$

Remark 7.9. The group $\mathrm{Aut}(\Gamma,\mu)$ is isomorphic to $(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes C_2$, which acts on P in the natural way.

Let ε be an Enriques involution belonging to $\mathrm{Aut}(X_{2,2,2}) = \mathrm{Aut}(X, D_X)$ (see Remark 7.2). Then $D_Y := \mathcal{P}_Y \cap D_X$ is an induced chamber of ι_Y, and since $h_8^{\varepsilon} = h_8$, the class h_8 is an interior point of D_Y. The group $\mathrm{Cen}(\varepsilon, D_Y)$ is equal to $\mathrm{Aut}(X_{2,2,2})$, and $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. The eigenvalues of 6 elements of $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ are $1^4(-1)^6$, whereas the eigenvalues of 9 elements of $\mathrm{Cen}(\varepsilon, D_Y)|\mathcal{P}_Y$ are $1^6(-1)^4$.

In the 40 walls of D_Y, 16 are outer and 24 are inner. There exists an indexing $\gamma \mapsto r_\gamma$ of the (-2)-vectors defining the walls of D_Y by the set $\Gamma = P \sqcup B$ above with the following properties.

- Suppose that $\gamma \in P$. Then $D_Y \cap (r_\gamma)^\perp$ is outer. There exists a unique pair of lines C, $\varepsilon(C)$ of $X_{2,2,2}$ such that $D_Y \cap (r_\gamma)^\perp = \mathcal{P}_Y \cap D_X \cap ([C])^\perp$, and we have $\pi^*(r) = [C] + [\varepsilon(C)]$.
- Suppose that $\gamma \in B$. Then $D_Y \cap (r_\gamma)^\perp$ is inner. There exists a unique wall $D_X \cap (v_\gamma)^\perp$ of D_X such that $D_Y \cap (r_\gamma)^\perp = \mathcal{P}_Y \cap D_X \cap (v_\gamma)^\perp$. This wall $D_X \cap (v_\gamma)^\perp$ is perpendicular to D_Y, and corresponds to a Hutchinson–Göpel involution g_γ (Theorem 5.5 (a)). The involution g_γ commutes with ε, and $g_\gamma|\mathcal{P}_Y$ maps D_Y to the induced chamber adjacent to D_Y across $D_Y \cap (r_\gamma)^\perp$. The eigenvalues of $g_\gamma|\mathcal{P}_Y$ are $1^6(-1)^4$.

Therefore the group $\mathrm{Cen}(\varepsilon)$ is generated by $\mathrm{Aut}(X_{2,2,2})$ and 24 Hutchinson–Göpel involutions.

Remark 7.10. In [38], we have determined conjugacy classes of Enriques involutions of singular $K3$ surfaces whose transcendental lattice is of discriminant ≤ 36.

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Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

Email address: `ichiro-shimada@hiroshima-u.ac.jp`