FROBENIUS SUMMANDS OF GRADED RINGS

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We are motivated by a question arising from commutative algebra, asking what kind of graded rings in characteristic p have finite F-representation type (FFRT). In geometric setting, this is related to the problem of looking out for Frobenius summands. Namely, given a line bundle L on a projective variety X, we want to know how many and what kind of indecomposable direct summands appear in the direct sum decomposition of the iterated Frobenius push-forwards $F^e_*(L^i)$, where e, i are non-negative integers with $0 \le i \le p^e - 1$. We will consider the problem in the following two cases.

(1) two-dimensional normal graded rings (a joint work with Ryo Ohkawa [HO])
(2) the anti-canonical ring of a quintic del Pezzo surface

After reviewing the preliminary results in Section 1, we will take a look at the result obtained in [HO] in Section 2. Our description here is based on the Pinkham–Demazure construction: A two-dimensional normal graded ring R is isomorphic to the graded ring $R(C, D) = \bigoplus_{n \ge 0} H^0(C, \mathcal{O}_C(\lfloor nD \rfloor))$, where D is an ample $\mathbb{Q}$-divisor on the smooth curve $C = \operatorname{Proj} R$. We introduce the invariant $\delta = \deg(K_C + D')$, the degree of the canonical divisor of C plus the "fractional part" D' of D. It is known that Spec R has a log terminal singularity if and only if $\delta < 0$, and in this case, R has FFRT (Proposition 2.2). On the other hand, we will see in Theorem 2.3 that if $\delta \ge 0$, then R has FFRT only in the exceptional cases where the characteristic p divides a denominator of the fractional coefficient of D.

In Section 3, we introduce an attempt to looking out for Frobenius summands on a quintic del Pezzo surface X and its anti-canonical ring $R(X, -K_X)$. Unlike case (1) above, the present situation in this case (2) is far from satisfactory, and we have not yet come to a conclusion whether the anti-canonical ring has FFRT or not. We give partial results and examples on the Frobenius summands of $F^e_*(\omega_X^{-i})$ mainly in the cases $i = 0$ and $i = \frac{p^e-1}{2}$.

1. Preliminaries

Throughout this note, we work over an algebraically closed field k of characteristic $p > 0$. For a noetherian commutative ring R over k, the Frobenius ring homomorphism sending $a \in R$ to $a^p \in R$ will be denoted by $F\colon R \to R$. For a k-scheme X, we denote the (absolute) Frobenius morphism $(\mathrm{id}_X, F)\colon (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ by $F\colon X \to X$ and its associated ring homomorphism by $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ as well.

From now on, We always assume that R is an F-finite (i.e., $F\colon R \to R$ is module-finite) integral domain. In this case, we can identify the e-times iterated Frobenius ring homomorphism $F^e\colon R \to R$ and the inclusion map $R \hookrightarrow R^{1/p^e}$ into the ring R^{1/p^e} of p^e-th roots of R, for all $e = 0, 1, 2, \dots$

When R is an $\mathbb{N}$-graded ring $R = \bigoplus_{n\geq 0} R_n$ over $R_0 = k$, the ring R^{1/p^e} has a natural $\mathbb{Q}$-grading (actually, a $\frac{1}{p^e}\mathbb{Z}$-grading) and the inclusion map $R \hookrightarrow R^{1/p^e}$ preserves the grading. Note that the category of finitely generated $\mathbb{Q}$-graded R-modules is a Krull–Schmidt category. For each $e = 0, 1, 2, \dots$, we have a decomposition

$$(*) \qquad R^{1/p^e} = M_1^{(e)} \oplus \cdots \oplus M_{m_e}^{(e)}$$

in the category of finitely generated $\mathbb{Q}$-graded R-modules with each $M_i^{(e)}$ indecomposable.

Definition 1.1 (Smith–Van den Bergh [SVdB])**.** Let R be an $\mathbb{N}$-graded ring over $R_0 = k$ such that each R^{1/p^e} has a decomposition as $(*)$. We say that R has *finite F-representation type* (FFRT) if the set

$$\{M_i^{(e)} \mid e = 0, 1, 2, \dots;\ i = 1, 2, \dots, m_e\}/\cong$$

is finite, where $\cong$ denotes isomorphism of graded R-modules admitting degree shift.

Example 1.2 (rings of FFRT)**.**

(1) Let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Then R has FFRT, since
$$R^{1/q} = k[x_1^{1/q}, \dots, x_n^{1/q}] = \bigoplus_{0\leq i_1,\dots,i_n\leq q-1} R x_1^{i_1/q}\cdots x_n^{i_n/q} \cong R^{\oplus q^n}$$
is a free R-module for all $q = p^e$.

(2) Two-dimensional rational double points have FFRT (Artin–Verdier [AV]).

(3) Tame quotient singularities have FFRT ([SVdB]). Namely, if $R = S^G$ is the invariant subring of finite group G of order not divisible by p acting on a polynomial ring S, then R has FFRT.

(4) A Cohen–Macaulay ring R is called a *Frobenius sandwich* if an iterated Frobenius ring homomorphism of a polynomial ring S factors through R, i.e., there exists a power q of p such that $S^q \subset R \subset S$. If R is a Frobenius sandwich, then it has FFRT. For example, $R = k[x, y, z]/(z^p - f(x, y))$ has FFRT.

Remark 1.2.1. Rings in (1), (2) have stronger property "finite representation type," i.e., there exist only finitely many isomorphism classes of maximal Cohen–Macaulay R-modules. On the other hand, rings in (3), (4) do not necessarily have this property.

Remark 1.2.2. Rings in (1)–(3) are F-regular, but Frobenius sandwiches are not F-regular in general. It seems natural to ask if F-regular implies FFRT, since this is true in dimension ≤ 2. But this implication fails in higher dimension ([SS], [TT]).

Section rings. The first example of a two-dimensional graded ring that does not have FFRT was found by Smith–Van den Bergh [SVdB]. Let us review their construction. Let X be a smooth projective variety over k, L an ample invertible sheaf on X and let

$$R = R(X, L) = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n})t^n$$

be the section ring associated to (X, L), where t is a homogeneous element of degree 1. In what follows we denote the n-times tensor power $L^{\otimes n}$ of L simply by L^n.

For each $q = p^e$, the $\frac{1}{q}\mathbb{Z}$-graded R-module $R^{1/q}$ decomposes as

$$R^{1/q} = \bigoplus_{n \geq 0} H^0(X, F^e_*(L^n))t^{n/q} = \bigoplus_{i=0}^{q-1} (R^{1/q})_{i/q \bmod \mathbb{Z}},$$

where the graded R-modules

$$(R^{1/q})_{i/q \bmod \mathbb{Z}} = \bigoplus_{0 \leq n \equiv i \bmod q} H^0(X, F^e_*(L^n))t^{n/q} \cong \bigoplus_{m \geq 0} H^0(X, F^e_*(L^i) \otimes L^m)$$

appearing as the direct summands are in one-to-one correspondence with the coherent sheaves $F^e_*(L^i)$ on X. Thus the decomposition of $(R^{1/q})_{i/q \bmod \mathbb{Z}}$ into indecomposable graded R-modules are described in terms of the decomposition

$$F^e_*(L^i) = \mathcal{F}_1^{(e,i)} \oplus \cdots \oplus \mathcal{F}_{m_{e,i}}^{(e,i)}$$

of the vector bundles $F^e(L^i)$ into indecomposable bundles $\mathcal{F}_j^{(e,i)}$ in $\mathrm{Coh}(X)$.

Proposition-Definition 1.3. *Let the notation be as above. Then $R = R(X, L)$ has FFRT if and only if the set of isomorphism classes in* $\mathrm{Coh}(X)$*,*

$$\{\mathcal{F}_j^{(e,i)} \mid e \in \mathbb{N};\ i = 0, 1, \ldots, p^e - 1;\ j = 1, \ldots, m_{e,i}\}/\cong$$

is finite. In this case, the pair (X, L) is said to have globally finite F-representation type (GFFRT).

The following proposition generalizes [SVdB, Example 3.1.7].

Proposition 1.4. *Let C be a smooth projective curve over k of genus $g(C) \geq 1$ and let L be an ample invertible sheaf on C. Then the section ring $R = R(C, L)$ does not have FFRT.*

Proof. In view of Proposition 1.3, it is sufficient to show that there appear infinitely many isomorphism classes of indecomposable direct summands of $F^e_*\mathcal{O}_C$ when e ranges over all non-negative integers. This is verified case by case as follows:

Case 1: $g(C) = 1$. If C is an ordinary elliptic curve, then $F^e_*\mathcal{O}_C$ splits into p^e distinct p^e-torsion line bundles. If C is supersingular, then $F^e_*\mathcal{O}_C$ is isomorphic to Atiyah's indecomposable vector bundle $\mathcal{F}_{p^e}$; see [A].

Case 2: $g(C) \geq 2$. In this case, the vector bundle $F^e_*\mathcal{O}_C$ is stable and so is indecomposable for all $e \geq 0$ (Sun [Su], see also Kitadai–Sumihiro [KS], Mehta–Pauly [MP]). □

2. FFRT property of two-dimensional graded rings

In this section, we consider the condition for two-dimensional normal graded rings to have FFRT. Specifically, we will answer the following question:

Question (H. Brenner). Does the ring $R = k[x, y, z]/(x^2 + y^3 + z^7)$ have FFRT?

It is known that two-dimensional F-regular rings have FFRT. On the other hand, due to Proposition 1.4 we could expect that a two-dimensional normal graded ring R has FFRT only if $\operatorname{Proj} R \cong \mathbb{P}^1$; see Theorem 2.1. So, Brenner's question is in a critical case, because the ring $R = k[x, y, z]/(x^2 + y^3 + z^7)$ is not F-regular and $\mathrm{Proj}\, R \cong \mathbb{P}^1$. In this case, however, it is known that R has FFRT if $p \le 7$, since it is a Frobenius sandwich in characteristic $p = 2, 3, 7$ (Shibuta [Sh]).

Pinkham–Demazure construction ([P], [D]). Let R be a two-dimensional normal graded ring over $R_0 = k$. Then there exists an ample $\mathbb{Q}$-Cartier divisor D on $C = \mathrm{Proj}\, R$ such that

$$R \cong R(C, D) = \bigoplus_{n \ge 0} H^0(C, \mathcal{O}_C(\lfloor nD \rfloor))t^n.$$

Let $g(C)$ denote the genus of the smooth projective curve C. We write

$$D = \lfloor D \rfloor + \sum_{i=1}^{m} \frac{s_i}{r_i} P_i$$

with closed points P_i of C and coprime integers $r_i \ge 2$ and s_i. We then put

$$D' := \sum_{i=1}^{m} \frac{r_i - 1}{r_i} P_i$$

and call it the *fractional part* of D.

We now state the main results of Hara–Ohkawa [HO]. Let the notation be as above.

Theorem 2.1 ([HO]). *If $g(C) \ge 1$, then $R = R(C, D)$ does not have FFRT.*

Proposition 2.2 ([HO]). *If $\deg(K_C + D') < 0$, then $R = R(C, D)$ has FFRT.*

Remark 2.2.1. Note that $\deg(K_C + D') < 0$ if and only if $C \cong \mathbb{P}^1$, and $m \le 2$ or $m = 3$ and $(r_1, r_2, r_3) = (2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)$. These are exactly the cases where $R(C, D)$ has a log terminal singularity.

Theorem 2.3 ([HO]). *Suppose $C = \mathbb{P}^1$, $\deg(K_C + D') \ge 0$ and $r_1, \dots, r_m$ are not divisible by p. Then $R = R(\mathbb{P}^1, D)$ does not have FFRT.*

Idea of proof. In what follows, we briefly sketch the idea of the proof of the theorems. When D is an integral divisor, then R is the section ring associated to the line bundle $L = \mathcal{O}_C(D)$, and we have the correspondence between the direct summands $(R^{1/q})_{i/q \bmod \mathbb{Z}}$ of $R^{1/q}$ and the vector bundles $F^e_*(L^i)$ on C as described in Section 1. The obstruction is that we do not have this correspondence in the case where D is not an integral divisor.

To overcome the above difficulty, we import notions from the theory of algebraic stacks [B], [Ol]. What we will use is the *orbifold curve*

$$\mathfrak{C} = C[\sqrt[r_1]{P_1}, \dots, \sqrt[r_m]{P_m}\,] \xrightarrow{\pi} C.$$

This is not a scheme (if D is not integral) but is a one-dimensional root stack of weight $(r_1, \dots, r_m)$ over $P_1, \dots, P_m \in C$. The orbifold curve $\mathfrak{C}$ is something like the "minimal covering" of C on which D becomes integral. We summarize properties of $\mathfrak{C}$ in the following lemma.

Lemma 2.4. *For each $i = 1, \dots, m$, there is a "stacky point" Q_i on $\mathfrak{C}$ lying over P_i satisfying the following properties.*

(1) $\pi\colon \mathfrak{C} \to C$ *is an isomorphism away from Q_i and P_i.*
(2) Q_i *is a Cartier divisor on $\mathfrak{C}$ and $\pi^* P_i = r_i Q_i$.*
(3) *If E is a $\mathbb{Q}$-divisor on C such that $\pi^* E$ is integral, then*

$$\pi_* \mathcal{O}_{\mathfrak{C}}(\pi^* E) \cong \mathcal{O}_C(\lfloor E \rfloor) \text{ and } R^1 \pi_* \mathcal{O}_{\mathfrak{C}}(\pi^* E) = 0.$$

(4) $\mathfrak{C}$ *has a dualizing sheaf*

$$\omega_{\mathfrak{C}} \cong \pi^* \omega_C \otimes \mathcal{O}_{\mathfrak{C}}(\sum_{i=1}^{m} (r_i - 1) Q_i).$$

It follows from the lemma that $\pi^* D$ is an integral Cartier divisor on $\mathfrak{C}$ and if we denote $\mathcal{L} = \mathcal{O}_{\mathfrak{C}}(\pi^* D)$, then

$$H^0(\mathfrak{C}, \mathcal{L}^{\otimes n}) \cong H^0(C, \mathcal{O}_C(\lfloor nD \rfloor))$$

for all $n \in \mathbb{Z}$. Thus

$$R = R(C, D) \cong R(\mathfrak{C}, \mathcal{L})$$

is the section ring associated to the line bundle $\mathcal{L}$ on $\mathfrak{C}$.

Corollary 2.5. *$R = R(C, D)$ has FFRT if and only if $(\mathfrak{C}, \mathcal{L})$ has GFFRT in the same sense as in Proposition-Definition 1.3.*

Now let $\delta_{\mathfrak{C}} = \deg \omega_{\mathfrak{C}}$. Then $\delta_{\mathfrak{C}} = \deg(K_C + D')$ by Lemma 2.4 (4). If $\delta_{\mathfrak{C}} < 0$, then $(\mathfrak{C}, \mathcal{L})$ has GFFRT by [CB, Theorem 1], from which Proposition 2.2 follows.

To prove that R does not have FFRT in Theorems 2.1 and 2.3, it is sufficient to show that infinitely many indecomposable summands appear in $F^e_* \mathcal{O}_{\mathfrak{C}}$, when e ranges over all non-negative integers. In case $g(C) \geq 1$ (Theorem 2.1), this follows as in the proof of Proposition 1.4, since $\pi_* F^e_* \mathcal{O}_{\mathfrak{C}} \cong F^e_* \mathcal{O}_C$ by Lemma 2.4 (3).

The proof of our Main Theorem 2.3 is again due to case-by-case verification.

Case $\delta_{\mathfrak{C}} = 0$. In this case, it follows that $m = 3$ or 4 and the weight $(r_1, \dots, r_m)$ ordered as $r_1 \leq \cdots \leq r_m = r$ is either one of the following: $(2,3,6)$, $(2,4,4)$, $(3,3,3)$, $(2,2,2,2)$. We have a separable r-fold covering $f\colon E \to C = \mathbb{P}^1$ from an elliptic curve E with assigned ramification indexes $(r_1, \dots, r_m)$. It factors through $\mathfrak{C}$ as

$$f\colon E \xrightarrow{\varphi} \mathfrak{C} \xrightarrow{\pi} C,$$

with φ unramified. We can use the unramified morphism $\varphi\colon E \to \mathfrak{C}$ to prove the following; see [HO] for details.

(1) If E is supersingular, then $\varphi^* F^e_* \mathcal{O}_{\mathfrak{C}}$ is isomorphic to the Atiyah's indecomposable bundle $\mathcal{F}_{p^e}$ of rank p^e and degree zero [A]. Hence $F^e_* \mathcal{O}_{\mathfrak{C}}$ itself is indecomposable.
(2) If E is ordinary, then $p \equiv 1 \pmod{r}$ and there are exactly $s = \frac{p^e - 1}{r}$ equivalence classes of non-trivial p^e-torsion line bundles on E with respect to the action of $\mathrm{Gal}(E/C)$. If $L_1, \dots, L_s$ are complete representatives thereof, then

$$F^e_* \mathcal{O}_{\mathfrak{C}} \cong \mathcal{O}_{\mathfrak{C}} \oplus \varphi_* L_1 \oplus \cdots \oplus \varphi_* L_s,$$

where $\varphi_* L_1, \cdots, \varphi_* L_s$ are non-isomorphic indecomposable r-bundles on $\mathfrak{C}$.

Case $\delta_{\mathfrak{C}} > 0$. In this case, we have the following theorem, which follows similarly as in the case of smooth projective curves of genus $g \geq 2$ [Su, Theorem 2.2].

Theorem 2.6. *If $\delta_{\mathfrak{C}} > 0$ and $r_1, \ldots, r_m$ are not divisible by p, then $F_*^e \mathcal{O}_{\mathfrak{C}}$ is stable and so indecomposable for all $e \geq 0$.*

Example 2.7. Let $R = k[x,y,z]/(x^2 + y^3 + z^7)$, the ring in Brenner's question. This is not a rational singularity but Proj $R \cong \mathbb{P}^1$ and $R \cong R(\mathbb{P}^1, D)$ for a $\mathbb{Q}$-divisor $D = \frac{1}{2}(\infty) - \frac{1}{3}(0) - \frac{1}{7}(1)$ on $\mathbb{P}^1$. By Theorem 2.3, R does not have FFRT if $p \neq 2, 3, 7$.

Example 2.8. Let $R = R(\mathbb{P}^1, D)$ for a $\mathbb{Q}$-divisor $D = \frac{1}{3}(\infty) + \frac{1}{3}(0) - \frac{1}{3}(1)$ on $\mathbb{P}^1$. This is a rational log canonical singularity but not log terminal. The ring R does not have FFRT if and only if $p \neq 3$. In the exceptional case when $p = 3$, the weighted projective line $\mathfrak{C}$ of weight $(3,3,3)$ is a Frobenius sandwich.

3. The anticanonical ring of the quintic Del Pezzo surface

The FFRT problem for graded rings is wide open yet in higher dimension (i.e., $\dim R \geq 3$). We do not know even the answer to the following question.

Question. Let X be the smooth quintic del Pezzo surface in characteristic $p > 0$ with anticanonical bundle $L = \omega_X^{-1}$. Does the section ring $R(X, -K_X) = R(X, L)$ have FFRT?

The setup in the question above is considered one of the simplest non-trivial cases because of the following reasons:

(1) Del Pezzo surfaces of degree $K^2 \geq 6$ are toric surfaces. In this case, the Frobenius push-forward of any line bundle splits into line bundles [To], and it is easy to see that the anticanonical ring has FFRT.
(2) In order to prove that $R(X, L)$ is FFRT, one has to know the decomposition of $F_*^e(L^i)$ for all i with $0 \leq i \leq p^e - 1$. However, when $L = \omega_X^{-1}$, it is enough to consider $0 \leq i \leq \frac{p^e-1}{2}$, since $F_*^e(L^i)$ is dual to $F_*^e(\omega_X^{1-p^e} \otimes L^{-i}) = F_*^e(L^{p^e-1-i})$.

In this section, we will study the structure of $F_*^e(L^i)$ mainly in the extremal cases $i = 0$ and $i = \frac{p^e-1}{2}$. Since the quintic del Pezzo surface X is obtained by blowing up the projective plane $\mathbb{P}^2$ at four points in general position, we work under the following notation throughout this section.

Notation. Let $\pi \colon X \to \mathbb{P}^2$ be the blow-up at four points $P_1, P_2, P_3, P_4 \in \mathbb{P}^2$ in general position. Let H be a line in $\mathbb{P}^2$ and $E_i = \pi^{-1}(P_i)$ the exceptional curve over P_i. Also let $E = E_1 + E_2 + E_3 + E_4$.

Theorem 3.1 (case $i = 0$ [H])**.** *Any indecomposable direct summand of $F_*^e \mathcal{O}_X$ ($e = 1, 2, \ldots$) coincides with one of the following vector bundles of rank ≤ 3.*

(1) *line bundles $\mathcal{O}_X$, $L_0 = \mathcal{O}_X(E - 2\pi^* H)$ and $L_i = \mathcal{O}_X(E_i - \pi^* H)$, $i = 1, 2, 3, 4$;*
(2) *an indecomposable rank two bundle $\mathcal{G}$ given by a unique non-trivial extension*
$$0 \to \mathcal{O}_X(-\pi^* H) \to \mathcal{G} \to L_0 \to 0;$$
(3) *an indecomposable rank three bundle $\mathcal{B}$ given by a non-trivial extension*
$$0 \to L_1 \oplus L_2 \to \mathcal{B} \to \mathcal{O}_X(E_3 + E_4 - \pi^* H) \to 0.$$

Furthermore, for any power $q = p^e$ of p with $e \geq 1$ one has

$$F^e_* \mathcal{O}_X \cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus(q-2)} \oplus \mathcal{B} \oplus \mathcal{G}^{\oplus \frac{(q-2)(q-3)}{2}}$$

Case $i = \frac{p^e-1}{2}$. Let $L = \omega_X^{-1}$ and assume that the characteristic p is an odd prime. We consider the decomposition of $F^e_*(L^i)$ in the other extremal case, i.e., $i = \frac{p^e-1}{2}$. Let $q = p^e$ for $e = 0, 1, 2, \dots$ Note that the vector bundle $F^e_*(L^{\frac{q-1}{2}})$ is self-dual.

We begin with constructing the L-stable bundle $\mathcal{F}$ of rank three which is supposed to be a unique non-trivial indecomposable summand of $F^e_*(L^{\frac{q-1}{2}})$. We require $\mathcal{F}$ to sit in an exact sequence

$$0 \to \mathcal{G}(2\pi^* H - E) \to \mathcal{F} \to \mathcal{O}_X(E - \pi^* H) \to 0,$$

where $\mathcal{G}$ is the rank two bundle given in Theorem 3.1 (2). To identify the isomorphism class of $\mathcal{F}$, we also need the following splitting condition: For $i = 1, 2, 3, 4$, the restriction of $\mathcal{F}$ to $U_i = X \setminus E_i$ splits into line bundles as

$$(\star) \qquad \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}(\pi^* H - E) \oplus \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i}(E - \pi^* H).$$

We fix an open covering $X = U \cup V$ with $U = X \setminus E_4$, $V = X \setminus E_1$ and let

$$\mathcal{F}_U = \mathcal{O}_U(\pi^* H - E) \oplus \mathcal{O}_U \oplus \mathcal{O}_U(E - \pi^* H),$$
$$\mathcal{F}_V = \mathcal{O}_V(\pi^* H - E) \oplus \mathcal{O}_V \oplus \mathcal{O}_V(E - \pi^* H).$$

Then $\mathcal{F}$ is given by gluing $\mathcal{F}_U$ and $\mathcal{F}_V$ via an isomorphism $\varphi_{UV} \colon \mathcal{F}_U|_{U\cap V} \to \mathcal{F}_V|_{U \cap V}$ corresponding to a transition matrix

$$T_{\alpha,\beta,\gamma} = \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta, \gamma \in k$.

Proposition 3.2. *Let $\mathcal{F}_{\alpha,\beta,\gamma}$ denote the vector bundle given by gluing $\mathcal{F}_U$ and $\mathcal{F}_V$ with the transition matrix $T_{\alpha,\beta,\gamma}$.*

(1) *$\mathcal{F}_{\alpha,\beta,\gamma}$ satisfies condition $(\star)$ for $i = 1, 2, 3, 4$ if and only if $\alpha\beta = 2\gamma$.*
(2) *If $\mathcal{F}$ is an indecomposable bundle satisfying condition $(\star)$ for $i = 1, 2, 3, 4$, then $\mathcal{F} \cong \mathcal{F}_{1,1,1/2}$.*

Conjecture 3.3. Assume that p is an odd prime and let $q = p^e$ for $e = 0, 1, 2, \dots$

(1) The rank of the maximal free summand of $F^e_*(L^{\frac{q-1}{2}})$ is $h^0(L^{\frac{q-1}{2}}) = \frac{5q^2+3}{8}$.
(2) $F^e_*(L^{\frac{q-1}{2}}) \cong \mathcal{O}_X^{\oplus \frac{5q^2+3}{8}} \oplus (\mathcal{F}_{1,1,1/2})^{\oplus \frac{q^2-1}{8}}$.

Proposition 3.4. *Conjecture 3.3* (1) *implies Conjecture 3.3* (2).

Proof. If Conjecture 3.3 (1) is true, then $F^e_*(L^{\frac{q-1}{2}}) \cong \mathcal{O}_X^{\oplus \frac{5q^2+3}{8}} \oplus \mathcal{E}$ for a vector bundle $\mathcal{E}$ of rank $3n$, where $n = (q^2-1)/8$. It follows that $\mathcal{E}$ is obtained by gluing

$$\mathcal{E}_U = \mathcal{O}_U(\pi^* H - E)^{\oplus n} \oplus \mathcal{O}_U^{\oplus n} \oplus \mathcal{O}_U(E - \pi^* H)^{\oplus n},$$
$$\mathcal{E}_V = \mathcal{O}_V(\pi^* H - E)^{\oplus n} \oplus \mathcal{O}_V^{\oplus n} \oplus \mathcal{O}_V(E - \pi^* H)^{\oplus n}$$

with the transition matrix

$$\begin{bmatrix} I_n & A & \Gamma \\ O & I_n & B \\ O & O & I_n \end{bmatrix}.$$

Here we note that $\mathcal{E}|_{U_i}$ splits into line bundles for each $i = 1, 2, 3, 4$, since $U_i = X \setminus E_i$ is isomorphic to an open set of the sextic del Pezzo surface, which is toric. As in Proposition 3.2, this splitting condition implies that $AB = 2\Gamma$. On the other hand, we see that the line bundles $\mathcal{O}_X(E - \pi^* H)$ and $\mathcal{O}_X(\pi^* H - E)$ are not direct summands of $F^e_*(L^{\frac{q-1}{2}})$ (and hence of $\mathcal{E}$). This implies that $\operatorname{rank} A = \operatorname{rank} B = n$. Then the transition matrix is transformed under elementary transformations *within row and column blocks* to

$$\begin{bmatrix} I_n & I_n & \frac{1}{2} I_n \\ O & I_n & I_n \\ O & O & I_n \end{bmatrix} = T^{\oplus n}_{1,1,1/2}.$$

It follows that $\mathcal{E} \cong (\mathcal{F}_{1,1,1/2})^{\oplus n}$. □

Remark 3.4.1. Conjecture 3.3 (1) holds if and only if the natural pairing

$$\operatorname{Hom}(\mathcal{O}_X, L^{\frac{q-1}{2}}) \times \operatorname{Hom}(L^{\frac{q-1}{2}}, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X)$$

is a perfect paring. Choosing appropriate affine coordinates x, y on $\mathbb{P}^2$, we can identify $\operatorname{Hom}(\mathcal{O}_X, L^{\frac{q-1}{2}}) \cong \operatorname{Hom}(L^{\frac{q-1}{2}}, \mathcal{O}_X) \cong H^0(X, L^{\frac{q-1}{2}})$ with a subspace of $V = \left\langle x^i y^j \mid 0 \le i, j \le q-1 \text{ and } \frac{q-1}{2} \le i + j \le \frac{3q-3}{2} \right\rangle$. Then the pairing above is identified with the pairing

$$\langle\ ,\ \rangle \colon H^0(X, L^{\frac{q-1}{2}}) \times H^0(X, L^{\frac{q-1}{2}}) \to k$$

given by

$$\langle \phi, \psi \rangle = \text{the coefficient of the product } \phi\psi \text{ in } (xy)^{q-1}$$

for $\phi, \psi \in H^0(X, L^{\frac{q-1}{2}}) \subset V$. Taking this into account, we can rephrase Conjecture 3.3 (1) into the assertion that a certain $\frac{q^2-1}{8} \times \frac{q^2-1}{8}$ matrix is invertible mod p. M. Tano has implemented a computer program to examine this assertion and verified that it is true up to $p^e < 100$.

Finally, we shall take a look at examples which we hope illustrate the behavior of the single Frobenius push-forwards $F_*(L^i)$ for all i in the range $0 \le i \le \frac{p-1}{2}$. In the following, we put $M_{i,j} = \mathcal{O}_X(E_i + E_j - \pi^* H)$ for $1 \le i < j \le 4$.

Example 3.5 ($p = 5$)**.**

$$F_* \mathcal{O}_X \cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus 3} \oplus \mathcal{B} \oplus \mathcal{G}^{\oplus 3},$$

$$F_* L \cong \mathcal{O}_X^{\oplus 6} \oplus \bigoplus_{i=1}^{4} \mathcal{O}_X(-E_i) \oplus \bigoplus_{\substack{(i,j)=(1,2),(1,3),(1,4),\\(2,3),(2,4),(3,4)}} M_{i,j} \oplus \mathcal{B}^{\oplus 3},$$

$$F_*(L^2) \cong \mathcal{O}_X^{\oplus 16} \oplus \mathcal{F}^{\oplus 3}$$

Example 3.6 ($p = 7$).

$$F_*\mathcal{O}_X \cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus 5} \oplus \mathcal{B} \oplus \mathcal{G}^{\oplus 10},$$

$$F_*L \cong \mathcal{O}_X^{\oplus 6} \oplus \bigoplus_{i=0}^{4} L_i^{\oplus 3} \oplus \bigoplus_{i=1}^{4} \mathcal{O}_X(-E_i) \oplus \bigoplus_{\substack{(i,j)=(1,2),(1,3),(1,4),\\(2,3),(2,4),(3,4)}} M_{i,j} \oplus \mathcal{B}^{\oplus 6},$$

$$F_*(L^2) \cong \mathcal{O}_X^{\oplus 16} \oplus \bigoplus_{i=1}^{4} \mathcal{O}_X(-E_i)^{\oplus 3} \oplus \bigoplus_{\substack{(i,j)=(1,2),(1,3),(1,4),\\(2,3),(2,4),(3,4)}} M_{i,j}^{\oplus 3} \oplus \mathcal{B},$$

$$F_*(L^3) \cong \mathcal{O}_X^{\oplus 31} \oplus \mathcal{F}^{\oplus 6}.$$

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