COHEN-MACAULAYNESS OF KLT SINGULARITIES IN POSITIVE CHARACTERISTICS

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Arguably klt singularities form the most important class of singularities from the viewpoint of birational geometry. It is well-known that in characteristic zero, klt singularities are Cohen-Macaulay. Recently many counterexamples to this fact in positive characteristics were constructed. In this note, we review them.

1. Klt, CM and rational

Throughout the note, we work over an algebraically closed field k of characteristic $p \geq 0$.

Let us first recall basic classes of singularities. Let X be a normal variety. We say that X is $\mathbb{Q}$-Gorenstein if its canonical divisor K_X is $\mathbb{Q}$-Cartier. For a normal modification $f\colon Y \to X$ (a proper birational morphism with Y normal), we can define the relative canonical divisor $K_{Y/X} = K_Y - f^*K_X$, which is a $\mathbb{Q}$-divisor with support contained in the exceptional locus of f. Let us write $K_{Y/X} = \sum_E a_E E$, where E runs over exceptional prime divisors and $a_E \in \mathbb{Q}$.

Definition 1.1. We say that X is terminal (resp. canonical, klt, log canonical) if for every normal modification $f\colon Y \to X$ and for every exceptional prime divisor E on Y, we have $a_E > 0$ (resp. ≥ 0, > -1, ≥ -1).

We also consider klt singularities of pairs and potentially klt singularities which does make sense even if X is not $\mathbb{Q}$-Gorenstein.

Definition 1.2. Let X be a normal variety and D an effective $\mathbb{Q}$-divisor on X. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that (X, D) is klt if for any normal modification $f\colon Y \to X$, $K_{Y/(X,D)} = K_Y - f^*(K_X + D)$ has coefficients > -1. We say that a normal variety X is potentially klt if there exists an effective $\mathbb{Q}$-divisor D such that (X, D) is klt.

Definition 1.3. We say that X is Cohen-Macaulay (for short, CM) if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is CM.

Definition 1.4. In characteristic zero, we say that a normal variety X has rational singularities if for any resolution $f\colon Y \to X$ and for $i > 0$, $R^i f_* \mathcal{O}_Y = 0$.

In characteristic zero, potentially klt singularities are rational [Elk81, KMM87] and rational singularities are CM [KKMSD73]. Thus:

Theorem 1.5. *In characteristic zero, potentially klt singularities are CM.*

In characteristic $p > 0$, being CM is often included in the definition of rational singularities rather than it is a property. Let us adopt the following definition by Kovács.

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Definition 1.6 ([Kov17]). In characteristic $p > 0$, we say that a normal variety X has rational singularities if for any modification $f\colon Y \to X$ such that Y is CM and for $i > 0$, $R^i f_* \mathcal{O}_Y = 0$.

Theorem 1.7 ([Kov17]). *Potentially klt singularities which are CM are rational. Namely, for potentially klt singularities, being CM is equivalent to being rational.*

2. Non-CM klt singularities

In positive characteristics, potentially klt singularities are not necessarily CM. The existence of non-CM potentially klt singularities had not appeared in the literature until recently, however some specialists had known that such singularities of dimension 7 and characteristic 2 can be constructed from examples of Fano varieties violating Kodaira vanishing from [HL93, LR97]. In the last 6 years, many examples of such singularities appeared in the literature, some in lower dimensions and some in arbitrary characteristics. We now briefly review them in the chronological order.

(1) Yasuda [Yas14] constructed canonical singularities which are not Cohen-Macaulay in arbitrary characteristic. His examples are quotient varieties of affine spaces by linear actions of the cyclic group of order p. The lowest dimension of such examples is 6 in characteristic 2, 5 in characteristic 3 and $\left\lceil \sqrt{2p+\frac{1}{4}}+\frac{1}{2} \right\rceil$ in characteristic $p \geq 5$.
(2) Gongyo, Nakamura and Tanaka [GNT15] constructed a 4-dimensional potentially klt variety X in characteristic 2 such that for a resolution $f\colon Y \to X$, $R^1 f_* \mathcal{O}_Y \neq 0$. (From Kovács' result, X is not CM.)
(3) Cascini and Tanaka [CT16] constructed 3-dimensional non-CM klt singularities in characteristic two.
(4) Kovács [Kov] constructed 7-dimensional non-CM canonical singularities in characteristic two.
(5) Bernasconi [Ber] constructed a 3-dimensional non-CM klt singularity in characteristic 3.
(6) Totaro [Tot] constructed an isolated non-CM terminal singularity in characteristic $p \geq 3$ of dimension $2p+2$.
(7) Yasuda [Yas] showed that quotients of affine spaces by the cyclic group of order p are often even terminal (and non-CM). The lowest dimension of non-CM terminal singularities by this construction is 6 in characteristic 2, 5 in characteristic 3 and $\left\lfloor \sqrt{2p+\frac{1}{4}}+\frac{1}{2} \right\rfloor + 1$ in characteristic $p \geq 5$.
(8) Totaro [Tot] construted an isolated 3-dimensional terminal singularity in characteristic 2. This is the quotient of a smooth 3-fold by a non-linear action of the cyclic group of order 2.

Note that normal surface singularities are always CM. Thus dimension 3 is the lowest possible for non-CM singularities. In the above list, constructions 1, 7 and 8 are by taking quotients of smooth varieties by the cyclic group of order p. The others are constructed from counterexamples of Kodaira or Kawamata-Viehweg vanishing theorem.

On the opposite direction, Hacon-Witaszek proved:

Theorem 2.1 ([HW]). *There exists $p_0 \in \mathbb{N}$ such that in characteristic $p \geq p_0$, every klt 3-fold is CM.*

Problem 2.2. (1) Can we generalize the theorem of Hacon-Witaszek to higher dimensions?
(2) If we could do so, determine the lowest possible p_0 for each dimension.

Results of Totaro and Yasuda mentioned above seem to suggest that the answer to the first problem would be positive.

3. Failure of Kodaira vanishing and cone construction

For a normal projective variety X and an ample line bundle L on X, let $C_a(X,L) := \operatorname{Spec} \bigoplus_{n\geq 0} H^0(X, L^n)$ be the associated affine cone. Whether $C_a(X,L)$ is CM or not is characterized as follows (for instance, see [Kol13]):

Proposition 3.1. *Suppose that X is CM. Then $C_a(X,L)$ is CM if and only if $H^0(X,L^n) = 0$ for every $n \in \mathbb{Z}$ and every $i \in \mathbb{Z}$ with $0 < i < \dim X$.*

Proposition 3.2. *Let D be an effective $\mathbb{Q}$-divisor on X and $\tilde{D}$ be the corresponding $\mathbb{Q}$-divisor on $C_a(X,L)$. Suppose that $-(K_X + D) \sim_{\mathbb{Q}} rL$ for some $r \in \mathbb{Q}$. Then $(C_a(X,L), \tilde{D})$ is terminal (resp. canonical, klt) if and only if (X,D) is terminal (resp. canonical, klt) and $r > -1$ (resp. ≥ -1, > 0).*

In particular, if X is a smooth Fano variety and L is a line bundle with $rL \sim_{\mathbb{Q}} -K_X$ for some integer $r > 1$ (resp. ≥ 1, > 0), then $C_a(X,L)$ is terminal (resp. canonical, klt). If $H^i(L^n) \neq 0$ for some $n \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $0 < i < \dim X$, then $C_a(X,L)$ is non-CM. Note that $H^i(L^n) \neq 0$ for some $n \in \mathbb{Z}$ violates the Kodaira vanishing.

If X is a smooth Fano variety and L is an ample line bundle some power of which violates Kodaira vanishing, then $C_a(X,L)$ is non-CM and potentially klt.

Constructions 2, 3, 4, 5 and 6 in Section 2 are based on Fano type varieties violating Kodaira or Kawamata-Viehweg vanishing and take the

4. Quotient singularities I

An alternative construction of non-CM klt singularities is by means of quotient singularities. Note that in characteristic zero, quotient singularities are klt and hence CM. For the construction, it is enough to consider the case of cyclic group of order p. We first recall construction by Yasuda.

Let $G = \langle g \rangle \cong \mathbb{Z}/p$ be the cyclic group of order p with a generator g. Suppose that G acts on an affine space $V = \mathbb{A}^d_k$ linearly. Considering the Jordan normal form of the action of a generator $g \in G$, we see that the action is determined by sizes of Jordan blocks. Note that the only eigenvalue of g is 1 and a Jordan block is determined by its size. Moreover sizes do not exceed p because of the order of g.

We say that g is a pseudo-reflection if the fixed point locus V^G has codimension one. This is the case exactly when there is one Jordan block of size 2 and all the other blocks have size 1. In this case, we can easily see that the quotient variety V/G is again isomorphic to $\mathbb{A}^d_k$. If g is not a pseudo-reflection, then V/G is singular.

Proposition 4.1 ([ES80])**.** *Suppose that g is not a pseudo-reflection. Then V/G is CM if and only if V^G has codimension 2.*

This holds only if either
(1) there are 2 Jordan blocks of size 2 and all the other blocks have size 1, or
(2) there is 1 Jordan block of size 3 and all the other blocks have size 1.

Thus V/G is almost always non-CM. Let $d_1, \dots, d_l$ be the sizes of Jordan blocks of g and let $D := \sum_i d_i(d_i - 1)/2$.

Proposition 4.2 ([Yas14, Yas])**.** *Suppose that g is not a pseudo-reflection. Then V/G is terminal (resp. canonical, log canonical) if and only if $D > p$ (resp. $\geq p$, $\geq p-1$).*

The outline of the proof is as follows. We can determine if V/G is terminal, canonical or log canonical by looking at the convergence/divergence of the stringy invariant of V/G, which is defined as a certain motivic integral on the arc space of V/G. By the wild McKay correspondence, this is equivalent to the convergence/divergence of weighted counts of Artin-Schreier extensions of $k((t))$. We can compute the latter quite explicitly thanks to the Artin-Schreier theory and determine whether the weighted count converges or diverges.

Example 4.3.
(1) Suppose $p = 2$. If $G = \mathbb{Z}/2$ acts on $V = \mathbb{A}_k^6$ by 3 Jordan blocks of size 2, then $D = 3 > 2$ and $\operatorname{codim} V^G = 3$. Thus V/G is non-CM and terminal.
(2) Suppose $p = 3$. If $G = \mathbb{Z}/3$ acts on $V = \mathbb{A}_k^5$ by one block of size 3 and one of size 2, then $D = 4 > 3$ and $\operatorname{codim} V^G = 3$. Thus V/G is non-CM and terminal.
(3) For $p \geq 5$, suppose that $G = \mathbb{Z}/p$ acts on $V = \mathbb{A}_k^d$ with $d \leq p$ by a single Jordan block. If $d \geq \left\lfloor \sqrt{2p + \frac{1}{4}} + \frac{1}{2} \right\rfloor + 1$, then $D \geq p + 1$ and $\operatorname{codim} V^G = d - 1 \geq 3$. Thus V/G is non-CM and terminal.

5. Quotient singularities II

Lastly we briefly review Totaro's construction of 3-dimensional non-CM terminal singularity in characteristic 2. Let $G = \mathbb{Z}/2$ be the cyclic group of order 2 and let G act on $\mathbb{G}_m^3$ by the involution $(x, y, z) \mapsto (1/x, 1/y, 1/z)$. The only fixed point is $(1, 1, 1)$. From [Fog81] (a result similar to one in [ES80] for (necessarily non-linear) actions which are free outside a unique fixed point), the quotient variety $\mathbb{G}_m^3/G$ is non-CM.

Theorem 5.1 ([Tot])**.** *The variety $\mathbb{G}_m^3/G$ is terminal.*

To prove this, he takes equivariant blowups of $\mathbb{G}_m^3$ until the associated quotient variety becomes smooth. When the quotient variety is smooth is determined by the following criterion.

Proposition 5.2 ([KL13])**.** *Suppose that $G = \mathbb{Z}/p$ acts on a smooth variety X. Then X/G is smooth if and only if the fixed point scheme X^G is a Cartier divisor of X.*

Remark 5.3. We have a $\mathbb{Z}/2$-equivariant embedding $\mathbb{G}_m^3 \hookrightarrow \mathbb{A}_k^6$, where $\mathbb{Z}/2$ acts on $\mathbb{A}_k^6$ by three Jordan blocks of size two. Thus $\mathbb{G}_m^3/(\mathbb{Z}/2)$ is a closed subvariety of the quotient variety $\mathbb{A}_k^6/(\mathbb{Z}/2)$ in Example 4.3, (1). The author expects that we can similarly construct 3-dimensional non-CM terminal singularities in characteristics 3 and 5 as closed subvarieties of $\mathbb{A}_k^5/(\mathbb{Z}/3)$ and $\mathbb{A}_k^4/(\mathbb{Z}/4)$ respectively.

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