ON THE NOETHERIAN PROPERTY OF SYMBOLIC REES RINGS

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1. Definitions of symbolic powers and symbolic Rees rings

Throughout this section, we assume that R is a Noetherian ring and I is a proper ideal of R. Moreover, $\operatorname{Min} I$ denotes the set of minimal prime ideals containing I. We put $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. For any $r \in \mathbb{Z}$, we set

$$\begin{aligned} I^{(r)} &= \bigcap_{P \in \operatorname{Min} I} (I^r R_P \cap R) \\ &= \{\, a \in R \mid \text{There exists } s \in R \setminus \textstyle\bigcup_{P \in \operatorname{Min} I} P \text{ such that } sa \in I^r \} \end{aligned}$$

and call it the r-th symbolic power of I. Then we obviously have $I^{(r)} \supseteq I^r$ and $I^{(r)} \supseteq I^{(r+1)}$ for any $r \in \mathbb{Z}$.

Proposition 1.1. *The following assertions hold.*

(1) *If R is Cohen-Macaulay and I is generated by a regular sequence, we have $I^{(r)} = I^r$ for any $r \in \mathbb{Z}$.*
(2) *If $\sqrt{I} = I$, we have $I^{(r)} = \bigcap_{P \in \operatorname{Min} I} P^{(r)}$ for any $r \in \mathbb{Z}$.*

Let t be an indeterminate. We put

$$\mathscr{R}_s(I) = \sum_{r \in \mathbb{N}_0} I^{(r)} t^r \subset R[t]$$

and call it the symbolic Rees ring of I. Since $I^{(r)} I^{(s)} \subseteq I^{(r+s)}$ for any $r, s \in \mathbb{Z}$, $\mathscr{R}_s(I)$ is a graded subring of $R[t]$. Moreover, we set

$$\mathscr{R}'_s(I) = \sum_{r \in \mathbb{Z}} I^{(r)} t^r \subset R[t, t^{-1}].$$

Theorem 1.2. *The following conditions are equivalent.*

(1) *$\mathscr{R}_s(I)$ is finitely generated.*
(2) *$\mathscr{R}'_s(I)$ is finitely generated.*
(3) *There exists $k \in \mathbb{N}$ such that $I^{(kr)} = (I^{(k)})^r$ for any $r \in \mathbb{Z}$.*

2. Historical background

First, let us recall Nagata's counterexample to Hilbert's 14th problem. Let $R = K[x, y, z]$ be a polynomial ring over a field K. Let $\{\, (\alpha_i : \beta_i : \gamma_i) \,\}_{i=1,\cdots,m}$ be a set of points in $\mathbb{P}^2_K$. We set

$$P_i = I_2 \begin{pmatrix} x & y & z \\ \alpha_i & \beta_i & \gamma_i \end{pmatrix} \in \operatorname{Spec} R$$

and $I_H = \bigcap_{i=1}^m P_i$. Then we have $I_H^{(r)} = \bigcap_{i=1}^m P_i^r$ for all $r \in \mathbb{Z}$. Now, let $K = \mathbb{C}$, and assume that H is consisting of independent generic points, i.e., $\{\alpha_i, \beta_i, \gamma_i\}_{i=1,\cdots,m}$ is algebraically independent over $\mathbb{Q}$.

Theorem 2.1. (Nagata [13], 1954) *The following assertions hold.*

(1) *There exists a polynomial ring S and a group G acting on S such that*

$$S^G \cong \mathscr{R}'_s(I_H).$$

(2) *$\mathscr{R}'_s(I_H)$ is not finitely generated if $m = 4^2, 5^2, 6^2, \ldots$.*

Next, let us recall Cowsik's question. Let $(R, \mathfrak{m})$ be a local ring such that $R/\mathfrak{m}$ is infinite and $\dim R = d > 0$. Let P be a prime ideal of R such that $\dim R/P = 1$.

Theorem 2.2. (Cowsik [2], 1984) *If $\mathscr{R}_s(P)$ is Noetherian, then P is a set theoretic complete intersection.*

Proof. As $\mathscr{R}_s(P)$ is Noetherian, there exists $k \in \mathbb{N}$ such that $P^{(kr)} = (P^{(k)})^r$ for any $r \in \mathbb{Z}$. Let $I = P^{(k)}$. Then we have depth $R/I^r > 0$ for any $r \in \mathbb{N}$. Let

$$\mathscr{F} = \oplus_{r \in \mathbb{N}_0} I^r/\mathfrak{m}I^r$$

be the fiber cone of I. Then by Burch's inequality (cf. [1]),

$$\dim \mathscr{F} \leq \dim R - \inf\{\operatorname{depth} R/I^r\}_{r=1,2,\ldots} \leq d-1.$$

Hence there exist $a_1, \cdots, a_{d-1} \in I$ such that $I^{r+1} = (a_1, \cdots, a_{d-1})I^r$ for $r \gg 0$. Thus we see $P = \sqrt{I} = \sqrt{(a_1, \cdots, a_{d-1})R}$. $\square$

Question 2.3. (Cowsik [2], 1984) *Is $\mathscr{R}_s(P)$ Noetherian if R is a regular local ring and $P \in \operatorname{Spec} R$?*

Example 2.4. *The following is a list of negative answers to Cowsik's question.*

(1) (cf. [15], 1985)) *Roberts gave the first counterexample in the case where* $\dim R = 3$ *using Nagata's counterexample to Hilbert's 14th problem. Unfortunately, in this example, $\hat{P}$ is not a prime ideal in $\hat{R}$.*
(2) (cf. [16], 1990) *Roberts gave another counterexample. In this example, R is complete,* $\dim R = 7$ *and* $\dim R/P = 4$.
(3) (cf. [7], 1994) *Goto, Nishida and Watanage found counterexamples among the ideals defining space monomial curves in the case where the base field has characteristic zero. In their examples, the minimum value of* $\mathrm{e}(R/P)$ *is* 25.
(4) (cf. [4], 2016) *González and Karu extended the class of ideals described in (3). In their examples, the minimum value of* $\mathrm{e}(R/P)$ *is* 7.
(5) (cf. [17], 2017) *Sannai and Tanaka constructed a counterexample in the polynomial ring with 12 variables over any field.*

3. Remarks on a system of parameters for a two dimensional regular local ring

In this section, we assume that $(R, \mathfrak{m})$ is a 2-dimensional regular local ring and a_1, a_2 is an sop for R such that $a_i \in \mathfrak{m}^{r_i}$ for $i = 1, 2$, where $r_i \in \mathbb{N}$. We set

$$\mathscr{R}(R) = \sum_{r=0}^{\infty} \mathfrak{m}^r t^r,$$

which is a graded subring of $R[t]$. Let $\mathscr{R}(R)_+$ be the ideal generated by $\{\mathfrak{m}^r t^r\}_{r=1,2,\ldots}$.

Lemma 3.1. *The following conditions are equivalent.*

(1) $\mathscr{R}(R)_+ = \sqrt{(a_1 t^{r_1}, a_2 t^{r_2})\mathscr{R}(R)}$.
(2) $\mathfrak{m}^r = a_1 \mathfrak{m}^{r-r_1} + a_2 \mathfrak{m}^{r-r_2}$ *for* $r \gg 0$.
(3) $\mathfrak{m}^{2r_1 r_2} = Q\mathfrak{m}^{r_1 r_2}$, *where* $Q = (a_1^{r_2}, a_2^{r_1})R \subset \mathfrak{m}^{r_1 r_2}$.

Proof. All implications (1) ⇒ (2) ⇒ (3) ⇒ (1) can be verified directly. □

The following fact plays a key role in this report.

Lemma 3.2. *We have* $\ell_R(\, R/(a_1, a_2)R)\,) \geq r_1 r_2$, *where the equality holds if and only if*

$$\mathscr{R}(R)_+ = \sqrt{(a_1 t^{r_1}, a_2 t^{r_2})\mathscr{R}(R)}\,.$$

Proof. We put $Q = (a_1^{r_2}, a_2^{r_1})R \subset \mathfrak{m}^{r_1 r_2}$. Then

$$r_1 r_2 \cdot \ell_R(\, R/(a_1, a_2)R\,) = \ell_R(\, R/Q\,) = \mathrm{e}(Q) \geq \mathrm{e}(\mathfrak{m}^{r_1 r_2}) = (r_1 r_2)^2 \cdot \mathrm{e}(\mathfrak{m}) = r_1^2 r_2^2.$$

Therefore the required inequality follows. Moreover,

$$\begin{aligned} \ell_R(\, R/(a_1, a_2)R\,) = r_1 r_2 \quad &\Leftrightarrow \quad \mathrm{e}(Q) = \mathrm{e}(\mathfrak{m}^{r_1 r_2}) \\ &\Leftrightarrow \quad Q \text{ is a reduction of } \mathfrak{m}^{r_1 r_2}. \end{aligned}$$

Consequently, we get the last assertion by 3.1. □

4. Radical ideals of regular local rings of dimension three

Throughout this section, we assume that $(R, \mathfrak{m})$ is a 3-dimensional regular local ring. Moreover, I is an ideal of R such that $\sqrt{I} = I$ and $\dim R/I = 1$. Then $I = \bigcap_{P \in \operatorname{Min} I} P$ and R/I is a CM ring. Furthermore, we have $\operatorname{ht} P = 2$ and $IR_P = PR_P$ for any $P \in \operatorname{Min} I$. Hence, for $f \in R$ and $r \in \mathbb{Z}$, we see that $f \in I^{(r)}$ if and only if $f \in P^r R_P$ for any $P \in \operatorname{Min} I$.

Theorem 4.1. *Let* $\xi_i \in I^{(r_i)}$ *for* $i = 1, 2$, *where* $r_i \in \mathbb{N}$. *Let* $u \in \mathfrak{m}$ *be an sop for* R/I *such that* $\sqrt{(x, \xi_1, \xi_2)R} = \mathfrak{m}$. *Then*

$$\mathrm{e}_{uR}(R/(\xi_1, \xi_2)R) \geq r_1 r_2 \cdot \mathrm{e}_{uR}(R/I).$$

Proof. If $P \in \operatorname{Min} I$, we have $\xi_i \in P^{r_i}R_P$ for $i = 1, 2$. We put $\mathscr{P} = \operatorname{Min}(\xi_1, \xi_2)R \supseteq \operatorname{Min} I$. Then, applying the additive formula of multiplicity and Lemma 3.2, we get

$$\begin{aligned} \mathrm{e}_{uR}(R/(\xi_1,\xi_2)R) &= \sum_{P \in \mathscr{P}} \ell(\, R_P/(\xi_1,\xi_2)R_P\,) \cdot \mathrm{e}_{uR}(R/P) \\ &\geq \sum_{P \in \operatorname{Min} I} \ell(\, R_P/(\xi_1,\xi_2)R_P\,) \cdot \mathrm{e}_{uR}(R/P) \\ &\geq \sum_{P \in \operatorname{Min} I} r_1 r_2 \cdot \mathrm{e}_{uR}(R/P) \\ &= r_1 r_2 \cdot \sum_{P \in \operatorname{Min} I} \ell(\, R_P/IR_P\,) \cdot \mathrm{e}_{uR}(R/P) \\ &= r_1 r_2 \cdot \mathrm{e}_{uR}(R/I). \end{aligned}$$

□

Now we introduce Huneke's Condition. Let $\xi_i \in I^{(r_i)}$ for $i = 1, 2$, where $r_i \in \mathbb{N}$.

Definition 4.2. *If there exists an sop $u \in \mathfrak{m}$ for R/I such that $\sqrt{(u, \xi_1, \xi_2)R} = \mathfrak{m}$ and*

$$(*) \qquad \mathrm{e}_{uR}(R/(\xi_1,\xi_2)R) = r_1 r_2 \cdot \mathrm{e}_{uR}(R/I),$$

we say that ξ_1 and ξ_2 satisfy **HC** *on I.*

Lemma 4.3. *The following conditions are equivalent.*

(1) *ξ_1 and ξ_2 satisfy* **HC** *on I.*
(2) *$\mathfrak{m} = \sqrt{(u, \xi_1, \xi_2)R}$ for any sop $u \in \mathfrak{m}$ for R/I and $(*)$ holds.*
(3) *$I = \sqrt{(\xi_1, \xi_2)R}$ and $\mathscr{R}(R_P)_+ = \sqrt{(\xi_1 t^{r_1}, \xi_2 t^{r_2})\mathscr{R}(R_P)}$ for any $P \in \operatorname{Min} I$.*

Proof. Let $u \in \mathfrak{m}$ be an sop for R/I such that $\sqrt{(u, \xi_1, \xi_2)R} = \mathfrak{m}$. From the proof of Theorem 4.1, we see that

$$\mathrm{e}_{uR}(R/(\xi_1,\xi_2)R) = r_1 r_2 \cdot \mathrm{e}_{uR}(R/I)$$

holds if and only if

$$\operatorname{Min}(\xi_1,\xi_2)R = \operatorname{Min} I \text{ and } \ell(\, R_P/(\xi_1,\xi_2)R_P\,) = r_1 r_2 \text{ for any } P \in \operatorname{Min} I.$$

Of course, $\operatorname{Min}(\xi_1, \xi_2)R = \operatorname{Min} I$ holds if and only if $\sqrt{(\xi_1, \xi_2)R} = I$. Moreover, Lemma 3.2 implies that, for any $P \in \operatorname{Min} I$, $\ell(\, R_P/(\xi_1,\xi_2)R_P\,) = r_1 r_2$ holds if and only if $\mathscr{R}(R_P)_+ = \sqrt{(\xi_1 t^{r_1}, \xi_2 t^{r_2})\mathscr{R}(R_P)}$. Therefore we get (1) ⇒ (3) and (3) ⇒ (2) of Lemma 4.3. The implication (2) ⇒ (1) holds obviously. □

The next result is called the Huneke's criterion.

Theorem 4.4. *(cf.* [11, 12]*) The following conditions are equivalent.*

(1) *$\mathscr{R}_s(I\,)$ is finitely generated.*
(2) *There exist $r_1, r_2 \in \mathbb{N}$ for which we can choose elements $\xi_1 \in I^{(r_1)}$ and $\xi_2 \in I^{(r_2)}$ satisfying* **HC** *on I.*

Huneke's criterion was first found by Huneke (cf. [11]) in the case where I is a prime ideal and $R/\mathfrak{m}$ is infinite. Kurano and Nishida (cf. [12]) gave the generalized version together with a totally different proof for (2) $\Rightarrow$ (1). The assumption that R *is local* is essential for (1) $\Rightarrow$ (2). There exists a *graded version* for (2) $\Rightarrow$ (1), which will be explained in the following. For that purpose, let us recall some basic facts on the localization by the irrelevant maximal ideal.

Let $S = K[x, y, z]$ be the polynomial ring over a field K. We regard S as an $\mathbb{N}_0$-graded ring putting suitable weight on each variable, and set $\mathfrak{n} = (x, y, z)S$. Suppose that $\mathfrak{a}$ is a homogeneous ideal of S such that $\sqrt{\mathfrak{a}} = \mathfrak{a}$ and $\dim S/\mathfrak{a} = 1$. We put $R = S_\mathfrak{n}$ and $I = \mathfrak{a}R$. Then, the basic assumptions on R and I of this section are satisfied. It is easy to see that, for any homogeneous ideal $\mathfrak{b}$ of S, we have $\ell(\, S/\mathfrak{b}\,) = \ell(\, R/\mathfrak{b}R\,)$. Moreover, the following assertions hold.

Proposition 4.5. *For any* $r \in \mathbb{Z}$, $\mathfrak{a}^{(r)}$ *is homogeneous and* $\mathfrak{a}^{(r)}R = I^{(r)}$. *Moreover,* $\mathscr{R}_s(\mathfrak{a}\,)$ *is finitely generated if and only if so is* $\mathscr{R}_s(I\,)$

Let $\xi_i \in \mathfrak{a}^{(r_i)}$ for $i = 1, 2$, where $r_i \in \mathbb{N}$. Then the image of ξ_i in R is in $I^{(r_i)}$.

Definition 4.6. *We say that* ξ_1 *and* ξ_2 *satisfy* **HC** *on* $\mathfrak{a}$, *if the images of those elements in* R *satisfy* **HC** *on* I.

Proposition 4.7. $\mathscr{R}_s(\mathfrak{a}\,)$ *is finitely generated if and only if there exist* $r_1, r_2 \in \mathbb{N}$ *for which we can choose elements* $\xi_1 \in \mathfrak{a}^{(r_1)}$ *and* $\xi_2 \in \mathfrak{a}^{(r_2)}$ *satisfying* **HC** *on* $\mathfrak{a}$.

Let us notice that the elements ξ_1 and ξ_2 satisfying **HC** on $\mathfrak{a}$ are not necessarily homogeneous.

5. Radical ideals of $K[x, y, z]$ generated by homogeneous polynomials

Throughout this section, we assume that $R = K[x, y, z]$ is a polynomial ring over a field K. We put $\mathfrak{m} = (x, y, z)R$ and regard R as an $\mathbb{N}_0$-graded ring setting $\deg x = \deg y = \deg z = 1$. Let I be a homogeneous ideal of R such that $\sqrt{I} = I$ and $\dim R/I = 1$. We put $e = \mathrm{e}(R/I)$. Then $I = \bigcap_{P \in \operatorname{Min} I} P$ and R/I is a homogeneous Cohen-Macaulay ring. Let us regard $\mathscr{R}_s(I\,)$ as an $\mathbb{N}^2$-graded ring. If $f \in [I^{(r)}]_d$, then the degree of $ft^r \in \mathscr{R}_s(I\,)$ is (r, d).

It is obvious that any $P \in \operatorname{Min} I$ is homogeneous and $\operatorname{ht} P = 2$. Hence, for any $P \in \operatorname{Min} I$, we have $IR_P = PR_P$, so

$$\ell_{R_P}(\, R_P/I^{(r)}R_P\,) = \ell_{R_P}(\, R_P/P^rR_P\,) = 1 + 2 + \cdots + r = \frac{r(r+1)}{2}\,.$$

Then, by additive formula of multiplicity, we see

$$\mathrm{e}(R/I^{(r)}) \,= \sum_{P \in \operatorname{Min} I} \ell(\, R_P/I^{(r)}R_P\,)\,\mathrm{e}(R/P) = \,\frac{r(r+1)}{2} \sum_{P \in \operatorname{Min} I} \mathrm{e}(R/P)\,.$$

Thus we get the following result.

Proposition 5.1. $\mathrm{e}(R/I^{(r)}) = \dfrac{r(r+1)}{2} \cdot e$ *for any* $r \in \mathbb{N}$ *.*

Let us notice that $R/I^{(r)}$ is a 1-dimensional graded Cohen-Macaulay ring. Hence $R/I^{(r)}$ has a homogeneous non-zero-divisor of degree one. Therefore

$$[R/I^{(r)}]_d \hookrightarrow [R/I^{(r)}]_{d+1}$$

for any $d \in \mathbb{N}$. By Propsosition 5.1, we see

$$\dim_K [R/I^{(r)}]_d = \frac{r(r+1)}{2} \cdot e$$

for $d \gg 0$, and so

$$\begin{aligned} \dim_K [I^{(r)}]_d &= \dim_K R_d - \dim_K [R/I^{(r)}]_d \\ &\geq \binom{d+2}{2} - e \cdot \frac{r(r+1)}{2} \\ &= \frac{1}{2}\{(d+2)(d+1) - er(r+1)\} \\ &= \frac{1}{2}\{d^2 + 3d + 2 - er(r+1)\}\,. \end{aligned}$$

If $d \geq \sqrt{e}(r+1)$, then $d^2 \geq e(r+1)^2 > er(r+1)$, so $\dim_K [I^{(r)}]_d > 0$. Consequently, we get the following result.

Proposition 5.2. $[I^{(r)}]_d \neq 0$ *for any* $(r, d) \in \mathbb{N}^2$ *satisfying* $d \geq \sqrt{e} \cdot r + \sqrt{e}$*.*

Here, let us introduce the condition **NC** as follows.

Definition 5.3. *We say that* I *satisfies* **NC** *if* $[I^{(r)}]_d = 0$ *for any* $(r, d) \in \mathbb{N}^2$ *satisfying* $d/r \leq \sqrt{e}$*.*

As is well known, if e is not a square number, then $\sqrt{e} \notin \mathbb{Q}$, and so we may replace the inequality $d/r \leq \sqrt{e}$ in Definition 5.3 with $d/r < \sqrt{e}$.

Conjecture 5.4. (Nagata's conjecture) *Let* $K = \mathbb{C}$ *and let* H *be a set of independent generic* m *points in* $\mathbb{P}^2_{\mathbb{C}}$*. If* $m \geq 10$*, then* I_H *satisfies* **NC**.

Nagata himself proved that his conjecture is true if $m = 4^2, 5^2, 6^2, \ldots$ (cf. [13]).

Theorem 5.5. *If* I *satisfies* **NC***, then* $\mathscr{R}_s(I\,)$ *is not finitely generated.*

Proof. Suppose that I satisfies **NC**. Let us take finitely many non zero homogeneous elements $f_1 \in [I^{(r_1)}]_{d_1}, f_2 \in [I^{(r_2)}]_{d_2}, \ldots, f_n \in [I^{(r_n)}]_{d_n}$ arbitrarily, where $r_i, d_i \in \mathbb{N}$ for $i = 1, \ldots, n$. Setting $T = R[\,f_1 t^{r_1}, f_2 t^{r_2}, \ldots, f_n t^{r_n}\,]$, We aim to show $T \subsetneq \mathscr{R}_s(I\,)$.

Let $a = \min\{\,d_1/r_1, d_2/r_2, \ldots, d_n/r_n\,\}$. Since I satisfies **NC**, we have $a > \sqrt{e}$. On the other hand, if $T_{(r,d)} \neq 0$, it follows that $d/r \geq a$. Let us notice that there exists $(r', d') \in \mathbb{N}^2$ such that $a > d'/r' > \sqrt{e}$ and $d' \geq \sqrt{e} \cdot r' + \sqrt{e}$. Then we have $T_{(r',d')} = 0$ and $[I^{(r')}]_{d'} \neq 0$ by Proposition 5.2. Therefore we see $T \subsetneq \mathscr{R}_s(I\,)$ $\square$

The next result is the homogeneous version of Theorem 4.1.

Theorem 5.6. *Suppose $\xi_i \in [I^{(r_i)}]_{d_i}$ for $i = 1, 2$, where $r_i, d_i \in \mathbb{N}$. Assume that ξ_1, ξ_2 is an R-regular sequence. Then we have*

$$\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} \geq e$$

Proof. We may assume that K is infinite. Let us choose sufficiently general element $u \in R_1$. Since $\xi_i \in (IR_{\mathfrak{m}})^{(r_i)}$ for $i = 1, 2$ and u is an sop for $R_{\mathfrak{m}}/(\xi_1, \xi_2)R_{\mathfrak{m}}$, by Theorem 4.1 we get

$$\mathrm{e}_{uR_{\mathfrak{m}}}(R_{\mathfrak{m}}/(\xi_1, \xi_2)R_{\mathfrak{m}}) \geq r_1 r_2 \cdot \mathrm{e}_{uR_{\mathfrak{m}}}(R_{\mathfrak{m}}/IR_{\mathfrak{m}}).$$

The left hand side coincides with $\mathrm{e}(R/(\xi_1, \xi_2)R) = d_1 d_2$. Moreover, we have

$$\mathrm{e}_{uR_{\mathfrak{m}}}(R_{\mathfrak{m}}/IR_{\mathfrak{m}}) = \mathrm{e}(R/I) = e\,.$$

Hence we get the required inequality. $\square$

Here, let us review the condition **HC**.

Lemma 5.7. *Let $\xi_i \in [I^{(r_i)}]_{d_i}$ for $i = 1, 2$, where $r_i, d_i \in \mathbb{N}$. We assume that ξ_1, ξ_2 is an R-regular sequence. Then ξ_1 and ξ_2 satisfy* **HC** *on I if and only if*

$$(*) \qquad \frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = e\,.$$

.
Therefore, by Huneke's criterion we get the next result.

Theorem 5.8. *$\mathscr{R}_s(I\,)$ is finitely generated if there exist $r_1, d_1, r_2, d_2 \in \mathbb{N}$ satisfying the following conditions ;*

(1) *the equality $(*)$ holds, and*
(2) *there exist $\xi_i \in [I^{(r_i)}]_{d_i}$ for $i = 1, 2$ such that ξ_1, ξ_2 is an R-regular sequence.*

Remark 5.9. *Let $\xi_i \in [I^{(r_i)}]_{d_i}$ for $i = 1, 2$, where $r_i, d_i \in \mathbb{N}$. We assume that ξ_1 and ξ_2 satisfy* **HC** *on I, i.e.,*

$$\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = e\,.$$

Then the following two cases can not happen;

$$\text{(i)} \quad \frac{d_i}{r_i} > \sqrt{e} \quad \textit{for } i = 1, 2 \quad ; \quad \text{(ii)} \quad \frac{d_i}{r_i} < \sqrt{e} \quad \textit{for } i = 1, 2.$$

Hence, replacing the subscripts 1 and 2 with each other if necessary, we have

$$\frac{d_1}{r_1} \leq \sqrt{e} \quad \textit{and} \quad \frac{d_2}{r_2} \geq \sqrt{e}.$$

6. Fermat Ideals

Throughout this section, we assume that $R = K[x,y,z]$ is a polynomial ring over a field K. We set $\mathfrak{m} = (x,y,z)R$ and regard R as an $\mathbb{N}_0$-graded ring setting $\deg x = \deg y = \deg z = 1$. Let $3 \le n \in \mathbb{N}$. We assume that $\operatorname{ch} K \nmid n$ if $\operatorname{ch} K > 0$ and there exists a primitive n-th root of unity θ in K.

Let H be the set of the following n^2+3 points in $\mathbb{P}^2_K$;

$$\{\,(1:0:0)\,,\ (0:1:0)\,,\ (0:0:1)\,\} \cup \{\,(1:\theta^i:\theta^j) \mid i,j = 1,2,\dots,n\,\}.$$

Then we have

$$I_H = (y,z)\cap(z,x)\cap(x,y)\cap \bigcap_{i,j=1}^{n} P_{ij}\,,$$

where $P_{ij} = (y-\theta^i x, z-\theta^j x)$. Here we set $f = y^n - z^n$, $g = z^n - x^n$ and $h = x^n - y^n$. Since $f+g+h=0$, we have $(f,g) = (g,h) = (h,f)$. Moreover, we can prove

$$I_H = (xf, yg, zh) \quad \text{and} \quad (f,g) = \bigcap_{i,j=1}^{n} P_{i,j}.$$

Therefore, the following assertion holds.

Lemma 6.1. $I_H^{(r)} = (y,z)^r \cap (z,x)^r \cap (x,y)^r \cap (f,g)^r$ *for any* $r \in \mathbb{Z}$.

Harbourne and Seceleanu proved that $\mathscr{R}_s(I_H)$ is finitely generated if $n=3$ (cf. [9]). Moreover, Nagel and Seceleanu proved that $\mathscr{R}_s(I_H)$ is still finitely generated even if $n \ge 4$ (cf. [14]). Here, we would like to give another proof using Huneke's criterion.

First, let us consider the case where $n=3$. We set

$$\xi_1 = fgh \in [I_H^{(3)}]_9 \ \text{and}\ \xi_2 = xf\cdot yg + yg\cdot zh + zh\cdot xf \in [I_H^2]_8\,.$$

Since $(9/3)\cdot(8/2) = 12 = \mathrm{e}(R/I_H)$, it follows that ξ_1 and ξ_2 satisfies **HC** on I_H by Lemma 5.7. Next, we consider the case where $n \ge 4$. Choosing $\alpha \in K \setminus \{0,1\}$, we set

$$\xi_1 = (fgh)(\alpha f + g)^{n-3}\,\text{and}\,\xi_2 = (xf)^2(yg)^{n-2} + (yg)^2(zh)^{n-2} + (zh)^2(xf)^{n-2} + f^{n-2}gh.$$

Then $\xi_1 \in I_H^{(n)}$ and $\xi_2 \in I_H^n$. Although ξ_2 is not homogeneous, we can prove that ξ_1 and ξ_2 satisfy **HC** on I_H using Lemma 4.3.

7. Ideals of $\mathbb{Z}[x,y,z]$ generated by quasihomogeneous polynomials of type (a,b,c)

Throughout this section, we assume that $S = \mathbb{Z}[x,y,z]$ is a polynomial ring over $\mathbb{Z}$. We put $\mathfrak{n} = (x,y,z)S$. Let K be a field. We set $S_K = K \otimes_{\mathbb{Z}} S = K[x,y,z]$, and for an ideal J of S, we denote JS_K by J_K. Moreover, for an element $\xi \in S$, we denote its image in S_K by ξ_K. Let us regard S and S_K as $\mathbb{N}_0$-graded rings setting $\deg x = a$, $\deg y = b$, $\deg z = c$, where $a,b,c \in \mathbb{N}$. We assume that I is a homogeneous ideal of S such that $\sqrt{xS+I} = \mathfrak{n}$, $\sqrt{I_K} = I_K$ and $\dim S_K/I_K = 1$ for any field K. Finally, throughout this section p denotes a prime number.

Definition 7.1. *Let K be a field, $k \in \mathbb{N}$ and $f \in I_K^{(k)}$. We define*

$$\mathrm{HC}(I_K;k,f) := \{\,\ell \in \mathbb{N} \mid \textit{There exists } g \in I_K^{(\ell)} \textit{ such that } f \textit{ and } g \textit{ satisfy } \mathbf{HC} \textit{ on } I_K\,\}\,.$$

Proposition 7.2. *Let $k = 1$ or 2, and let $f \in I_K^{(k)}$. We assume that there exists $i \in \mathbb{N}$ such that $f \equiv y^i \mod xS_K$ and $\mathrm{HC}(I_K; k, f) \neq \phi$. We set $m = \min \mathrm{HC}(I_K; k, f)$. Then the following assertions hold.*

(1) $\mathrm{HC}(I_K; k, f) = \{m, 2m, 3m, \dots\}$.

(2) $S_K[\, I_K t, I_K^{(2)} t^2, \dots, I_K^{(m-1)} t^{m-1}\,] \subsetneq \mathscr{R}_s(I_K\,)$.

Definition 7.3. *For any $r \in \mathbb{Z}$, we set $I^{(r,x)} = \bigcup_{i=1}^{\infty} (I^r :_R x^i)$.*

If $\xi \in I^{(r,x)}$, it is easy to see $\xi_K \in I_K^{(r)}$ for any field K.

Proposition 7.4. *The following assertions hold.*

(1) $(I_{\mathbb{Q}})^{(r)} = (I^{(r,x)})_{\mathbb{Q}}$ *and* $(I_{\mathbb{F}_p})^{(r)} = (I^{(r,x)})_{\mathbb{F}_p}$ *for* $p \gg 0$.

(2) *Let $\xi \in I^{(k,x)}$ and $\eta \in I^{(\ell,x)}$, where $k, \ell \in \mathbb{N}$. Suppose that $\xi_{\mathbb{Q}}$ and $\eta_{\mathbb{Q}}$ satisfy* **HC** *on $I_{\mathbb{Q}}$. Then $\xi_{\mathbb{F}_p}$ and $\eta_{\mathbb{F}_p}$ satisfy* **HC** *on $I_{\mathbb{F}_p}$ for $p \gg 0$.*

(3) *Suppose that $k \in \mathbb{N}$, $\xi \in I^{(k,x)}$ and $\xi \equiv y^i \mod xS$ for some $i \in \mathbb{N}$. Then we have* $\mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}}) = \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ *for* $p \gg 0$.

Theorem 7.5. *Let $k = 1$ or 2. Let $\xi \in I^{(k,x)}$ and $\xi \equiv y^i \mod xS$ for some $i \in \mathbb{N}$. Suppose that there exists $r \in \mathbb{N}$ such that, for any $p \gg 0$, $rp^{e_p} \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ holds for some $e_p \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) $\mathscr{R}_s(I_{\mathbb{Q}}\,)$ *is finitely generated.*

(2) $\mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}}) \neq \phi$.

(3) $r \in \mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}})$.

(4) $r \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ *for* $p \gg 0$.

Under the assumption of Theorem 7.5, it follows that $\mathscr{R}_s(I_{\mathbb{Q}}\,)$ is not finitely generated if $r \notin \mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}})$.

8. Ideals defining space monomial curves

Throughout this section we assume that $S = \mathbb{Z}[x, y, z]$ is a polynomial ring over $\mathbb{Z}$. We put $\mathfrak{n} = (x, y, z)S$. Let K be a field. We set $S_K = K \otimes_{\mathbb{Z}} S = K[x, y, z]$. Moreover, for an element $\xi \in S$, we denote its image in S_K by ξ_K. Let us regard S and S_K as $\mathbb{N}_0$-graded rings setting $\deg x = a$, $\deg y = b$, $\deg z = c$, where $a, b, c \in \mathbb{N}$.

Let $\varphi : S_K \to K[t]$ be the homomorphism of K-algebras such that $\varphi(x) = t^a$, $\varphi(y) = t^b$ and $\varphi(z) = t^c$. We set

$$\mathfrak{p}_K(a, b, c) = \operatorname{Ker} \varphi\,,$$

which is a homogeneous prime ideal of S_K of height 2. If $\mathfrak{p}_K(a, b, c)$ is not a complete intersection, then it is generated by the maximal minors of a matrix of the following form;

$$(\sharp) \quad \begin{pmatrix} y^{t_3} & z^{u_1} & x^{s_2} \\ z^{u_2} & x^{s_3} & y^{t_1} \end{pmatrix},$$

where $s_2, s_3, t_1, t_3, u_1, u_2$ are positive integers which are determined without depending on the field K (cf. [10]). Let $\mathfrak{p}(a, b, c)$ be the ideal of S generated by the maximal minors of $(\sharp)$. Then we have $\sqrt{xS + \mathfrak{p}(a, b, c)} = \mathfrak{n}$ and $\mathfrak{p}(a, b, c)S_K = \mathfrak{p}_K(a, b, c)$ for any field K.

The ideals defining space monomial curves explained above are deeply related to the defining ideals of certain finite set of points in $\mathbb{P}^2_K$. Let us verify this fact in the case where $K = \mathbb{C}$. We put $\theta_n = e^{2\pi i/n} \in \mathbb{C}$ for $n \in \mathbb{N}$. Let $H(a,b,c)$ be the set of the following points in $\mathbb{P}^2_{\mathbb{C}}$;

$$\{ (\theta_a^i : \theta_b^j : \theta_c^k) \mid i = 1, \dots, a \ ; \ j = 1, \dots, b \ ; \ k = 1, \dots, c \}.$$

Taking new variables X, Y, Z, we set $T = \mathbb{C}[X, Y, Z]$. We consider the defining ideal of $H(a,b,c)$ in T, i.e.,

$$I_{H(a,b,c)} = \bigcap_{i,j,k} I_2 \begin{pmatrix} X & Y & Z \\ \theta_a^i & \theta_b^j & \theta_c^k \end{pmatrix}.$$

Let us regard $S_{\mathbb{C}}$ as a subring of T setting $x = X^a, y = Y^b, z = Z^c$. Then the equality

$$I_{H(a,b,c)}^{(r)} = \mathfrak{p}_{\mathbb{C}}(a,b,c)^{(r)} T$$

holds for any $r \in \mathbb{Z}$ and we have the following.

Proposition 8.1. *$\mathscr{R}_s(I_{H(a,b,c)})$ is finitely generated if and only if so is $\mathscr{R}_s(\mathfrak{p}_{\mathbb{C}}(a,b,c))$.*

It is not so difficult to find concrete examples of $\mathfrak{p}_K(a,b,c)$ whose symbolic Rees algebras are finitely generated by using Huneke's criterion. For example, Huneke himself proved that $\mathscr{R}_s(\mathfrak{p}_K(a,b,c))$ is finitely generated if $\min\{a,b,c\} \le 4$ and $\operatorname{ch} K \neq 2$ (cf. [11]). On the other hand, constructing infinitely generated $\mathscr{R}_s(\mathfrak{p}_K(a,b,c))$ is hard. Goto, Nishida and Watanabe found concrete examples of $\mathfrak{p}_K(a,b,c)$ with infinitely generated symbolic Rees rings for the first time (cf. [7]), and later González and Karu extended such class of ideals much wider (cf. [4]). In the following, we give examples of infinitely generated $\mathscr{R}_s(\mathfrak{p}(a,b,c))$ of new type.

First, we choose $\alpha \in \mathbb{Q}$ with $1 < \alpha < 5/4$ arbitrary. Then, as $2 < (17 - 10\alpha)/(6 - 3\alpha)$, we can choose $\beta \in \mathbb{Q}$ so that $2 < \beta < (17 - 10\alpha)/(6 - 3\alpha)$. Next, we write $\alpha = u_2/u_1$ and $\beta = s_2/s_3$, taking $u_2, u_1, s_2, s_3 \in \mathbb{N}$ suitably. Let $t_1 = t_3 = 1$ and $a = 2u_1 + u_2$, $b = s_3u_2 + s_2u_1 + s_2u_2$, $c = s_2 + 2s_3t_1$.

Example 8.2. ([12]) *If* $\mathrm{GCD}\{a,b,c\} = 1$, *then $\mathfrak{p}_K(a,b,c)$ is minimally generated by the maximal minors of the matrix ($\sharp$) stated above for any field K. We can find $\xi \in \mathfrak{p}(a,b,c)^{(2,x)}$ satisfying the following conditions ;*

(i) $\xi \equiv y^3$ *mod* xS,
(ii) *for any prime number p, $3p^{e_p} \in \mathrm{HC}(\mathfrak{p}_{\mathbb{F}_p}(a,b,c); 2, \xi_{\mathbb{F}_p})$ if $e_p \gg 0$, and*
(iii) $3 \notin \mathrm{HC}(I_{\mathbb{Q}}; 2, \xi_{\mathbb{Q}})$.

Consequently, it follows that $\mathscr{R}_s(\mathfrak{p}_{\mathbb{Q}}(a,b,c))$ is not Noetherian.

The simplest case is $\alpha = 6/5$ and $\beta = 49/24$. In this case $a = 16$, $b = 683$ and $c = 97$. In order to explain what is new about the example stated above, let us recall the notion of negative curve (cf. [3]). First, we have to consider the irreducible decomposition of elements in $[\,\mathfrak{p}_K(a,b,c)^{(r)}\,]_d$, where $r, d \in \mathbb{N}$. We put $R = S_K$ and $P = \mathfrak{p}_K(a,b,c)$. Let $\xi \in [P^{(r)}]_d \setminus P^{(r+1)}$.

Lemma 8.3. *Let $\xi = \xi_1\xi_2 \cdots \xi_s$, where $\xi_i \in [R]_{d_i}$ for $i = 1, 2, \cdots, s$. We set $r_i = \max\{\ell \mid \xi_i \in P^{\ell} R_P\}$. Then the following assertions hold.*

(1) $\xi_i \in [P^{(r_i)}]_{d_i}$ *for* $i = 1, 2, \cdots, s$.
(2) $r_1 + r_2 + \cdots + r_s = r$ *and* $d_1 + d_2 + \cdots + d_s = d$.
(3) *If* $d/r < \alpha \in \mathbb{R}$, *then* $d_i/r_i < \alpha$ *for some* $i = 1, 2, \ldots, s$.

Proof. The assertion (1) and $d_1+d_2+\cdots+d_s = d$ is obvious. We get $r_1+r_2+\cdots+r_s = r$ by considering the initial forms of $\xi_1, \cdots, \xi_s$ in the associated graded ring of R_P, which is an integral domain. Suppose that $d_i/r_i \ge \alpha$ for any $i = 1, 2, \cdots, s$. Then we have

$$d = d_1 + d_2 + \cdots + d_s \ge \alpha(r_1 + r_2 + \cdots + r_s) = \alpha r\,,$$

which means $d/r \ge \alpha$. □

Definition 8.4. *Let* $\xi \in [\,\mathfrak{p}_K(a,b,c)^{(r)}\,]_d$, *where* $r, d \in \mathbb{N}$. *If* ξ *is irreducible and*

$$\frac{d}{r} < \sqrt{abc}\,,$$

ξ *is called a negative curve.*

Theorem 8.5. *Assume that* a, b, c *are pairwise coprime and* abc *is not a square number. Then there exists a negative curve if* $\mathscr{R}_s(\mathfrak{p}_K(a,b,c)\,)$ *is finitely generated.*

Proof. Let us give an algebraic proof in the case where $K = \mathbb{C}$. We put $R = S_{\mathbb{C}}$, $P = \mathfrak{p}_{\mathbb{C}}(a,b,c)$ and $H = H(a,b,c) \subset P^2_{\mathbb{C}}$. Let us define I_H to be an ideal of $T = \mathbb{C}[X,Y,Z]$. We assume that $\mathscr{R}_s(P\,)$ is Noetherian. Then $\mathscr{R}_s(I_H\,)$ is also Noetherian. Hence I_H does not satisfy **NC** by Theorem 5.5. This means that there exist $r, \delta \in \mathbb{N}$ such that $[I_H^{(r)}]_\delta \ne 0$ and $\delta/r < \sqrt{abc}$ ($\delta/r = \sqrt{abc}$ can not happen as abc is not a square number). Since $I_H^{(r)} = P^{(r)}T$, we have $[P^{(r)}]_d \ne 0$ for some $d \in \mathbb{N}$ with $d \le \delta$. Let us notice $d/r \le \delta/r < \sqrt{abc}$.

Now we take an element $0 \ne \xi \in [P^{(r)}]_d$. Let $\xi = \xi_1\xi_2\cdots\xi_s$ be the irreducible decomposition, where $\xi_i \in [R]_{d_i}$. We set $r_i = max\{\ell \mid \xi \in P^\ell R_P\}$ for $i = 1, 2, \cdots, s$. By Lemma 8.3, we have $d_i/r_i < \sqrt{abc}$ for some $i = 1, 2, \cdots, s$. Then ξ_i is a negative curve as $\xi_i \in [P^{(r_i)}]_{d_i}$. □

Cutkosky proved that the converse of Theorem 8.5 holds if $\operatorname{ch} K > 0$.

Theorem 8.6. *We assume that* a, b, c *are pairwise coprime. Let* $\xi_i \in [\,\mathfrak{p}_K(a,b,c)^{(r_i)}\,]_{d_i}$ *for* $i = 1, 2$, *where* $r_i, d_i \in \mathbb{N}$. *Let* ξ_1, ξ_2 *be an* S_K*-regular sequence. Then the following assertions hold.*
(1) $\dfrac{d_1}{r_1} \cdot \dfrac{d_2}{r_2} \ge abc$.
(2) *The equality holds in* (1) *if and only if* ξ_1 *and* ξ_2 *satisfies* **HC** *on* $\mathfrak{p}_K(a,b,c)\,$.

If $K = \mathbb{C}\,$, Theorem 8.6 follows from Theorem 5.5, Lemma 5.6 and Proposition 8.3 as $\mathrm{e}(T/I_{H(a,b,c)}) = \sharp H(a,b,c) = abc\,$.

The following result explains the uniqueness of negative curve.

Theorem 8.7. *We assume that* a, b, c *are pairwise coprime. Let* $\xi_i \in [\mathfrak{p}_K(a,b,c)^{(r_i)}]_{d_i}$ *for* $i = 1, 2$, *where* $r_i, d_i \in \mathbb{N}$. *If both* ξ_1 *and* ξ_2 *are negative curves, then* $\xi_1 \sim \xi_2$.

Proof. Suppose that ξ_i is a negative curve for $i = 1, 2$ and $\xi_1 \not\sim \xi_2$. Then, as $d_i/r_i < \sqrt{abc}$ for $i = 1, 2$, we have $(d_1/r_1)(d_2/r_2) < abc$. On the other hand, ξ_1, ξ_2 is S_K-regular as ξ_i is irreducible for $i = 1, 2$. So, by TheoremrefT8.7 we have $(d_1/r_1)(d_2/r_2) \geq abc$, which is impossible. Therefore the required assertion follows. □

Let $\mathfrak{p}_{\mathbb{Q}}(a, b, c)$ be one of the examples found by Goto, Nishida, Watanabe (cf. [7]) and González, Karu (cf. [4]). Then it has a negative curve in the first symbolic power.

Example 8.8. (cf. [12]) *Let $\mathfrak{p}_{\mathbb{Q}}(a, b, c)$ be the example given in Example 8.2. Let ξ be the element in $\mathfrak{p}(a, b, c)^{(2, x)}$ used for proving that $\mathscr{R}_s(\mathfrak{p}_{\mathbb{Q}}(a, b, c)\,)$ is infinitely generated. Then, for any field K, $\xi_K \in \mathfrak{p}_K(a, b, c)^{(2)}$ and it is a negative curve.*

For example, if $P = \mathfrak{p}_K(16, 683, 97)$, then $\xi_K \in [P^{(2)}]_{2049}$. One can check $2049/2 < \sqrt{16 \cdot 683 \cdot 97}$ directly.

Recently, for any $k \in \mathbb{N}$, González and Karu found examples of $\mathfrak{p}_{\mathbb{Q}}(a, b, c)$ such that $\mathscr{R}_s(\mathfrak{p}_{\mathbb{Q}}(a, b, c)\,)$ is infinitely generated and there exists a negative curve in $\mathfrak{p}_{\mathbb{Q}}(a, b, c)^{(k)}$ (cf. [5]).

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