NONCOMMUTATIVE QUOTIENT SINGULARITIES

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Abstract. Tilting objects play a key role in the study of triangulated categories. A remarkable result due to Iyama and Takahashi asserts that the stable categories of graded maximal Cohen-Macaulay modules over quotient singularities have tilting objects. This paper proves a noncommutative generalization of Iyama and Takahashi's theorem using noncommutative algebraic geometry. Namely, if S is a noetherian AS-regular Koszul algebra and G is a finite group acting on S such that S^G is a "Gorenstein isolated singularity", then the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object. In particular, the category $\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)$ is triangle equivalent to the derived category of a finite dimensional algebra.

1. Preliminaries

This is a report on the joint work with K. Ueyama [6], [7]. Throughout this paper, k denotes an algebraically closed field. Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded algebra. The category of graded right A-modules is denoted by $\mathrm{GrMod}\, A$ and the full subcategory of $\mathrm{GrMod}\, A$ consisting of finitely presented modules is denoted by $\mathrm{grmod}\, A$. The category of graded left A-modules is identified with $\mathrm{GrMod}\, A^o$ where A^o is the opposite graded algebra of A. For $M = \oplus_{i\in\mathbb{Z}} M_i \in \mathrm{GrMod}\, A$ and $n \in \mathbb{Z}$, we define $M_{\geq n} \in \mathrm{GrMod}\, A$ by $M_{\geq n} := \oplus_{i\geq n} M_i$, and $M(n) \in \mathrm{GrMod}\, A$ by $M(n)_i = M_{n+i}$. For $M, N \in \mathrm{GrMod}\, A$, we write $\mathrm{Ext}^i_A(M,N) := \mathrm{Ext}^i_{\mathrm{GrMod}\, A}(M,N)$ and $\underline{\mathrm{Ext}}^i_A(M,N) := \oplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_A(M, N(n))$.

If A is graded right coherent, then $\mathrm{grmod}\, A$ is an abelian category, and, in this case, we define the quotient category $\mathrm{tails}\, A := \mathrm{grmod}\, A/\mathrm{tors}\, A$ where $\mathrm{tors}\, A$ is the full subcategory of $\mathrm{grmod}\, A$ consisting of finite dimensional modules over k. Following [2], we will view $\mathrm{tails}\, A$ as the noncommutative projective scheme associated to A, since if A is commutative and generated in degree 1 over k, then $\mathrm{tails}\, A$ is equivalent to the category of coherent sheaves on $\mathrm{Proj}\, A$ by Serre. If A is not commutative, then we will write $X = \mathrm{Proj}_{nc}\, A$ so that $\mathrm{tails}\, A = \mathrm{coh}\, X$ is the category of "coherent sheaves" on an "imaginary ringed space" X.

We say that a graded algebra A is connected graded if $A_0 = k$, and locally finite if $\dim_k A_i < \infty$ for all $i \in \mathbb{N}$. An AS-regular algebra defined below is the most important algebra studied in noncommutative algebraic geometry.

Definition 1.1. [1] A locally finite connected graded algebra A is called AS-regular (resp. AS-Gorenstein) over k of dimension d and of Gorenstein parameter ℓ if the following conditions are satisfied:

(1) $\mathrm{gldim}\, A = d < \infty$ (resp. $\mathrm{injdim}_A\, A = \mathrm{injdim}_{A^o}\, A = d < \infty$), and

(2) $\underline{\mathrm{Ext}}^i_A(k,A) \cong \underline{\mathrm{Ext}}^i_{A^o}(k,A) \cong \begin{cases} k(\ell) & \text{if } i = 0 \\ 0 & \text{if } i \neq d. \end{cases}$

If S is a right noetherian AS-regular algebra of dimension d, then we may view $\mathrm{tails}\, S$ as a quantum projective space of dimension $d-1$ since S is a commutative noetherian AS-regular algebra of dimension d if and only if $S \cong k[x_1, \dots, x_d]$.

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Let A be a graded algebra. We denote by $\operatorname{GrAut} A$ the group of all graded algebra automorphisms of A. If $G \leq \operatorname{GrAut} A$ is a subgroup, then the fixed subalgebra $A^G := \{a \in A \mid g(a) = a \text{ for every } g \in G\}$ of A by G is a graded subalgebra of A. If S is a right noetherian AS-regular algebra and $G \leq \operatorname{GrAut} S$ is a finite subgroup, then we may view S^G as a (homogeneous coordinate ring of) a noncommutative quotient singularity, since if $S = k[x_1, \dots, x_d]$ is the polynomial algebra (the only noetherian commutative AS-regular algebra), then $\operatorname{Spec} S^G \cong \mathbb{A}^d/G$ is a quotient singularity. The purpose of this paper is to study noncommutative quotient singularities.

Triangulated categories are increasingly important in many areas of mathematics including algebraic geometry and representation theory. There are two major classes of triangulated categories, namely, (bounded) derived categories $\mathcal{D}^b(\mathcal{C})$ of abelian categories $\mathcal{C}$, and stable categories $\underline{\mathcal{C}}$ of Frobenius categories $\mathcal{C}$. For example, derived categories $\mathcal{D}^b(\operatorname{coh} X)$ of coherent sheaves on schemes X have been extensively studied in algebraic geometry, and stable module categories $\underline{\operatorname{mod}}\Lambda$ of self-injective algebras Λ have been extensively studied in representation theory of finite dimensional algebras. Let A be a noetherian AS-Gorenstein algebra. A graded right A-module M is called maximal Cohen-Macaulay if $\underline{\operatorname{Ext}}^i_A(M, A) = 0$ for every $i \geq 1$. The full subcategory of $\operatorname{grmod} A$ consisting of graded maximal Cohen-Macaulay modules is denoted by $\operatorname{CM}^{\mathbb{Z}}(A)$. Note that $\operatorname{CM}^{\mathbb{Z}}(A)$ is a Frobenius category, so the stable category $\underline{\operatorname{CM}}^{\mathbb{Z}}(A)$ is a triangulated category.

In the study of triangulated categories, tilting objects play a key role.

Definition 1.2. Let $\mathcal{T}$ be a triangulated category. An object $T \in \mathcal{T}$ is called tilting if

(1) $\operatorname{thick}(T) = \mathcal{T}$, and
(2) $\operatorname{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i \neq 0$.

A tilting object plays an essential role in this paper due to the following result. It often enables us to realize abstract triangulated categories as concrete derived categories of modules over algebras.

Theorem 1.3. (cf. [4, Theorem 2.2]) *Let $\mathcal{T}$ be an algebraic Krull-Schmidt triangulated category and $T \in \mathcal{T}$ a tilting object. If $\operatorname{gldim} \operatorname{End}_{\mathcal{T}}(T) < \infty$, then $\mathcal{T} \cong \mathcal{D}^b(\operatorname{mod} \operatorname{End}_{\mathcal{T}}(T))$.*

One of the remarkable results on the existence of tilting objects has been obtained by Iyama and Takahashi.

Theorem 1.4. [4, Theorem 2.7, Corollary 2.10] *Let $S = k[x_1, \dots, x_d]$ be a polynomial algebra over an algebraically closed field k of characteristic 0 such that $\deg x_i = 1$ for all i and $d \geq 2$. Let G be a finite subgroup of $\operatorname{SL}(d, k)$ acting linearly on S, and S^G the fixed subalgebra of S. Assume that S^G is an isolated singularity. If we define the graded S^G-module*

$$T := \bigoplus_{i=1}^{d} \Omega^i_S k(i),$$

then the stable category $\underline{\operatorname{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object $[T]_{\operatorname{CM}}$, where $[T]_{\operatorname{CM}}$ is the maximal direct summand of T which is a graded maximal Cohen-Macaulay module. As a consequence, there exists a finite dimensional algebra Γ of finite global dimension such that

$$\underline{\operatorname{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^b(\operatorname{mod} \Gamma).$$

The stable categories of graded maximal Cohen-Macaulay modules are crucial objects studied in representation theory of algebras and also attract attention from the viewpoint of Kontsevich's homological mirror symmetry conjecture. The aim of the present paper is to generalize Theorem 1.4 to the noncommutative setting using noncommutative algebraic geometry.

2. Main Results

Let A be a graded algebra, and $G \le \operatorname{GrAut} A$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$. In this case, the group algebra kG is a semisimple algebra. The skew group algebra of A by G is a graded algebra $A * G = \oplus_{i \in \mathbb{N}} A_i \otimes_k kG$ with the multiplication $(a \otimes g)(b \otimes h) := ag(b) \otimes gh$. An element of $A * G$ is often denoted by $a * g := a \otimes g$. Two idempotent elements

$$e := \frac{1}{|G|} \sum_{g \in G} g, \quad \text{and} \quad e' := 1 - e$$

of kG play crucial roles in this paper. Since $kG \subset A * G$, we often view e, e' as idempotent elements of $A * G$. Moreover, since $e(A * G)e \cong A^G$ as graded algebras, we usually identify $e(A * G)e$ with A^G.

In [5], Jørgensen and Zhang defined the homological special linear group $\operatorname{HSL}(S) \le \operatorname{GrAut} S$ for a noetherian AS-regular algebra S, and proved that if $G \le \operatorname{HSL}(S)$ is a finite subgroup, then S^G is a noetherian AS-Gorenstein algebra. On the other hand, in [9], Ueyama introduced a notion of graded isolated singularity for noncommutative graded algebras, which agrees with the usual notion of isolated singularity if the algebra is commutative and generated in degree 1. For a noetherian AS-regular algebra S over k and a finite subgroup $G \le \operatorname{GrAut} S$, it was shown in [6] that the condition that $S * G/(e)$ is finite dimensional over k is closely related to the noncommutative graded isolated singularity property of S^G.

Theorem 2.1. [6, Theorem 3.10] *Let S be a noetherian AS-regular algebra over k. For a finite subgroup $G \le \operatorname{HSL}(S)$, $S * G/(e)$ is finite dimensional over k if and only if S^G is a noncommutative graded isolated singularity and $S * G \cong \underline{\operatorname{End}}_{S^G}(S)$.*

If $S = k[x_1, \dots, x_d]$ is the polynomial algebra with $\deg x_i = 1$ for all i, then $\operatorname{tails} S \cong \operatorname{coh} \mathbb{P}^{d-1}$, and it is well-known that $\mathcal{O}_{\mathbb{P}^{d-1}}, \mathcal{O}_{\mathbb{P}^{d-1}}(1), \dots, \mathcal{O}_{\mathbb{P}^{d-1}}(d-1)$ is a full strong exceptional sequence for $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^{d-1})$ so that $\bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^{d-1}}(i)$ is a tilting object for $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^{d-1})$. Suppose that a finite group G acts on a noetherian AS-regular algebra S over k of Gorenstein parameter ℓ so that the noncommutative projective scheme $X = \operatorname{Proj}_{nc} S$ associated to S is viewed as a quantum projective space. The inclusion map $f : S^G \to S$ induces a functor $f_* : \operatorname{tails} S \to \operatorname{tails} S^G$. If G is non-trivial, then $f_*\mathcal{O}_X, f_*\mathcal{O}_X(1), \dots, f_*\mathcal{O}_X(\ell-1)$ is no longer an exceptional sequence for $\mathcal{D}^b(\operatorname{tails} S^G)$ where $\mathcal{O}_X$ is the "structure sheaf" on X, however, the following result shows that $\bigoplus_{i=0}^{\ell-1} f_*\mathcal{O}_X(i)$ is a tilting object for $\mathcal{D}^b(\operatorname{tails} S^G)$ if S^G is an "isolated singularity".

Theorem 2.2. [6, Theorem 3.14] *Let S be a noetherian AS-regular algebra over k of dimension $d \ge 2$ and of Gorenstein parameter ℓ, and $G \le \operatorname{GrAut} S$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k, then*

$$\bigoplus_{i=0}^{\ell-1} f_*\mathcal{O}_X(i)$$

is a tilting object in $\mathcal{D}^b(\operatorname{tails} S^G)$ where $X = \operatorname{Proj}_{nc} S$ and $f : S^G \to S$ is the inclusion.

There exists another full strong exceptional sequence $\Omega^{d-1}_{\mathbb{P}^{d-1}}(d-1), \dots, \Omega^1_{\mathbb{P}^{d-1}}(1), \Omega^0_{\mathbb{P}^{d-1}}$ for $\mathcal{D}^b(\operatorname{coh} \mathbb{P}^{d-1})$ so that $\bigoplus_{i=0}^{d-1} \Omega^i_{\mathbb{P}^{d-1}}(i)$ is a tilting object. In the setting of the above theorem, if G is non-trivial, then $f_*\Omega^{d-1}_X(d-1), \dots, f_*\Omega^1_X(1), f_*\Omega^0_X$ is no longer an exceptional sequence for $\mathcal{D}^b(\operatorname{tails} S^G)$ where Ω^i_X is the "sheaf of differential i-forms" on X, however, we will show in this paper that $\bigoplus_{i=0}^{d-1} f_*\Omega^i_X(i)$ is a tilting object for $\mathcal{D}^b(\operatorname{tails} S^G)$ if S is Koszul.

Definition 2.3. Let A be a graded algebra. A linear resolution of $M \in \operatorname{GrMod} A$ is a graded projective resolution of M

$$\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$$

where each P^i is a graded projective right A-module generated in degree i.

A locally finite graded algebra A is called Koszul if A_0 is a semisimple algebra, and $A_0 := A/A_{\geq 1} \in \operatorname{GrMod} A$ has a linear resolution.

Note that an AS-regular algebra of dimension d and of Gorenstein parameter ℓ is Koszul if and only if $d = \ell$. In particular, the polynomial algebra $k[x_1, \dots, x_d]$ is Koszul if and only if $\deg x_i = 1$ for all i.

Theorem 2.4. [7, Theorem 3.20] *Let S be a noetherian AS-regular Koszul algebra over k of dimension $d \geq 2$, and $G \leq \operatorname{GrAut} S$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k, then*

$$\bigoplus_{i=0}^{d-1} f_* \Omega^i_X(i)$$

is a tilting object in $\mathcal{D}^b(\operatorname{tails} S^G)$ where $X = \operatorname{Proj}_{nc} S$ and $f : S^G \to S$ is the inclusion. As a consequence, there exists a finite dimensional algebra Λ of finite global dimension such that

$$\mathcal{D}^b(\operatorname{tails} S^G) \cong \mathcal{D}^b(\operatorname{mod} \Lambda).$$

We define a graded right $S * G$-module U by

$$U := \bigoplus_{i=1}^{d} \Omega^i_{S*G} kG(i).$$

Using Theorem 2.4, we will show the existence of a tilting object of the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)$ if S^G is a "Gorenstein isolated singularity". The main result of this paper is as follows.

Theorem 2.5. [7, Theorem 4.10, Theorem 4.17] *Let S be a noetherian AS-regular Koszul algebra over k of dimension $d \geq 2$, and $G \leq \operatorname{HSL}(S)$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k, then*

$$e'Ue$$

is a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)$. As a consequence, there exists a finite dimensional algebra $\Gamma = e'\Lambda e'$ of finite global dimension such that

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^b(\operatorname{mod} \Gamma).$$

If S is a commutative AS-regular Koszul algebra of dimension d, then $S \cong k[x_1, \dots, x_d]$ with $\deg x_i = 1$ for all i. In this case,

- $\operatorname{HSL}(S)$ coincides with $\operatorname{SL}(d, k)$.
- $S * G/(e)$ is finite dimensional over k if and only if S^G is a (graded) isolated singularity (see [6, Corollary 3.11]).
- $e'Ue = [T]_{\mathrm{CM}}$ (see [4, Proof of Theorem 2.9]).

Thus it follows that our result is a noncommutative generalization of Theorem 1.4. However, our proof is different from the original one given in [4]. Thanks to Theorem 2.4, we can give a more conceptual proof using a diagram of triangulated categories.

3. Flow of the Proofs

In this last section, we give a list of results needed and a flow diagram for the proofs of Theorem 2.4 and Theorem 2.5.

First, we will give properties of a "Gorenstein singularity". Let A be a noetherian graded algebra. The graded singularity category is defined by the Verdier localization $\mathcal{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) := \mathcal{D}^b(\operatorname{grmod} A)/\mathcal{D}^b(\operatorname{grproj} A)$ where $\operatorname{grproj} A$ is the full subcategory of $\operatorname{grmod} A$ consisting of projective modules. We denote the localization functor by $\upsilon : \mathcal{D}^b(\operatorname{grmod} A) \to \mathcal{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$.

Theorem 3.1. *Let A be a noetherian AS-Gorenstein algebra over k of Gorenstein parameter ℓ.*

(1) (Buchweitz equivalence [3]) $\mathcal{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \cong \underline{\mathrm{CM}}^{\mathbb{Z}} A$.
(2) (Orlov embedding [8]) *If $\ell > 0$, then there exists an embedding $\Phi : \mathcal{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \to \mathcal{D}^b(\operatorname{tails} A)$.*

Next, we give a property of a graded Frobenius algebra.

Definition 3.2. A locally finite graded algebra A is called graded Frobenius of Gorenstein parameter $-\ell$ if $\underline{\operatorname{Hom}}_k(A, k) \cong A(\ell)$ in $\operatorname{GrMod} A$.

The tilting object below was essentially obtained by Yamaura [10].

Lemma 3.3. (Yamaura tilting object [7, Lemma 3.11]) *If A is a graded Frobenius algebra of Gorenstein parameter $-\ell$ such that $\operatorname{gldim} A_0 < \infty$, then $\oplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1}$ is a tilting object for $\underline{\operatorname{grmod}} A$.*

Next, we give the following generalization of BGG correspondence. For a graded algebra A, we define $A^! := \oplus_{i\in\mathbb{N}} \underline{\operatorname{Ext}}^i_A(A_0, A_0)$. Note that if A is a Koszul algebra, then $(A^!)^! \cong A$ as graded algebras.

Proposition 3.4. (BGG correspondence [7, Proposition 3.3]) *If A is a Frobenius Koszul algebra of Gorenstein parameter $-\ell$ such that $A^!$ is graded right coherent, then there exists an equivalence*

$$K : \mathcal{D}^b(\operatorname{grmod} A) \to \mathcal{D}^b(\operatorname{grmod} A^!)$$

of triangulated categories, which induces an equivalence

$$\overline{K} : \underline{\operatorname{grmod}} A \to \mathcal{D}^b(\operatorname{tails} A^!)$$

of triangulated categories.

Finally, we give a property of an "isolated singularity".

Proposition 3.5. [7, Proposition 3.18] *Let A be a right noetherian connected graded algebra, and $G \leq \operatorname{GrAut} A$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$. Then the following are equivalent:*

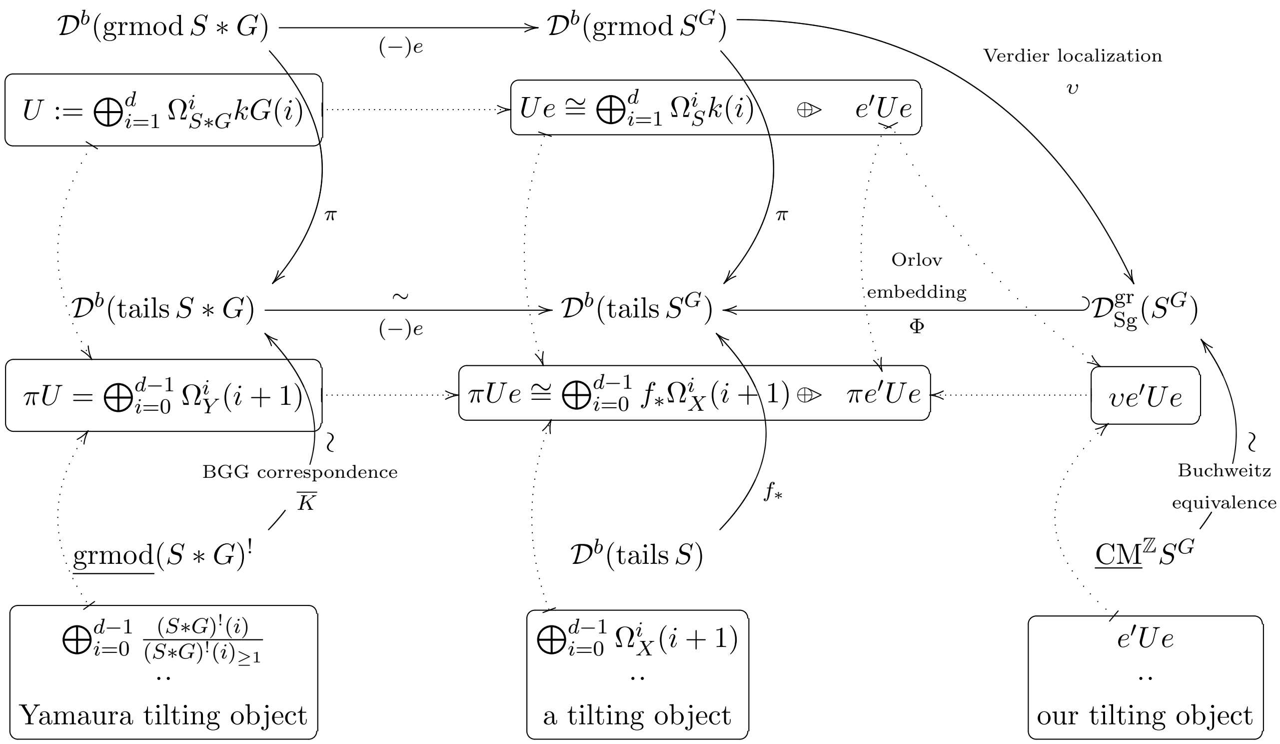
(1) *$A * G/(e)$ is finite dimensional over k.*
(2) *$(-)e : \operatorname{tails} A * G \to \operatorname{tails} A^G$ is an equivalence functor.*

In the setting of Theorem 2.5, $(S * G)^!$ is a Frobenius Koszul algebra of Gorenstein parameter $-d$ ([7, Proposition 2.27]) such that $((S * G)^!)^! \cong S * G$ is noetherian. The key point of our proof for Theorem 2.4 is to show that under the equivalences

$$\underline{\operatorname{grmod}}(S * G)^! \xrightarrow[\overline{K}]{\sim} \mathcal{D}^b(\operatorname{tails} S * G) \xrightarrow[(-)e]{\sim} \mathcal{D}^b(\operatorname{tails} S^G),$$

the tilting object in $\underline{\operatorname{grmod}}(S * G)^!$ obtained in Lemma 3.3 corresponds to the object $\bigoplus_{i=0}^{d-1} f_* \Omega^i_X(i)$ in $\mathcal{D}^b(\operatorname{tails} S^G)$ ([7, Corollary 3.15]).

The following is a diagram of triangulated categories which are essential for the proofs of Theorem 2.4 and Theorem 2.5.



where $X = \operatorname{Proj}_{nc} S$, $Y = \operatorname{Proj}_{nc} S * G$, and $M \oplus\!\!> N$ means that N is a direct summand of M.

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