Unbounded polytopes and toric type cusp singularities

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Introduction

In the theory of toric varieties, a fundamental result is the fact that a toric variety of dimension r with an ample invertible sheaf corresponds to a convex polytope with integral vertices in $\mathbf{R}^r$. In this note, we define quasi-polyhedral sets in $\mathbf{R}^r$ as a generalization of convex polytopes. For a quasi-polyhedral set P, the cc-dimension is defined as the dimension of the characteristic cone of P. A convex polytope is the case of cc-dimension zero. We call P a quasi-polytope if every proper face of P is bounded.

We recall the theory of cusp singularities defined by Tsuchihashi in this view point. This cusp singularity is defined for a pair of an open convex cone C and a discrete linear group Γ acting on it. Since the cusp singularity is constructed by contracting a toric divisor, it is important to consider the singularity with the toric resolution. In Section 4, we describe the construction over an arbitrary field by using a formal scheme, and algebraize the toric resolution to a scheme morphism. In Section 5, we consider a quasi-polytope of maximal cc-dimension with a group action. Such a quasi-polytope gives a cusp singularity if the action satisfies some conditions. Finally, in Section 6, we introduce beautiful examples obtained by Tsuchihashi recently. The four-dimensional example has a simple normal crossing exceptional divisor consisting of four irreducible components with 48 quadruple points.

1 Quasi-polyhedral sets

Let r be a non-negative integer and let M, N be mutually dual free $\mathbf{Z}$-modules of rank r. We denote $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, which are real spaces of dimension r. Then there exists a natural perfect bilinear map

$$\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \longrightarrow \mathbf{R}\ .$$

Although M and N have standard roles in the theory of toric varieties, we may exchange the roles in the study of dualities (cf. [I3]). Points in M and N are called *lattice points*, and those in $M_{\mathbf{Q}}$ and $N_{\mathbf{Q}}$ are *rational points*.

Since each $u \in N_{\mathbf{R}}$ is a linear function of $M_{\mathbf{R}}$ by the bilinear map, $\{x \in M_{\mathbf{R}} ; \langle x, u\rangle \geq a\}$ is a closed half space of $M_{\mathbf{R}}$ for every $a \in \mathbf{R}$ if $u \neq 0$. We denote it by $(u \geq a)$. The closed half space $(u \leq a)$ and open half spaces $(u > a)$, $(u < a)$ as well as the hyperplane $(x = a)$ are defined similarly. We will use this notation also for $u = 0$, where the set is not a half space nor a hyperplane.

A non-empty subset $C \subset M_{\mathbf{R}}$ is called a *polyhedral cone* if there exist $x_1, \ldots, x_s \in M_{\mathbf{R}}$ with $C = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_s$, where $\mathbf{R}_0 = \{c \in \mathbf{R} ; c \geq 0\}$. It is known that C is also expressed as $(u_1 \geq 0) \cap \cdots \cap (u_t \geq 0)$ with $u_1, \ldots, u_t$ in $N_{\mathbf{R}}$ (cf. [O, A.1]). We say C is *rational* if $x_1, \ldots, x_s$, or equivalently $u_1, \ldots, u_t$, are rational points. In this case, we can take these points in M and N, respectively.

For a subset $E \subset M_{\mathbf{R}}$ and $x \in M_{\mathbf{R}}$, we denote $E - x = \{y - x ; y \in E\}$. A subset E is said to be *locally polyhedral at* x if $E - x$ is equal to a polyhedral cone in a neighborhood of the origin. This is equivalent to the condition that $x \in E$ and E is equal to $(u_1 \geq a_1) \cap \cdots \cap (u_t \geq a_t)$ for some $u_1, \ldots, u_t \in N_{\mathbf{R}}$ and $a_1, \ldots, a_t \in \mathbf{R}$ in a neighborhood of x. If $\dim E = r$, then we may assume that $\langle x, u_i\rangle = a_i$ and u_i defines an $(r-1)$-dimensional face of the polyhedral cone for every i by reducing redundant members. A non-empty convex subset P is called a *quasi-polyhedral set* if P is locally polyhedral at every point $x \in \overline{P}$. Then it follows that $P = \overline{P}$, i.e., P is closed. A quasi-polyhedral set P is *rational* if $u_1, \ldots, u_t$ and $a_1, \ldots, a_t$ above are rational for all x. A non-empty subset Q of a quasi-polyhedral set $P \subset M_{\mathbf{R}}$ is called a *face* if there exist $u \in N_{\mathbf{R}}$ and $a \in \mathbf{R}$ such that $P \subset (u \geq a)$ and $Q = P \cap (u = a)$. If $\dim P = r$ and P has an irredundant expression $(u_1 \geq a_1) \cap \cdots \cap (u_t \geq a_t)$ at a point $x \in P$, then P is contained in $(u_i \geq a_i)$ and $P \cap (u_i = a_i)$ is a face of dimension $r - 1$ for each i. We call P a *quasi-polytope* if every proper face of P is bounded.

For a non-empty closed convex set D, the *characteristic cone* $\mathrm{cc}(D)$ is defined by

$$\mathrm{cc}(D) = \{y \in M_{\mathbf{R}} ; x + \mathbf{R}_0 y \subset D\}$$

for $x \in D$ (cf. [G, p.24]). This is a closed convex cone which does not depend on the choice of x since we assume D closed. We define *cc-dimension* by $\text{cc-dim}(D) = \dim \mathrm{cc}(D)$ which has a value between 0 and r, and is zero if and only if D is bounded (cf. [G, p.24]).

2 Open cone with lattice

We fix Euclidean metrics on the real spaces $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$. The metrics are used in the proof of Theorem 2.3 and in the definition of the characteristic function of a cone.

Let C be an open convex cone in $N_{\mathbf{R}}$, i.e., C is the interior of a full-dimensional closed convex cone in $N_{\mathbf{R}}$. We assume the closure $\overline{C}$ of C is strongly convex. Then the dual cone $\overline{C}^{\vee}$ in $M_{\mathbf{R}}$ is also a strongly convex closed cone of dimension r. We set $C^* = \mathrm{int}(\overline{C}^{\vee})$, which is an open convex cone in $M_{\mathbf{R}}$. Note that if $x \in C^*$ then $\{u \in \overline{C} ; \langle x, u\rangle \leq a\}$ is bounded for any $a \geq 0$. Actually, there exist linearly independent $x_1, \ldots, x_r \in C^*$ with $x = x_1 + \cdots + x_r$ since C^* is an open convex cone. Then the set is contained in $(x_1 \geq 0) \cap \cdots \cap (x_r \geq 0) \cap (x_1 + \cdots + x_r \leq a)$ which is clearly bounded.

For a subset $S \subset C^*$, we set

$$K(S) = \{u \in N_{\mathbf{R}} \; ; \; \langle x, u\rangle \geq 1 \text{ for all } x \in S\} = \bigcap_{x\in S}(x \geq 1) \; .$$

Clearly, $K(S)$ is a closed convex set of $N_{\mathbf{R}}$ which might be empty.

Lemma 2.1 *Assume that $S \subset C^*$ is discrete in $M_{\mathbf{R}}$. Let u be a point of C. Then (1) u is outside $K(S)$ if $S \cap (u < 1) \neq \emptyset$, (2) $K(S)$ is locally equal to the convex set $\bigcap_{x\in S\cap(u=1)}(x \geq 1)$ at u if $S \cap (u < 1) = \emptyset$. A point u is in the interior of $K(S)$ if $S \cap (u \leq 1) = \emptyset$. In particular, $K(S)$ is locally polyhedral at every point of $\overline{K(S)} \cap C$.*

Proof If there exists $x \in S \cap (u < 1)$, then u is outside $K(S) \subset (x \geq 1)$. Assume $S \cap (u < 1) = \emptyset$. Set $S_1 = S \cap (u = 1)$ and $S_2 = S \cap (u > 1)$. Since $u \in C = \operatorname{int}(\overline{C})$, $C^* \cap (u < c)$ is bounded and $S \cap (u < c)$ is a finite set for any $c > 0$. Hence S_1 is finite, and there exists $a = \min\{\langle x, u\rangle \; ; \; x \in S_2\} > 1$ if $S_2 \neq \emptyset$. Then $a^{-1}u \in K(S_2)$ and $a^{-1}u + C \subset K(S_2)$. Since $a^{-1}u + C$ is an open set which contains $u = a^{-1}u + (1 - a^{-1})u$ and $K(S) = K(S_1) \cap K(S_2)$, $K(S)$ is equal to $K(S_1) = \bigcap_{x\in S_1}(x \geq 1)$ in this neighborhood of u. It is locally polyhedral since S_1 is finite. QED

Lemma 2.2 *Let $A \subset N_{\mathbf{R}}$ be a bounded closed convex subset and $u_0 \in N_{\mathbf{R}}$ a point such that the convex hull $B = \operatorname{conv}(A \cup \{u_0\})$ is of dimension r. Let D be the cone generated by $A - u_0$. Then, for any subsets $E \subset N_{\mathbf{R}}$ and $F \subset (D + u_0) \setminus B$, we have*

$$B \cap \operatorname{conv}(A \cup E) = B \cap \operatorname{conv}(A \cup E \cup F) \; .$$

In particular, if $\operatorname{conv}(A \cup E)$ is a polyhedron, then $P = \operatorname{conv}(A \cup E \cup F)$ is locally polyhedral at each point of $\overline{P} \cap \operatorname{int}(B)$.

Proof By a translation, we may assume $u_0 = 0$. If $0 \in A$, then $B = A$ and the assertion is obvious. We assume $0 \notin A$. Then $B \setminus A$ is an open subset of D (see Remark 2.3). Let u be a point of $B \cap \operatorname{conv}(A \cup E \cup F)$. It suffices to show that u is in $\operatorname{conv}(A \cup E)$. We may assume $u \notin A$. Since $u \in \operatorname{conv}(A \cup E \cup F)$, there exist $s > 0$, $v_1, \ldots, v_s \in A \cup E \cup F$ and $a_1, \ldots, a_s > 0$ with

$$a_1(v_1 - u) + \cdots + a_s(v_s - u) = 0 \; .$$

If $v_i \in F$ for an i, then take the maximal $c_i \geq 0$ with $v'_i = u + c_i(v_i - u) \in B$. Clearly $c_i < 1$ since $v_i \notin B$. Since $u + c'(v_i - u) \in D \setminus B$ for $c_i < c' \leq 1$, $v'_i \in B$ is in the closure of $D \setminus B$, and is in A since $B \setminus A$ is open in D. In particular, c_i is positive. Namely, we can replace $a_i(v_i - u)$ by $(a_i/c_i)(v'_i - u)$ in the equality. If we do it for all v_i's in F, we get an equality which says that u is in $\operatorname{conv}(A \cup E)$. QED

Remark 2.3 Here we prove this fact. Let w be a point in $B \setminus A$. Then there exist $p \in A$ and $0 \leq a < 1$ with $w = ap$. Since A is closed, there exists an open convex neighborhood U of w in $N_{\mathbf{R}}$ which does not intersect A. Suppose that U contains a point z of $D \setminus B$. Then there exist $q \in A$ and $b > 1$ with $z = bq$. For the real numbers $0 \leq a < 1$, $b > 1$, the equation $ta + (1 - t)b = 1$ has a solution $0 < t < 1$. Then $tw + (1 - t)z = tap + (1 - t)bq$ is in $U \cap A$, which is a contradiction since $U \cap A = \emptyset$. Hence $w \in U \cap D \subset B \setminus A$, which means $B \setminus A$ is open in D.

Let S_C be the set of elements $m \in M \cap C^*$ such that there exists $u \in C$ with $\langle m, u\rangle = 1$ and $\langle m', u\rangle > 1$ for $m' \in (M \cap C^*) \setminus \{m\}$. If $(m_1 \geq 1) \cap \cdots \cap (m_t \geq 1)$ is the irredundant expression of $K(M \cap C^*)$ at a point u, then $m_1, \ldots, m_t$ are in S_C. We see easily that $K(M \cap C^*) \cap C = K(S_C) \cap C$, and hence $K(M \cap C^*) = K(S_C)$ as closures. We set $Q_m = K(S_C) \cap (m = 1)$ for $m \in S_C$, which is locally equal to the hyperplane $(m = 1)$ at u in the definition of S_C. Then, we get one-to-one correspondences $m \mapsto Q_m$ between S_C and the set of codimension one faces of $K(S_C)$. $N \cap (\overline{C} \setminus \{0\})$ is contained in $K(S_C)$ since $\langle x, u\rangle$ is a positive integer for $x \in M \cap C^*$ and $u \in N \cap (\overline{C} \setminus \{0\})$. In particular, $K(S_C)$ is not necessarily contained in C (cf. [AMRT, II, 5.3]).

Theorem 2.4 *Let Θ be the convex hull of $N \cap C$. If C contains $K(S_C)$ and Θ contains $K_d = dK(S_C)$ for a positive integer d, then Θ is a locally polyhedral closed subset of $N_{\mathbf{R}}$. The vertices of Θ are in $N \cap C$.*

Proof Since $N \cap C \subset K(S_C)$, Θ is a subset of the closed convex set $K(S_C)$. Since we assume $K(S_C) \subset C$, the closure of Θ is contained in C. Hence it suffices to show that Θ is locally polyhedral at every point $u \in \overline{\Theta}$ with assuming $u \in C \setminus \mathrm{int}(K_d)$. Set $S_1 = \{m \in S_C \ ; \ \langle m, u\rangle \leq d\}$ and $S_2 = S_C \setminus S_1$. Then $a = \min\{\langle m, u\rangle \ ; \ m \in S_2\}$ is greater than d.

We set $b = d/a$, which is a positive number less than 1. Since $(1/d)u$ is outside $\mathrm{int}\, K(S_C)$, S_1 is not empty by Lemma 2.1. Let A_0 be the union of $(r-1)$-dimensional polytopes $K_d \cap (m = d)$ for $m \in S_1$.

We set $u' = bu$ and will show that $E = \{v \in N \cap C \ ; \ \overline{u'v} \cap K_d = \emptyset\}$ is finite. If it failed, we get a sequence $\{v_i\}$ from this set such that $\lim_{i\to\infty} |v_i| = \infty$ and $|v_i|^{-1} v_i$ converges to a unit vector w. Since v_i's are in C, w is in $\overline{C} \setminus \{0\}$. Hence $\langle m, w\rangle > 0$ for every $m \in S_C$. Since $\langle m, u' + cw\rangle = \langle m, u'\rangle + c\langle m, w\rangle$ and $\langle m, u'\rangle < d$ for $m \in S_1$, there exists $c > 0$ such that $\min\{\langle m, u' + cw\rangle \ ; \ m \in S_1\} = d$. Note that $\langle m, u'\rangle = b\langle m, u\rangle \geq ba = d$ and $\langle m, u' + cw\rangle > d$ for $m \in S_2$. Hence $u' + cw$ is a point of A_0 which is not on $K_d \cap (m = d)$ for any $m \in S_2$. Furthermore, $u' + c'w$ is a point of $\mathrm{int}(K_d)$ for $c' > c$. Since w is also the limit of $w_i = |v_i|^{-1}(v_i - u')$, and since $v_i = u' + |v_i| w_i$, the segment $\overline{u'v_i}$, which contains $u' + c'w_i$ if $c' < |v_i|$, intersects K_d for large i. This is a contradiction.

Assume that $v \in N \cap C$ and $\overline{u'v}$ intersects K_d. Let $c \geq 0$ be the minimal number with $v' = u' + c(v - u') \in K_d$. Since $K_d = dK(S_1) \cap dK(S_2)$ and $\overline{u'v'} \subset dK(S_2)$, there exists $m \in S_1$ with $\langle m, v'\rangle = d$, and hence v' is an intersection point of $\overline{u'v}$ and A_0. Let $A = \mathrm{conv}(A_0)$ and $B = \mathrm{conv}(A \cup \{u'\})$. Then u is in the interior of B unless $S_1 = \{m_0\}$ and u is on $K_d \cap (m_0 = d)$. In this case u is in the interior of Θ or locally defined by $(m_0 \geq d)$ at u. We assume u is in the interior of B. Set $F = (N \cap C) \setminus E$. Then by applying Lemma 2.2 for $u_0 = u'$, $\Theta = \mathrm{conv}(E \cup F)$ is locally polyhedral at each point of $\mathrm{int}\, B \cap \overline{\Theta}$, in particular, at u. If u is a vertex of Θ, then $u \in E \subset N \cap C$. QED

The *characteristic function* ϕ of a strongly convex open cone C is defined by

$$\phi(u) = \int_{C^*} \exp(-\langle x, u\rangle) dx$$

for $u \in C$. Important properties of ϕ are written and proved in Vinberg [V1, §2]. In particular, $\phi(u)$ is a positive valued differentiable convex function satisfying $\phi(\lambda u) =$

$\lambda^{-r}\phi(u)$ for $\lambda > 0$, here r is the dimension of $N_{\mathbf{R}}$. Furthermore, by defining $\phi(u) = \infty$ for $u \in \overline{C} \setminus C$, the map $\phi : \overline{C} \to (0, \infty]$ is continuous, and $\{u \in C \ ; \ \phi(u) \leq a\}$ is a closed convex subset of $N_{\mathbf{R}}$ for every $a > 0$.

A subgroup Γ of $\mathrm{GL}(N)$ is also considered as the subgroup $\{{}^{\mathrm{t}}g^{-1} \ ; \ g \in \Gamma\}$ of $\mathrm{GL}(M)$, and acts on both $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ linearly from the left. Namely, the equality $\langle g(x), g(u)\rangle = \langle x, u\rangle$ holds for $g \in \Gamma$, $x \in M_{\mathbf{R}}$ and $u \in N_{\mathbf{R}}$. We say Γ acts on C if $g(C) = C$ for all $g \in \Gamma$. Then Γ acts also on C^*. Since $g(M) = M$, Γ acts on $K(S_C)$ and $K(S_C) \cap C$. Note that $\det(g) = \pm 1$ is uniquely defined for $g \in \Gamma$.

Let $C/\mathbf{R}_+$ be the set of half lines $\{\mathbf{R}_+ u \ ; \ u \in C\}$ with the topology as the quotient space of C. For any $\lambda > 0$, $\{u \in C \ ; \ \phi(u) = \lambda\}$ is homeomorphic to $C/\mathbf{R}_+$. The following projective transformation maps C to the cylindrical area $\mathbf{R}_+ \times (C/\mathbf{R}_+)$.

Take points $n_0 \in C$ and $x_0 \in C^*$ with $\langle x_0, u_0\rangle = 1$. Let $p : N_{\mathbf{R}} \to N_{\mathbf{R}}/\mathbf{R}n_0$ be the natural surjection, and let $D_C = p(C \cap (x_0 = 1))$. Then, the map

$$q : C \longrightarrow \mathbf{R}_+ \times D_C \ , \ \ q(u) = \left(\frac{1}{\langle x_0, u\rangle}, \frac{p(u)}{\langle x_0, u\rangle}\right) \ ,$$

is a homeomorphic projective transformation. For $x \in C^*$ and $a > 0$, the subset $C \cap (x \geq a)$ is mapped to $\{(t, v) \in \mathbf{R}_+ \times D_C \ ; \ at \leq l_x(v)\}$, where l_x is the affine function on $N_{\mathbf{R}}/\mathbf{R}n_0$ such that $l_x(p(u)) = \langle x, u\rangle$ for $u \in (x_0 = 1)$. Since $x \in C^*$, there exist $m_x, M_x > 0$ with $m_x \leq l_x \leq M_x$. For $S \subset C^*$, $q(K(S) \cap C) = \{(t, v) \ ; \ t \leq l_x$ for all $x \in S\}$. We regard D_C as the quotient $C/\mathbf{R}_+$ through this homeomorphism. If a linear automorphism g of $N_{\mathbf{R}}$ fixes the cone C, then g induces a homeomorphism on D_C which is compatible with that on C. We see ϕq^{-1} is also a differentiable convex function, which satisfies $\phi q^{-1}(\lambda t, v) = \lambda^r \phi q^{-1}(t, v)$ for $\lambda > 0$. In particular, $\{u \in C \ ; \phi(u) = a\}$ is homeomorphic to D_C for any $a > 0$. Furthermore, q extends to a homeomorphism $\overline{C} \setminus \{0\} \to \mathbf{R}_+ \times \overline{D_C}$.

Lemma 2.5 *If a subgroup $\Gamma \subset \mathrm{GL}(N)$ acts on C and the quotient D_C/Γ is compact, then the condition of Theorem 2.4 is satisfied.*

Proof Let $\partial K(S_C)$ be the boundary of $K(S_C)$. Then Γ acts on $\partial K(S_C) \cap C$ which is naturally homeomorphic to $C/\mathbf{R}_+$. Since ϕ is constant on each orbit of Γ and $(C/\mathbf{R}_+)/\Gamma$ is compact, ϕ is bounded on $\partial K(S_C) \cap C$ and has the maximum λ. Let u be a point in the interior of $K(S_C)$. Then $a = \min\{\langle m, u\rangle \ ; \ m \in S_C\} > 1$ and $(1/a)u \in \partial K(S_C) \cap C$. Hence $\phi(u) = a^{-r}\phi((1/a)u) < \lambda$. Since $K(S_C)$ is the closure of its interior as a convex set, ϕ is at most λ on $K(S_C)$. Hence $K(S_C) \subset \phi^{-1}((0, \lambda])$ is contained in C. The vertices of $K(S_C)$ are rational points and form a finite number of orbits since D_C/Γ is compact. Hence there exists $d > 0$ such that all vertices of $dK(S_C)$ are in N. Since $K(S_C)$ is the convex hull of the union of proper faces, and hence of vertices, $dK(S_C)$ is contained in the convex hull of $N \cap C$. QED

3 Toric type cusp singularity

Let $C \subset N_{\mathbf{R}}$ be an open convex cone such that $\overline{C}$ is strongly convex. In this section, we assume that a group $\Gamma \subset \mathrm{GL}(N)$ acts on C and the quotient D_C/Γ is compact. When the action of Γ on C is free, a singularity, which we call a *toric type cusp singularity*, is constructed by Tsuchihashi [T1, Proposition 1.7] (see also [AMRT, p.162, Appendix]). In this paper, we call such pair (C, Γ) a *Tsuchihashi pair* if the action is free. We will discuss on Γ-invariant fans and their blowups.

We define the *canonical fan* Π by the convex closure Θ of $N \cap C$ as follows. Let $F(\Theta)$ be the set of proper faces of Θ. We get the following lemma by Theorem 2.4.

Lemma 3.1 *Each $Q \in F(\Theta)$ is a polytope whose vertices are points of $N \cap C$.*

Since $Q \in F(\Theta)$ is a polytope in a hyperplane $(x = a)$ with $x \in C^*$ and $a > 0$, $\mathbf{R}_0 Q$ is a rational polyhedral cone generated by the set of the vertices of Q. Define

$$\Pi = \{\mathbf{R}_0 Q \ ; \ Q \in F(\Theta)\} \cup \{\mathbf{0}\}$$

where $\mathbf{0} = \{0\}$, i.e., the zero cone. The following lemma is easy.

Lemma 3.2 *Under the assumption of this section, Π is a fan of $N_{\mathbf{R}}$ with the support $C \cup \{0\}$. The action of Γ on C induces an action on $\Pi \setminus \{\mathbf{0}\}$ such that $(\Pi \setminus \{\mathbf{0}\})/\Gamma$ is finite and every stabilizer is finite. The action is free if and only if that on C is free.*

Similarly, we can also define a fan Π_0 from the Γ-invariant quasi-polytope $K(S_C)$. We consider the case that Γ acts on a fan Σ of $N_{\mathbf{R}}$ with the support $C \cup \{0\}$ which is locally finite at each point of C. Then Σ is said to be Γ-*invarinat* or Γ-*admissible* if $(\Sigma \setminus \{\mathbf{0}\})/\Gamma$ is finite (cf. [AMRT, Chapter 2]). In our case, this finite condition follows from the compactness of D_C/Γ.

Let Σ be a Γ-invariant fan. A *support function* of Σ is a real-valued function h on $C \cup \{0\}$ such that the restriction to each $\sigma \in \Sigma$ is linear and $\mathbf{Z}$-valued on $N \cap \sigma$, i.e., there exists $m_\sigma \in M$ with $h = m_\sigma$ as functions on σ. We call it a support $\mathbf{Q}$-function if we weaken the last condition to $\mathbf{Q}$-valued on $N \cap \sigma$. A support function h is continuous on C since Σ is locally finite. We say h is *convex* if $h(u + v) \geq h(u) + h(v)$ for any $u, v \in C$, and *strictly convex* if $h(u + v) > h(u) + h(v)$ for u, v which are not in a common cone of Σ. For example, $h(u) = \min\{\langle x, u\rangle \ ; \ x \in M \cap C^*\}$ is a strictly convex support function of Π_0.

For an element $\rho \in \Sigma \setminus \{\mathbf{0}\}$, we set $\Sigma(\rho\prec) = \{\tau \in \Sigma \ ; \ \rho \prec \tau\}$. Let $\rho \in \Sigma$ be an element of dimension at least two such that $\Sigma(\rho\prec) \cap g(\Sigma(\rho\prec)) = \emptyset$ for every $g \in \Gamma \setminus \{1\}$. Let u be an element of $N \cap \operatorname{rel.int} \rho$. For each $\tau \in \Sigma(\rho\prec)$, we set

$$F(\tau, \rho) = \{\sigma \prec \tau \ ; \ u \notin \sigma, (\mathbf{R}_0 u + \sigma) \cap \operatorname{rel.int} \tau \neq \emptyset\} \ ,$$

which does not depend on the choice of u. Then, the Γ-*equivariant blowup*, or *star subdivision*, $\mathrm{Bl}_{\Gamma,u} \Sigma$ of Σ at u is defined by

$$\mathrm{Bl}_{\Gamma,u} \Sigma = (\Sigma \setminus \bigcup_{g \in \Gamma} g(\Sigma(\rho\prec))) \cup (\bigcup_{g \in \Gamma} g(\Delta)) \ ,$$

where

$$\Delta = \{\mathbf{R}_0 u + \sigma \ ; \ \sigma \in F(\tau, \rho), \tau \in \Sigma(\rho\prec)\} \ .$$

Note that, if Γ acts on Σ freely, then $\rho \in \Sigma$ of dimension r satisfies the condition since $\Sigma(\rho\prec) = \{\rho\}$. Clearly, $\mathrm{Bl}_{\Gamma,u}\Sigma$ is Γ-invariant. The barycentric subdivision of Σ is done by iterating the blowups for all elements of dimensions from r to 2 in Σ in this order. Namely, let $\overline{\Sigma}$ be a set of representatives of Σ/Γ, and take a primitive element $u_\rho \in N \cap \mathrm{rel.\,int}\,\rho$ for all $\rho \in \overline{\Sigma}$. The Γ-equivariant blowups at u_ρ for all $\rho \in \overline{\Sigma}(r)$ do not depend on the order, and all cones of $\overline{\Sigma}\setminus\overline{\Sigma}(r)$ remain in the obtained fan. Furthermore, cones in $\overline{\Sigma}(r-1)$ satisfy the condition in the new fan. Thus we can blowup Σ at all cones of dimension greater than one. A subdivision of Σ to a non-singular fan can also be done by these blowups if we take u_ρ's properly.

Lemma 3.3 *If Σ has a strictly convex support function h, then $\mathrm{Bl}_{\Gamma,u}\Sigma$ has also a strictly convex support function.*

Proof Let $U = \bigcup_{\tau\in\Sigma(\rho\prec)} \mathrm{rel.\,int}\,\tau$. If $v \in U$ is in $\mathrm{rel.\,int}\,\tau$ and $v = au + u'$ with $a \in \mathbf{R}_0$ and $u' \in \sigma \in F(\tau,\rho)$, then define $l(v) = a$. For $v \in C$, we set $l(v) = l(g^{-1}(v))$ if there exists $g \in \Gamma$ with $v \in g(U)$, and define $l(v) = 0$, otherwise. Then l is a support $\mathbf{Q}$-function on $\mathrm{Bl}_{\Gamma,u}\Sigma$ which is strictly convex on the subdivision of each $\tau \in \Sigma(\rho\prec)$, while h is linear on these cones. Now, we replace l by a multiple cl of an integer $c > 0$ so that l has integral values on $N \cap C$. Then the finiteness of Σ/Γ implies that, for a sufficiently large positive integer d, $dh + l$ is a strictly convex support function of $\mathrm{Bl}_{\Gamma,u}\Sigma$. QED

We assume that Σ has a Γ-invariant strictly convex support function h. For each $\gamma \in \Sigma(1)$, denote the associated prime divisor by $V(\gamma)$. Since the toric variety $Z(\Sigma)$ is not of finite type, a divisor on it may be an infinite sum. Namely, an infinite sum $D = \sum a_\gamma V(\gamma)$ is a *Cartier divisor* if the restriction $D|U(\sigma)$ to the affine toric variety $U(\sigma)$ is principal for every $\sigma \in \Sigma \setminus \{\mathbf{0}\}$. A Cartier divisor $D = \sum a_\gamma V(\gamma)$ is Γ-invariant if $a_\gamma = g(a_\gamma)$ for all $\gamma \in \Sigma(1)$ and $g \in \Gamma$. The associated invertible sheaf $\mathcal{O}_{Z(\Sigma)}(D)$ is also Γ-invariant if D is so. For a support function h of Σ, the associated Cartier divisor is defined by $D_h = -\sum h(n_\gamma)V(\gamma)$, where n_γ is the primitive generator of γ (cf. [O, p.69]). When h has only non-negative values, the coefficients of D_h are non-positive, i.e., $\mathcal{O}_{Z(\Sigma)}(D_h)$ is an ideal sheaf.

Lemma 3.4 *For a strictly convex support function h, the restriction of $\mathcal{O}_{Z(\Sigma)}(D_h)$ to $V(\gamma)$ is ample for all $\gamma \in \Sigma(1)$.*

Proof We set $N(\gamma) = N \cap (\gamma + (-\gamma))$ and $N[\gamma] = N/N(\gamma)$. Then $V(\gamma)$ is the $(r-1)$-dimensional complete toric variety defined by the complete fan $\Sigma[\gamma] = \{\sigma[\gamma]\ ;\ \sigma \in \Sigma(\gamma\prec)\}$ of $N[\gamma]_{\mathbf{R}}$, where $\sigma[\gamma]$ is the image of σ in $N[\gamma]_{\mathbf{R}} = N_{\mathbf{R}}/N(\gamma)_{\mathbf{R}}$ (cf. [O, Corollary 1.7]). There exists an element $m_0 \in M$ such that $h = m_0$ on γ. Then $\{(h - m_0)|\sigma\ ;\ \sigma \in \Sigma(\gamma\prec)\}$ induces a strictly convex support function $\bar{h}$ of $\Sigma[\gamma]$ which defines an invertible sheaf isomorphic to $\mathcal{O}_{Z(\Sigma)}(D_h)|V(\gamma)$. Hence it is ample by [O, Corollary 2.14]. QED

4 Power series ring

We fix a field k of an arbitrary characteristic from this section.

Let σ be a strongly convex rational polyhedral cone of $N_{\mathbf{R}}$, and let $\{n_1, \dots, n_s\}$ be the set of primitive generators of the one-dimensional faces. In particular, $\sigma^\vee = (n_1 \geq 0) \cap \dots \cap (n_s \geq 0)$. We consider the topology of the ring $k[M \cap \sigma^\vee]$ defined by the ideals

$$I_d = \langle \mathbf{e}(m) \; ; m \in M \cap (n_1 \geq d) \cap \dots \cap (n_s \geq d) \rangle_k$$

for $d \geq 0$. We denote by $k[M \cap \sigma^\vee]^\wedge$ the completion of $k[M \cap \sigma^\vee]$ with respect to this topology.

We denote by $\langle\langle M \rangle\rangle_k$ the k-vector space $\prod_{m \in M} k\mathbf{e}(m)$, which is not a ring if $r \geq 1$. An element of $\langle\langle M \rangle\rangle_k$ is written as an infinite sum $\sum a_m \mathbf{e}(m)$. We regard $k[M \cap \sigma^\vee]^\wedge$ a vector subspace for every cone σ. Then $\sum a_m \mathbf{e}(m)$ is in $k[M \cap \sigma^\vee]^\wedge$ if and only if $a_m = 0$ for $m \notin M \cap \sigma^\vee$ and there exist only finite m with $a_m \neq 0$ outside $m_0 + M \cap \sigma^\vee$ for every $m_0 \in M \cap \sigma^\vee$. Note that $\langle\langle M \rangle\rangle_k$ has a structure of $k[M]$-module.

Let (C, Γ) be a Tsuchihashi pair. We consider a Γ-invarinat fan Σ satisfying the following conditions.

(1) For any $\sigma, \tau \in \Sigma \setminus \{\mathbf{0}\}$, there exist at most one $g \in \Gamma$ with $g(\sigma) \cap \tau \neq \mathbf{0}$. In particular, $g(\sigma) \neq \sigma$ if $g \neq 1$.

(2) There exists a strictly upper convex Γ-invariant support $\mathbf{Q}$-function h on Σ, i.e., $h(g(u)) = h(u)$ for $u \in C$ and $g \in \Gamma$, $h(u + u') \geq h(u) + h(u')$ for $u, u' \in C$ and the equality holds if and only if u and u' are in a common cone $\sigma \in \Sigma$, and $h(u)$ are rational for all $u \in N \cap C$.

Since $(\Sigma \setminus \{\mathbf{0}\})/\Gamma$ is finite, we may assume that $h(u) \in \mathbf{Z}$ for every $u \in N \cap C$ by replacing h by dh for a positive integer d, if necessary. For each $\gamma \in \Sigma(1)$, let n_γ be the primitive generator and $V(\gamma)$ the associated prime divisor of the toric variety $Z(\Sigma)$. Then $D_h = -\sum_\gamma h(n_\gamma) V(\gamma)$ is a Cartier divisor. The restriction of the line bundle $\mathcal{O}_{Z(\Sigma)}(D_h)$ to each prime divisor $V(\gamma)$ is ample by Lemma 3.4.

We consider the reduced divisor $D(\Sigma) = Z(\Sigma) \setminus T_N$, and let $\widehat{Z}(\Sigma)$ be the formal completion of $Z(\Sigma)$ along $D(\Sigma)$. The formal scheme $\widehat{Z}(\Sigma)$ is covered by affine formal schemes $\widehat{U}_\sigma = \operatorname{Spf} k[M \cap \sigma^\vee]^\wedge$ for $\sigma \in \Sigma \setminus \{\mathbf{0}\}$.

The quotient $\widehat{Z}(\Sigma)/\Gamma$ is defined naturally. Namely, $\widehat{W} = \widehat{Z}(\Sigma)/\Gamma$ is covered by $\widehat{U}_\sigma$ for σ in the set of representatives $\overline{\Sigma}$ of $(\Sigma \setminus \{\mathbf{0}\})/\Gamma$, and $\widehat{U}_\sigma \cap \widehat{U}_\tau$ is $\widehat{U}_\rho$ if there exist $g_1, g_2 \in \Gamma$ with $\rho = g_1(\sigma) \cap g_2(\tau) \in \overline{\Sigma}$ and empty if otherwise. Note that the ρ here exists uniquely by the property (1). It follows also that $\widehat{W}$ is separated.

Let $A(C^*)$ be the completion of the semigroup ring $k[M \cap C^*]$ with respect to the topology defined by all monomial ideals of finite codimensions. $A(C^*)$ is described as $\prod_{m \in M \cap C^*} k\mathbf{e}(m)$, and each element is denoted as an infinite sum $\sum_{m \in M \cap C^*} a_m \mathbf{e}(m)$ or simply $\sum a_m \mathbf{e}(m)$. For $g \in \Gamma$, we define the automorphism g^* of $A(C^*)$ by

$$g^*(\sum a_m \mathbf{e}(m)) = \sum a_m \mathbf{e}(g^{-1}(m)) \; . \tag{4}$$

Note that $(g_1 g_2)^* = g_2^* g_1^*$, i.e., Γ acts on $A(C^*)$ from the right. We denote the invariant subring $A(C^*)^\Gamma$ by $B(C^*, \Gamma)$, which is integrally closed since so is $A(C^*)$.

Proposition 4.1 *$\widehat{Z}(\Sigma)$ is a formal scheme over* $\operatorname{Spf} A(C^*)$*, and* $\mathrm{H}^0(\widehat{W}, \mathcal{O}_{\widehat{W}}) = B(C^*, \Gamma)$*.*

Proof The action of Γ on $A(C^*)$ can be extended to $\langle\langle M\rangle\rangle_k$ by applying (4). Since $\widehat{W}$ is covered by open subspaces $\widehat{U}_\sigma$ for $\sigma \in \overline{\Sigma}$, a section of $\mathcal{O}_{\widehat{W}}$ is written as $(s_\sigma)_{\sigma\in\overline{\Sigma}}$ with $s_\sigma \in k[M \cap \sigma^\vee]^\wedge$. We will show that each s_σ is in $B(C^*, \Gamma)$. Let g be an arbitrary element of Γ. Take a point x in the relative interior of σ. Since $\sigma \neq \mathbf{0}$, x and $g(x)$ are in C. Hence the segment $E = \overline{xg(x)}$ is contained in C. Since C is the disjoint union of $\operatorname{rel.int}\sigma'$ for $\sigma' \in \Sigma \setminus \{\mathbf{0}\}$ and the intersection $E \cap \sigma'$ is a closed segment or a point if non-empty, there exist a sequence

$$\sigma = \sigma_0, \sigma_1, \ldots, \sigma_l = g(\sigma_0) \in \Sigma \setminus \{\mathbf{0}\}$$

such that $x \in \operatorname{rel.int}\sigma_0$, $g(x) \in \operatorname{rel.int}\sigma_l$ and $E \cap \sigma_{i-1} \cap \sigma_i \neq \emptyset$ for $i = 1, \ldots, l$. Since $\sigma_{i-1} \cap \sigma_i$ is in $\Sigma \setminus \{\mathbf{0}\}$, by adding this cone if necessary, we may assume $\sigma_{i-1} \prec \sigma_i$ or $\sigma_i \prec \sigma_{i-1}$ for all i. Then we can take $\tau_0, \ldots, \tau_l \in \overline{\Sigma}$ and $g_0, \ldots, g_l \in \Gamma$ with $\sigma_i = g_i(\tau_i)$ for all i since $\overline{\Sigma}$ is a set of representatives. We have $\tau_l = \tau_0$ since $g(\sigma_0) = \sigma_l$. By assumption, $g_i^{-1}(g_{i-1}(\tau_{i-1})) \prec \tau_i$ or $g_{i-1}^{-1}(g_i(\tau_i)) \prec \tau_{i-1}$, and hence $(g_{i-1}^{-1}g_i)^*(s_{\tau_{i-1}}) = s_{\tau_i}$ as an element of $\langle\langle M\rangle\rangle_k$ for each i. Hence

$$s_{\tau_l} = (g_{l-1}^{-1}g_l)^* \cdots (g_0^{-1}g_1)^*(s_{\tau_0}) = (g_0^{-1}g_l)^*(s_{\tau_0}) .$$

Since $\sigma_0 = \sigma \in \overline{\Sigma}$, we have $\tau_0 = \tau_l = \sigma$, $g_0 = 1$ and $g_l = g$. Hence $g^*(s_\sigma) = s_\sigma$. Since g is arbitrary, s_σ is in $B(C^*, \Gamma)$.

If $\sigma, \tau \in \overline{\Sigma}$ has the relation $g(\sigma) \prec \tau$ for an element $g \in \Gamma$, there exists a restriction map $\mathcal{O}_{\widehat{W}}(\widehat{U}_\tau) \to \mathcal{O}_{\widehat{W}}(\widehat{U}_\sigma)$ which is given by g^*. Hence $s_\sigma = g^*(s_\tau) = s_\tau$. Since any two elements of $\overline{\Sigma}$ is connected by this relation, all s_σ's are equal. Thus we know $\mathrm{H}^0(\widehat{W}, \mathcal{O}_{\widehat{W}}) \subset B(C^*, \Gamma)$.

Conversely, for any element $s \in B(C^*, \Gamma)$, $(s_\sigma)_{\sigma\in\overline{\Sigma}}$ defined by $s_\sigma = s$ for all σ is clearly an element of $\mathrm{H}^0(\widehat{W}, \mathcal{O}_{\widehat{W}})$. We are done. QED

Assume that Σ is non-singular and has a positive valued strictly convex Γ-invariant support function h. For each $\sigma \in \Sigma(r)$, there exists a unique $m_\sigma \in M$ with $h = m_\sigma$ on σ. The toric variety $Z(\Sigma)$ is covered by $U_\sigma = \operatorname{Spec}(k[M \cap \sigma^\vee])$ for $\sigma \in \Sigma(r)$, and the invertible sheaf $\mathcal{O}_{Z(\Sigma)}(D_h)$ is the associated sheaf of the ideal $k[M \cap \sigma^\vee]\mathbf{e}(m_\sigma)$ on each affine open set $U(\sigma)$. Hence the induced sheaf $\mathcal{O}_{\widehat{Z}(\Sigma)}(D_h)$ on the formal scheme $\widehat{Z}(\Sigma)$ is that of the ideal $k[M \cap \sigma^\vee]^\wedge\mathbf{e}(m_\sigma) \subset k[M \cap \sigma^\vee]^\wedge$ on each $\widehat{U}(\sigma)$.

Proposition 4.2 *Let $\hat{p} : \widehat{Z}(\Sigma) \to \widehat{W}$ be the natural morphism. Then there exists an invertible ideal sheaf $\widehat{\mathcal{L}} \subset \mathcal{O}_{\widehat{W}}$ such that $p^*\widehat{\mathcal{L}} = \mathcal{O}_{\widehat{Z}(\Sigma)}(D_h)$.*

Proof It is enough to show that $k[M \cap \sigma^\vee]^\wedge\mathbf{e}(m_\sigma)^\sim$ on each $\widehat{U}(\sigma)$ for $\sigma \in \overline{\Sigma}$ form an invertible sheaf on $\widehat{W}$. For $\sigma, \tau \in \overline{\Sigma}$, the intersection $\widehat{U}(\sigma) \cap \widehat{U}(\tau)$ is covered by $\widehat{U}(\rho)$ such that there exist $g_1, g_2 \in \Gamma$ with $\rho = g_1(\sigma) \cap g_2(\tau) \in \overline{\Sigma}$. Since $\mathbf{e}(g_1(m_\sigma))$, $\mathbf{e}(g_2(m_\tau))$ and $\mathbf{e}(m_\rho)$ defines a same Cartier divisor on $U(\rho)$, $\mathbf{e}(g_1(m_\sigma) - g_2(m_\tau))$ is invertible in $k[M \cap \rho^\vee]^\wedge$. Hence the restriction of $k[M \cap \sigma^\vee]^\wedge\mathbf{e}(m_\sigma)^\sim$ and $k[M \cap \tau^\vee]^\wedge\mathbf{e}(m_\tau)^\sim$ to $\widehat{U}(\rho)$

through $(g_1^{-1})^*$ and $(g_2^{-1})^*$, respectively, are equal. Hence these invertible sheaves on the affine formal schemes are patched together to an invertible sheaf $\widehat{\mathcal{L}}$. The relation $\hat{p}^*\widehat{\mathcal{L}} = \mathcal{O}_{\widehat{Z}(\Sigma)}(D_h)$ is clear by the construction. QED

Here we omit the proof of the following theorem (cf. [I4, Theorem 2.4]).

Theorem 4.3 *The ring $B(C^*, \Gamma)$ is a quotient of a formal power series ring of finite variables, i.e., a complete noetherian local ring with the residue field k.*

Lemma 4.4 *The morphism $\hat{q} : \widehat{W} \to \widehat{S} = \operatorname{Spf} B(C^*, \Gamma)$ of formal schemes is adic of finite type* (*cf.* [EGA, I, 10.12, 10.13]).

Proof For $m \in M \cap C^*$, the infinite sum $\sum_{g\in\Gamma} \mathbf{e}(g(m))$ is an element of the maximal ideal of $B(C^*, \Gamma)$. Take a positive valued Γ-invariant strictly convex support function h of Σ. Then $P = \{x \in M_{\mathbf{R}} \; ; \; \langle x, n_\gamma\rangle \geq h(n_\gamma)\}$ is a quasi-polytope contained in C^*. For each $\sigma \in \Sigma(r)$, $m_\sigma \in M \cap C^*$ with $h = m_\sigma$ on σ is a vertex of P such that $P - m_\sigma$ is locally equal to $\sigma^\vee$ at the origin. We set $f_\sigma = \sum_{g\in\Gamma} \mathbf{e}(g(m_\sigma))$. Let $\{\gamma_1, \ldots, \gamma_r\}$ be the set of edges of σ. We set $x_i = \mathbf{e}(\gamma_i)$ for $i = 1, \ldots, r$, then $k[M \cap \sigma^\vee]^\wedge$ is the completion of the polynomial ring $k[x_1, \ldots, x_r]$ by the monomial ideal $I = (x_1 \cdots x_r)$. For $g \in \Gamma \setminus \{1\}$, $g(m_\sigma)$ is not on the face $P \cap (n_{\gamma_i} = h(n_{\gamma_i}))$ of P for $i = 1, \ldots, r$ by the condition (1), and hence $\mathbf{e}(g(m_\sigma) - m_\sigma)$ is in the ideal I. If we write $f_\sigma = u\mathbf{e}(m_\sigma)$ in the $k[M]$-module $\langle\langle M\rangle\rangle_k$, then u is in $1 + I^\wedge \subset k[M \cap \sigma^\vee]^\wedge$. Hence u is a unit on the affine formal scheme $\widehat{U}_\sigma$, and f_σ generates a defining ideal of $\widehat{U}(\sigma)$ with the residue $k[x_1, \ldots, x_r]/(\mathbf{e}(m_\sigma))$. Hence the morphism $\hat{q} : \widehat{W} \to \widehat{S}$ is adic of finite type. QED

Let $q_0 : \widehat{W}_0 \to \operatorname{Spec} k$ be the fiber over the residue field. By this lemma, $\widehat{W}_0$ is a k-scheme and $(\widehat{W}_0)_{\mathrm{red}}$ is a union of $V(\gamma)$ for $\gamma \in \overline{\Sigma}(1)$.

Lemma 4.5 *The morphism $\hat{q}$ is proper and $\widehat{\mathcal{L}}|\widehat{W}_0$ is ample.*

Proof Since each $V(\gamma)$ is a compact toric variety and $\overline{\Sigma}(1)$ is finite, $\widehat{W}_0$ is also complete. Hence $\hat{q}$ is proper (cf. [EGA, III, 3.4]). Since the restriction $\widehat{\mathcal{L}}|V(\gamma)$ is isomorphic to $\mathcal{O}_{\widehat{Z}(\Sigma)}(D_h)|V(\gamma)$, it is ample by Lemma 3.4. Hence $\widehat{\mathcal{L}}|\widehat{W}_0$ is ample.
QED

By this lemma, $\hat{q}$ is algebraizable to a scheme morphism [EGA, III, Théorèm 5.4.5]. Namely, there exists a proper morphism $q : W \to \operatorname{Spec} B(C^*, \Gamma)$ such that $\widehat{W}$ is the completion of W along the closed fiber. Furthermore, there exists an ample invertible sheaf $\mathcal{L}$ on W such that $\widehat{\mathcal{L}}$ is the pull-back to $\widehat{W}$. We have $q_*\mathcal{O}_W = \mathcal{O}_{\operatorname{Spec} B(C^*,\Gamma)}$ by Proposition 4.1. If regard $-D_h = D_{-h}$ as a closed subscheme of $Z(\Sigma)$, then the quotient $\overline{D} = D_{-h}/\Gamma$ is a scheme with the structure sheaf $\mathcal{O}_{\widehat{W}}/\widehat{\mathcal{L}}$. The exact sequence

$$0 \longrightarrow \widehat{\mathcal{L}} \longrightarrow \mathcal{O}_{\widehat{W}} \longrightarrow \mathcal{O}_{\overline{D}} \longrightarrow 0$$

is algebraized to

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_{\overline{D}} \longrightarrow 0 \,.$$

Let s_0 be the closed point of $S = \operatorname{Spec} B(C^*, \Gamma)$. Then $W_0 = q^{-1}(p_0)$ is equal to $\widehat{W}_0$, and is a subscheme of $\overline{D}$ with the same support $(\widehat{W}_0)_{\mathrm{red}}$.

Theorem 4.6 *Then k-scheme $S \setminus \{s_0\}$ is geometrically regular at every point, i.e., s_0 is an isolated singularity of S.*

Proof Since the toric variety $Z(\Sigma)$ is smooth over k, the local rings of $\widehat{W}$ are geometrically regular. Hence every point of W is also geometrically regular. The proper morphism $q' : W \setminus W_0 \to S \setminus \{s_0\}$ is isomorphic since $\mathcal{L}|(W \setminus W_0)$ is trivial and q'-ample. In other words, q is the contraction of W_0 to the point s_0. Hence each point of $S \setminus \{s_0\}$ is geometrically regular. QED

By Theorem 4.6, we can apply Artin's algebraization theorem [A, Theorem 3.8]. Namely, there exists a closed point v of an algebraic variety V, and $B(C^*, \Gamma)$ is isomorphic to the completion of the local ring $\mathcal{O}_v$ by the maximal ideal. Namely, the cusp singularity is realized as a k-rational isolated singularity of an algebraic variety.

5 Quasi-polytope with group action

We say a quasi-polyhedral set $P \subset M_{\mathbf{R}}$ *non-degenerate* if P contains an interior point, and *strongly convex* if P contains no line (cf. [I5, §1]).

Let P be a non-degenerate strongly convex rational quasi-polyhedral set. For each point $x \in P$, we denote by C_x the cone generated by $P - x$. Since P is non-degenerate, locally polyhedral and rational, C_x is a rational polyhedral cone of dimension r. Hence the dual cone $C_x^\vee \subset N_{\mathbf{R}}$ is a strongly convex rational polyhedral cone. We set

$$\Sigma(P) = \{C_x^\vee \,;\, x \in P\} \ .$$

Then $\Sigma(P)$ is a fan of $N_{\mathbf{R}}$ with the support $|\Sigma(P)|$ such that

$$\mathrm{int}(\mathrm{cc}(P)^\vee) \subset |\Sigma(P)| \subset \mathrm{cc}(P)^\vee$$

(cf. [I5, Theorem 1.4]). There exists a one-to-one correspondence $Q \mapsto \sigma_Q$ from the set of faces of P to $\Sigma(P)$ such that $x \in \mathrm{rel.\,int}\, Q$ gives $\sigma_Q = C_x^\vee$, and $\mathrm{rel.\,int}\, \sigma_Q$ is contained in $\mathrm{int}(\mathrm{cc}(P)^\vee)$ if and only if Q is bounded (cf. [I5, Theorem 1.5, Proposition 1.6]). If P is a quasi-polytope, i.e., if every proper face of P is bounded, then $|\Sigma(P)| = \mathrm{int}(\mathrm{cc}(P)^\vee) \cup \{0\}$ [I5, Lemma 3.2].

Let P be a quasi-polytope of cc-dimension r. We consider the case where an affine transformation group $\widetilde{\Gamma}$ of M is acting on P. Namely, each $\tilde{g} \in \widetilde{\Gamma}$ is an affine transformation $x \mapsto g(x) + m_g$ for $x \in M_{\mathbf{R}}$ with $g \in \mathrm{GL}(M)$ and $m_g \in M$. We denote also g the element ${}^{\mathrm{t}}g^{-1} \in \mathrm{GL}(N)$. Then the group $\Gamma = \{g \,;\, \tilde{g} \in \widetilde{\Gamma}\}$ acts on both M and N from the left. The corresponding ring isomorphism $g^* : k[M] \to k[M]$ is defined by the map $\mathbf{e}(m) \mapsto \mathbf{e}(g^{-1}(m))$.

We define

$$\widehat{P} = \{(x,t) \in M_{\mathbf{R}} \times \mathbf{R} \,;\, t \geq 0, x \in tP\} \ ,$$

where $0P = \mathrm{cc}(P)$. Then $\widehat{P}$ is a strongly convex closed cone (cf. [I5, Lemma 2.2]). For $\mathcal{A}(P) = (M \oplus \mathbf{Z}) \cap \widehat{P}$, the semigroup ring $A(P) = k[\mathcal{A}(P)]$ has a grading defined by

$$A(P)_d = \bigoplus_{m \in M \cap dP} k\mathbf{e}(m,d)$$

for $d \geq 0$. Here we denote $\mathbf{e}(m, d)$ for $\mathbf{e}((m, d))$. The action of $\widetilde{\Gamma}$ on M induces a linear action on $M \oplus \mathbf{Z}$ such that $\tilde{g}(x, t) = (g(x) + tm_g, t)$ for $(x, t) \in M_{\mathbf{R}} \times \mathbf{R}$, which fix the cone $\widehat{P}$. Then $Z(P) = \operatorname{Proj} A(P)$ is equal to the toric variety on which $\widetilde{\Gamma}$ acts (cf. [I5, Proposition 2.5]).

Now we assume that $\widetilde{\Gamma}$ acts on the set of proper faces of P freely, and it has only finite orbits. If we set $C = \operatorname{int} \operatorname{cc}(P)^{\vee}$ and $\Gamma = \{g \,;\, \tilde{g} \in \widetilde{\Gamma}\}$, then (C, Γ) is a Tsuchihashi pair and we get a cusp singularity (cf. [I5, Proposition 3.3]).

The rational support function h_P on $|\Sigma(P)|$ is defined by

$$h_P(u) = \min\{\langle x, u\rangle \,;\, x \in P\}$$

for $u \in |\Sigma(P)| = C^* \cup \{0\}$. If Q is a face of P, then $h(u) = \langle x, u\rangle$ for $x \in Q$ and $u \in \sigma_Q$. We have $h_P(g(u)) = h_P(u) + \langle m_g, g(u)\rangle$ for $\tilde{g} \in \widetilde{\Gamma}$. Hence $h_P(g(u)) - h_P(u)$ is an integer if $u \in N \cap C$. Since $\Sigma(P) \setminus \{\mathbf{0}\}$ has only finite cones modulo Γ, there exists a positive integer d such that dh_P is integral on $N \cap C$. We take the minimal d. Then dh_P defines a Cartier divisor $D_P = D_{h_P} = -\sum_{\gamma \in \Sigma(P)} dh_P(n_\gamma)V(\gamma)$ and an invertible sheaf $\mathcal{O}_{Z(P)}(D_P)$. Since $g^{-1}(D_P) = D_P - (\mathbf{e}(dg^{-1}(m_g)))$ as divisor, we have an isomorphism

$$g^*(\mathcal{O}_{Z(P)}(D_P)) \simeq \mathcal{O}_{Z(P)}(D_P)$$

of invertible sheaves by multiplying $\mathbf{e}(dg^{-1}(m_g))$. We denote the formal completion of $Z(P)$ along $Y(P) = Z(P) \setminus T_N$ by $\widehat{Z}(P)$, and pull-back of this invertible sheaf by $\mathcal{O}_{\widehat{Z}(P)}(D_P)$. If the morphism $\hat{p} : \widehat{Z}(P) \to \widehat{Z}(P)/\Gamma$ is defined, there exists an invertible sheaf $\widehat{\mathcal{L}}_P$ such that $\hat{p}^*\widehat{\mathcal{L}}_P = \mathcal{O}_{\widehat{Z}(P)}(D_P)$ by the above isomorphisms for $g \in \Gamma$.

Although the fan $\Sigma(P)$ might be singular and does not satisfy the condition (1) in Section 4, the algebraization of the quotient of $\widehat{Z}(P)$ by Γ to a scheme morphism $q : W \to \operatorname{Spec} B(C^*, \Gamma)$ by $\widehat{\mathcal{L}}_P$ is possible as in Section 4. Namely, the condition (1) is satisfied if we replace Γ by a sufficiently small normal subgroup Γ' of finite index. The assertion corresponding to Lemma 4.4 is also proved by taking a sufficiently small Γ', while Σ being non-singular is not necessary. Thus we get a projective morphism $q' : W' \to \operatorname{Spec} B(C^*, \Gamma')$ for Γ', then q is obtained by taking the quotient by the action of the finite group Γ/Γ'. Then q is the contraction of $Y(P)/\Gamma$ to the closed point s_0 of $S = \operatorname{Spec} B(C^*, \Gamma)$. The algebraization $\mathcal{L}_P$ of $\widehat{\mathcal{L}}_P$ defines an invertible sheaf on $S \setminus \{s_0\}$.

Example 5.1 Let $\{p_i \,;\, i \in \mathbf{Z}\}$ be a set of points in $M_{\mathbf{R}} = \mathbf{R}^2$ defined by the recurrence relation

$$p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad p_{i+1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} p_i + \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

and let P be the convex closure of this set. Then P is a quasi-polytope of cc-dimension two with a cyclic group action.

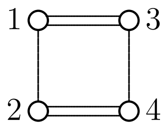
Also, for any Γ-invariant subdivision Σ' of Σ, we can algebraize $\hat{q}' : \widehat{Z(\Sigma')}/\Gamma \to \widehat{S} = \operatorname{Spf} B(C^*, \Gamma)$ to a scheme morphism as a toroidal modifications of $q : W \to S$ if the toroidal embedding $(W, S \setminus \{s_0\})$ is without self-intersection [KKMS, II, §2].

6 Examples by Tsuchihashi

Cusp singularities in arithmetic quotient spaces of **Q**-rank one are classified by Satake [S, §3]. In particular, there are 3- and 4-dimensional examples obtained from quaternion algebras over **Q** or an imaginary quadratic field. Some explicit calculations are done in [Ch]

A beautiful 4-dimensional example of cusp singularity is obtained by Tsuchihashi [T2, §6]. The Dynkin diagram

Λ :



,

which we denote by Λ, gives an infinite Coxeter group.

This group is realized as a linear Coxeter group [V2, Definition 2] as follows. Let K be the simplicial cone generated by the standard basis $\{e_1, \ldots, e_4\}$ of $\mathbf{R}^4$. For the vertices of this diagram labeled from 1 to 4, define the matrices by

$$s_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which operate on $\mathbf{R}^4$ with the coordinates (x_1, x_2, x_3, x_4) from the left. Then the subgroup $G = \langle s_1, s_2, s_3, s_4 \rangle \subset \mathrm{GL}(4, \mathbf{Z})$ is isomorphic to the Coxeter group. Namely, the relations

$$s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1 \ , \ \ (s_1 s_4)^2 = (s_2 s_3)^2 = 1 \ ,$$
$$(s_1 s_2)^3 = (s_3 s_4)^3 = 1 \ , \ \ (s_1 s_3)^4 = (s_2 s_4)^4 = 1$$

are checked easily. Each s_i fixes the facet $K \cap (x_i = 0)$ of K for $i = 1, \ldots, 4$. Then by Vinberg's result [V2, Theorem 2], G is a linear Coxeter gorup, and these are actually the defining relations of the group. We denote the set $\{s_1, s_2, s_3, s_4\}$ by $S = S_\Lambda$. Then the parabolic subgroup H_i generated by $S \setminus \{s_i\}$ is a finite group of order 48 for each i. On the other hand, the Dynkin diagram obtained by removing the edge connecting 1 and 3 (resp. 2 and 4) defines a Coxeter group of order 1152, which is isomorphic to the automorphism group of a regular 24-cell (cf. [C2, p.148]). These groups have an important role in the construction. It follows from [V2, Theorem 2] that there exists an open convex cone C, and we have

$$\bigcup_{g \in G} g(K) = C \cup \{0\} \ .$$

Tsuchihashi found a subgroup $\Gamma \subset G$ of index 48 such that $H_i \cap \Gamma = \{1\}$ for every i. Then Γ acts on C freely, and (C, Γ) is a Tsuchihashi pair. The cone $\sigma_0 = K$ is non-singular and the 4-dimensional cones $g(\sigma_0)$ and their faces for $g \in \Gamma$ form a Γ-invariant non-singular fan with the support $C \cup \{0\}$. There exists a strictly positive h, and we get $q : W \to \operatorname{Spec} B(C^*, \Gamma)$ as in previous sections. This is a resolution of the cusp singularity since Σ is non-singular.

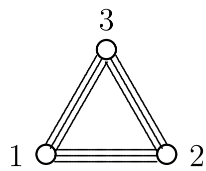
Let $\gamma_i = \mathbf{R}_0 e_i$ for $i = 1, \ldots, 4$. Then $\Sigma_i = \Sigma[\gamma_i]$ is a 3-dimensional complete non-singular fan on which H_i acts. Let V_i be the complete non-singular toric variety associated to Σ_i for each i. Although the canonical divisor of W is not q-ample, $-(4V_1 + 3V_2 + 3V_3 + 4V_4)$ is q-ample.

These 3-dimensional fans are described as follows. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbf{R}^3$. For an order (i, j, k) of $\{1, 2, 3\}$ and $\epsilon_i, \epsilon_j, \epsilon_k = \pm 1$, the cone generated by $\{\epsilon_i e_i, \epsilon_i e_i + \epsilon_j e_j, \epsilon_i e_i + \epsilon_j e_j + \epsilon_k e_k\}$ is a nonsingular cone in $\mathbf{R}^3$ with the lattice Z^3. There are exactly 48 such cones and form a non-singular complete fan Δ_1. Then, Σ_2 and Σ_3 are isomorphic to Δ_1. The fan Δ_2 consists of the same set of cone but in $\mathbf{R}^3$ with the lattice $\mathbf{Z}^3 + \mathbf{Z}(1/2, 1/2, 1/2)$, which is also a non-singular complete fan. The fans Σ_1 and Σ_4 are isomorphic to Δ_2. Hence, each of V_i has 48 torus action invariant points corresponding to the 48 maximal cones.

The exceptional divisor of q is a simple normal crossing divisor consisting of these four toric varieties. There are 48 quadruple points, and all four components go through each of these points at an invariant point. For a choice of the group Γ, I have calculated the intersection of the four irreducible components. By cutting the fan Σ_i with a cube with the center at the origin, each square face is triangulated to six triangles. Figures 1 through 4 are the nets of the cubes and each triangle on the net presents an invariant point of the component. The invariant points of each component are numbered from 0 to 47, and the four points labeled a same number form a quadruple point of the normal crossing exceptional divisor.

Tsuchihashi also found a very nice example in dimension three. Let Ω be the Dynkin diagram:

Ω :



.

Define the matrices

$$s_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $S = S_\Omega = \{s_1, s_2, s_3\}$ generates a linear Coxeter group acting on $\mathbf{R}^3$, with the base cone K generated by the standard basis of $\mathbf{R}^3$. Let $\widetilde{G}$ be the linear group generated by S and the order 3 rotation

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let Σ be the fan consisting of the 3-dimensional cones $g(K)$ and their faces for all $g \in \widetilde{G}$. Then the subgroup

$$H = \langle rs_1s_2s_3, rs_2s_3s_1, rs_3s_1s_2 \rangle \subset \widetilde{G}$$

of index 12 acts on $\Sigma \setminus \{\mathbf{0}\}$ freely. Thus we get a 3-dimensional toric type cusp singularity. This example is analyzed more precisely in [K].

In this example, $\Sigma(1)$ has only one orbit and $\Sigma(3)$ has four. Hence the exceptional set E of this cusp is a normal crossing irreducible divisor with four triples points. Furthermore, $h : C \cup \{0\} \to \mathbf{R}$ defined by $h(u) = \min\{\langle x, u\rangle \ ; \ x \in M \cap C^*\}$ is a strictly convex support function of Σ. In this case, $h(n_\gamma) = 1$ for the primitive generator n_γ for all $\gamma \in \Sigma(1)$. Hence D_h is the canonical divisor of the toric variety $Z(\Sigma)$ (cf. [O, 2.1]). The canonical divisor is defined by the Euler form

$$\omega = \frac{d\mathbf{e}(m_1)}{\mathbf{e}(m_1)} \wedge \frac{d\mathbf{e}(m_2)}{\mathbf{e}(m_2)} \wedge \frac{d\mathbf{e}(m_3)}{\mathbf{e}(m_3)} \ ,$$

where $\{m_1, m_2, m_3\}$ is a basis of M. Hence $g^*\omega = \det(g)\omega$ for $g \in \mathrm{GL}(N)$. Since $\det(g) = \pm 1$, $\omega^{\otimes 2}$ is invariant by the action of Γ. Hence we know $\mathcal{O}_W(-2E) \simeq \omega_W^{\otimes 2}$, where ω_W is the canonical invertible sheaf of W. Since $\mathcal{O}_W(-E)$ is relatively ample for the contraction morphism $q : W \to \operatorname{Spec} B(C^*, \Gamma)$, so is the canonical sheaf ω_W.

References

[A] M. Artin, Algebraic approximation of structures over complete local rings, Inst. Hautes Études Sci. Publ. Math. **36**, (1969), 23–58.

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. Tai, Smooth compactification of locally symmetric varieties, Lie Groups: History Frontiers and Applications IV, Math. Sci. Press, Brookline, Mass. 1975; Second edition, Cambridge University Press, 2010.

[C1] H. S. M. Coxeter, Discrete groups generated by reflections, Ann. of Math. (2) **35** (1934), 588–621.

[C2] H. S. M. Coxeter, Regular Polytopes, the third edition, Dover Publications, Inc, New York, 1973.

[Ch] K. Chinda, Cusp singularities constructed from maximal orders of a quaternion algebra (in Japanese), Master's thesis at Tohoku University, in preparation.

[EGA] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique I, II, III, IV, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32**, (1960-1967).

[G] B. Grünbaum, Convex Polytopes, Second edition, Graduate Texts in Math. **221**, Springer-Verlag, New York, Berlin, Heidelberg, 2003.

[I1] M. Ishida, Convex bodies and algebraic geometry - toric varieties and applications, II, in Proc. of the Seminar in Algebraic Geometry, Singapore 1987, World Scientific Publishing CO., Singapore, 1988, 15–32.

[I2] M. Ishida, Cusp singularities given by reflections of stellable cones, International J. Math. **2**, (1991), 635–657.

[I3] M. Ishida, The duality of cusp singularities, Math. Ann. **294**, (1992), 81–97.

[I4] M. Ishida, The graded rings associated to cusp singularities, preprint, (included in the proceedings of Symposium of Algebraic Geometry in Kinosaki 2012)

[I5] M. Ishida, Cusp singularities and quasi-polyhedral sets, to appear in ASPM 75.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal embeddings I, Lecture Notes in Math. 339, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[K] Y. Komatsu, Construction of cusp singularities by Coxeter groups (in Japanese), Master's thesis at Tohoku University, in preparation.

[N1] Y. Namikawa, Toroidal compactification of Siegel spaces, Lecture Notes in Math. **812**, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[O] T. Oda, Convex Bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties, Ergebnisse der Math. (3), **15**, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1988.

[S] I. Satake, On numerical invariants of arithmetic varieties (The case of **Q**-rank one), Sugaku Expo., **1**, (1988), Amer. Math. Soc., 59–74.

[T1] H. Tsuchihashi, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tohoku Math. J. **35**, (1983), 607–639.

[T2] H. Tsuchihashi, Examples of four dimensional cusp singularities, to appear in J. of MSJ.

[V1] È. B. Vinberg, The theory of convex homogeneous cones, Trans. Moscaw Math. Soc. **12** (1967), 303–368.

[V2] È. B. Vinberg, Discrete linear groups generated by reflections, Izv. Akad. Nauk SSSR, Ser. Mat. **35** (1971), 1083–1119.

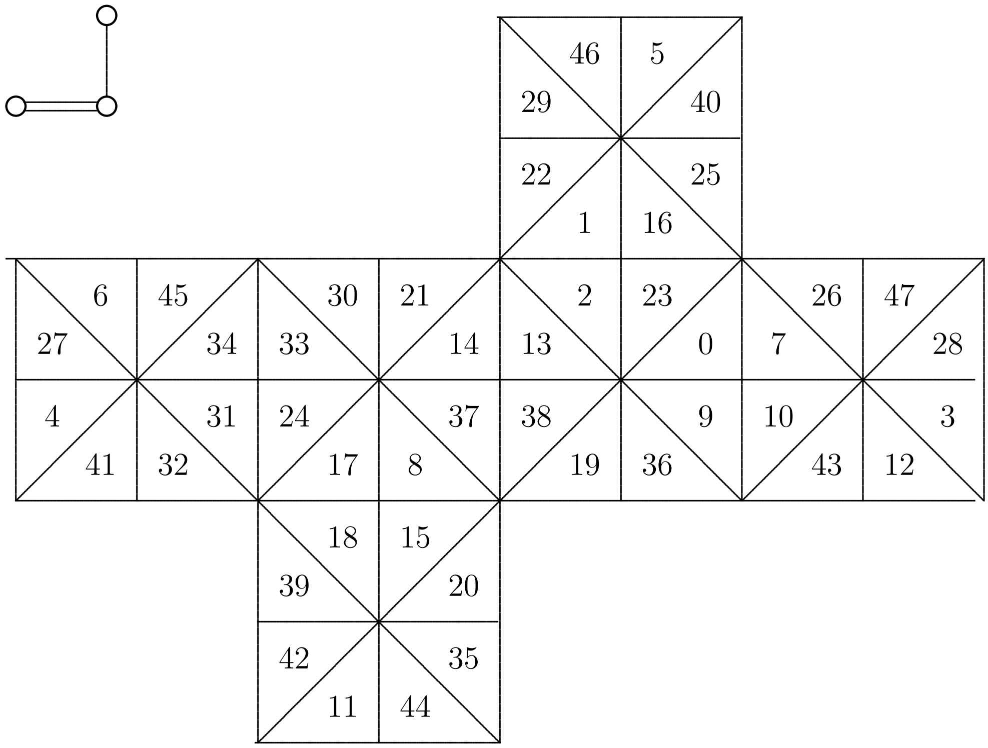


Figure 1:

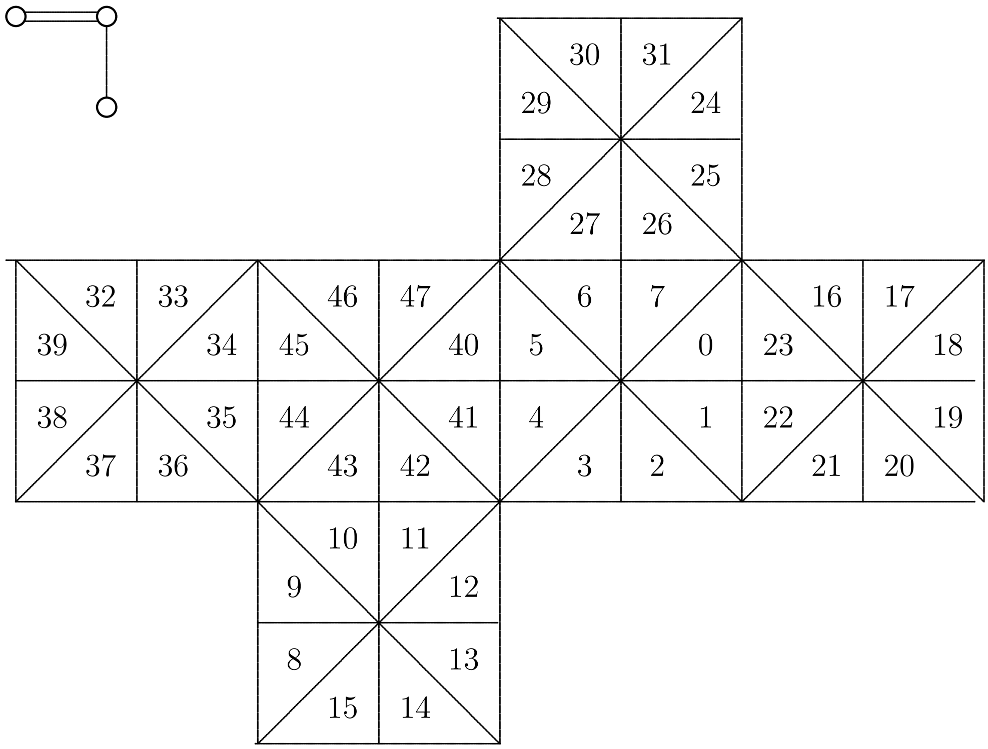


Figure 2:

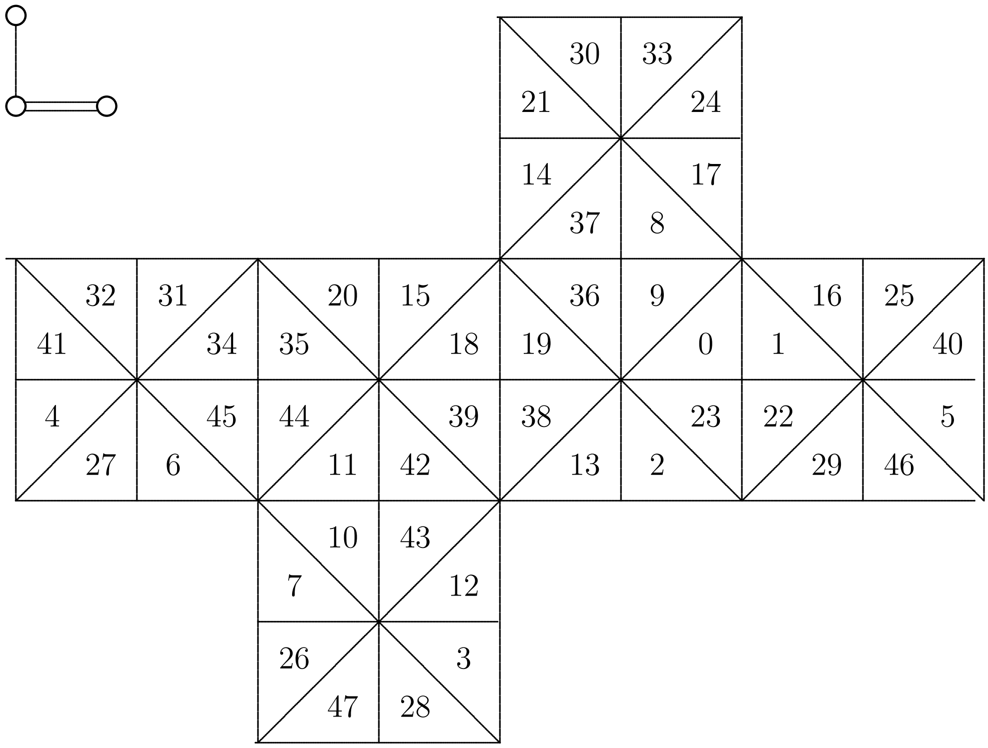


Figure 3:

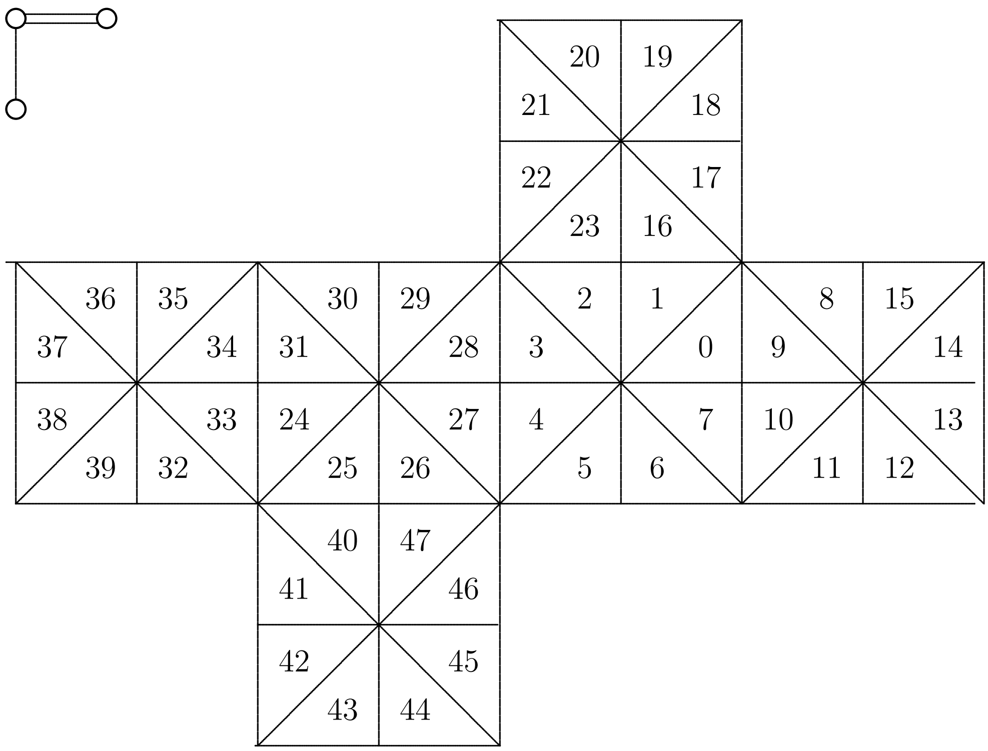


Figure 4: