ALMOST GORENSTEIN RINGS

NAOKI TANIGUCHI

Contents

1. Introduction

This article is based on the works jointly with Shiro Goto, Ryo Takahashi, Naoyuki Matsuoka, and Ken-ichi Yoshida ([8, 9, 10, 11, 15]).

There are known numerous examples of Cohen-Macaulay rings and among the progress of the theory of Cohen-Macaulay rings, we often encounter non-Gorenstein Cohen-Macaulay rings in the field of not only commutative algebra, but also algebraic geometry, representation theory, invariant theory, and combinatorics. On all such occasions, we have a natural query of why there are so many Cohen-Macaulay rings which are not Gorenstein.

The goal of this research is to find a new class of Cohen-Macaulay rings, which may not be Gorenstein, but sufficiently good next to the Gorenstein rings. One of the candidates for such a class is *almost Gorenstein rings*, which was originally studied by V. Barucci and R. Fröberg [2] in 1997. One can refer to [2] for a beautiful theory of almost symmetric numerical semigroups. Nevertheless, since the notion given by [2] was not flexible for the analysis of analytically ramified case, in 2013 S. Goto, N. Matsuoka and T. T. Phuong [7] extended the notion over one-dimensional Cohen-Macaulay local rings, using the first Hilbert coefficients of canonical ideals. More recently, in 2015 S. Goto, R. Takahashi and the author [15] finally gave the definition of almost Gorenstein *graded/local* rings of higher dimension.

Let us start on the definition of almost Gorenstein ring in the sense of [15].

The author was partially supported by JSPS Grant-in-Aid for Scientific Research 26400054.

Definition 1.1. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then R is said to be *an almost Gorenstein local ring*, if the following conditions are satisfied.

(1) R is a Cohen-Macaulay local ring, which possesses the canonical module K_R and
(2) there exists an exact sequence
$$0 \to R \to \mathrm{K}_R \to C \to 0$$
of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Here $\mu_R(C)$ denotes the number of elements in a minimal system of generators for C,
$$\mathrm{e}^0_{\mathfrak{m}}(C) = \lim_{n\to\infty} \frac{\ell_R(C/\mathfrak{m}^{n+1}C)}{n^{d-1}} \cdot (d-1)!$$
is the multiplicity of C with respect to $\mathfrak{m}$, and $d = \dim R$.

Notice that every Gorenstein local ring is almost Gorenstein (take $C = (0)$) and the converse is also true, if R is Artinian. In the exact sequence quoted in Definition 1.1 (2), if $C \neq (0)$, then C is a Cohen-Macaulay R-module with $\dim_R C = d-1$ and one has the equality
$$\mathfrak{m}C = (f_2, f_3, \dots, f_d)C$$
for some elements $f_2, f_3, \dots, f_d \in \mathfrak{m}$, provided the residue class field $R/\mathfrak{m}$ of R is infinite. Hence C is a *maximally generated Cohen-Macaulay* R-module in the sense of [3], which is called in the present article an *Ulrich* R-module. Roughly speaking, Definition 1.1 requires that if R is an almost Gorenstein local ring, then R might be a non-Gorenstein local ring but the ring R can be embedded into its canonical module K_R so that the difference K_R/R should be tame and well-behaved.

In the case where $d = 1$, if R is an almost Gorenstein local ring, then $\mathfrak{m}C = (0)$ and R is an almost Gorenstein local ring exactly in the sense of [7, Definition 3.1]. The converse is also true, if $R/\mathfrak{m}$ is infinite. We will later show that many results of [7] of dimension one are extendable over higher-dimensional local rings, which supports the appropriateness of our definition.

In what follows, unless otherwise specified, let R denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators for M (resp. the length of M). We denote by $\mathrm{e}^0_{\mathfrak{m}}(M)$ the multiplicity of M with respect to $\mathfrak{m}$.

2. Survey on one-dimensional almost Gorenstein rings

Throughout this section, let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let K_R stand for the canonical module of R. Then an ideal I of R is called *canonical*, if $I \neq R$ and $I \cong \mathrm{K}_R$ as an R-module. Notice that this definition implicitly assume the existence of the canonical module. By the result [16, Satz 6.21] of J. Herzog and E. Kunz, R possesses a canonical ideal if and only if the total ring of fractions $\mathrm{Q}(\widehat{R})$ of $\widehat{R}$ is Gorenstein, where we denote by $\widehat{R}$ the $\widehat{\mathfrak{m}}$-adic completion of R. Hence the ring R contains a canonical ideal I if it is analytically unramified. Since I

is faithful and $\dim R = 1$, I is an $\mathfrak{m}$-primary ideal of R. Therefore there exist integers $\mathrm{e}_0(I) > 0$ and $\mathrm{e}_1(I)$ such that

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I)$$

for all integers $n \gg 0$. The integers $\mathrm{e}_i(I)$'s are called *the Hilbert coefficients of R with respect to I*. These integers describe the complexity of given local rings, and there are a huge number of preceding researches about them, e.g., [5, 6, 7, 12, 13]. In particular, the integer $\mathrm{e}_0(I) > 0$ is called *the multiplicity of R with respect to I* and has been explored very intensively.

Let $\mathrm{r}(R)$ stand for the Cohen-Macaulay type of R ([16, Definition 1.20]). Then the almost Gorenstein ring is defined as follows.

Definition 2.1 ([7])**.** We say that R is *an almost Gorenstein local ring*, if R possesses a canonical ideal I of R such that $\mathrm{e}_1(I) \le \mathrm{r}(R)$.

Remember that if R is Gorenstein, then any parameter ideal Q of R is canonical and hence $\mathrm{e}_1(Q) < \mathrm{r}(R) = 1$, which implies that every Gorenstein local ring is an almost Gorenstien ring.

We now assume that I contains a parameter ideal $Q = (a)$ as a reduction, namely $I^{r+1} = QI^r$ for some integer $r \ge 0$. Note that this assumption is automatically satisfied, if the residue class field $R/\mathfrak{m}$ of R is infinite. We set

$$K = \frac{I}{a} = \left\{\frac{x}{a} \mid x \in I\right\} \subseteq \mathrm{Q}(R).$$

Notice that K is a fractional ideal of R such that

$$R \subseteq K \subseteq \overline{R} \ \text{ and } \ K \cong \mathrm{K}_R$$

where $\overline{R}$ denotes the integral closure of R in $\mathrm{Q}(R)$. Then the result [7, Theorem 3.11] says that R is an almost Gorenstein ring if and only if $\mathfrak{m}K \subseteq R$, or equivalently $\mathfrak{m}I = \mathfrak{m}Q$. The latter condition is the original definition of almost Gorenstein ring in the sense of [2]. Therefore if R is analytically unramified, that is $\widehat{R}$ is reduced, then the these two definitions of almost Gorenstein ring coincides, provided the residue class field $R/\mathfrak{m}$ of R is infinite.

Before entering the higher dimensional case, let us give examples of almost Gorenstein local rings of dimension one.

Example 2.2. Let k be an infinite field. The following are almost Gorenstein rings.

(1) $k[[t^3, t^4, t^5]]$
(2) $k[[t^a, t^{a+1}, \ldots, t^{2a-3}, t^{2a-1}]]$ $(a \ge 4)$
(3) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$
(4) $k[[X, Y, Z, U, V, W]]/I$, where

$$\begin{aligned} I &= (X^3 - Z^2, Y^2 - ZX) + (U, V, W)^2 \\ &\quad + (YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW). \end{aligned}$$

3. Almost Gorenstein local rings of higher dimension

In this section we summarize some basic results on almost Gorenstein rings of arbitrary dimension. Almost all the results are proved in [15].

In what follows, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R$ possessing the canonical module K_R. For simplicity we assume that the residue class field $R/\mathfrak{m}$ of R is infinite.

Definition 3.1 ([15]). We say that R is *an almost Gorenstein local ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where $\mu_R(C)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(C)$) stands for the number of elements in a minimal system of generators for C (resp. the multiplicity of C with respect to $\mathfrak{m}$).

We look at an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules. Here we do not need to assume that R is almost Gorenstein. If $C \neq (0)$, then C is a Cohen-Macaulay R-module of dimension $d-1$. Set $\overline{R} = R/[(0) :_R C]$ and let $\overline{\mathfrak{m}}$ denote the maximal ideal of $\overline{R}$. Choose elements $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ such that $(f_1, f_2, \ldots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}}$. Then we have

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \ldots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Therefore $\mathrm{e}^0_{\mathfrak{m}}(C) \geq \mu_R(C)$ and we say that C is *an Ulrich R-module* if $\mathrm{e}^0_{\mathfrak{m}}(C) = \mu_R(C)$, since C is *a maximally generated maximal Cohen-Macaulay $\overline{R}$-module* in the sense of B. Ulrich ([3]). Thus C is an Ulrich R-module if and only if

$$\mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C.$$

Therefore if $\dim R = 1$, then the Ulrich property for C is equivalent to saying that C is a vector space over $R/\mathfrak{m}$. Therefore we have the following, which ensures that Definition 3.1 is a natural generalization of the definition of almost Gorenstein rings given by S. Goto, N. Matsuoka, and T. T. Phuong [7].

Remark 3.2. Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Then the following conditions are equivalent.

(1) R is an almost Gorenstein local ring in the sense of Definition 3.1
(2) R is an almost Gorenstein local ring in the sense of [7, Definition 3.1].

One can construct many examples of almost Gorenstein rings of higher dimension. The significant examples of almost Gorenstein rings are one-dimensional Cohen-Macaulay local rings of finite Cohen-Macaulay representation type and two-dimensional rational singularity. Therefore every two-dimensional finite Cohen-Macaulay representation type is almost Gorenstein. Furthermore, for all the known examples of finite

Cohen-Macaulay representation type are almost Gorenstein. Thus, it might be true that for any finite Cohen-Macaulay representation type is almost Gorenstein for arbitrary dimension, which we leave as an open question.

Let us recall the fundamental results on almost Gorenstein rings. We begin with the following, which is called *non-zerodivisor characterization.*

Theorem 3.3. *Let $f \in \mathfrak{m}$ and assume that f is R-regular.*

(1) *If $R/(f)$ is an almost Gorenstein local ring, then R is an almost Gorenstein local ring. Moreover if R is not a Gorenstein ring, then $f \notin \mathfrak{m}^2$.*
(2) *Conversely, suppose that R is a non-Gorenstein almost Gorenstein local ring. Consider the exact sequence*
$$0 \to R \to \mathrm{K}_R \to C \to 0$$
of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If f is superficial for C with respect to $\mathfrak{m}$ and $d \geq 2$, then $R/(f)$ is an almost Gorenstein local ring.

Proof. We set $\overline{R} = R/(f)$. Remember that $\mathrm{K}_R/f\mathrm{K}_R = \mathrm{K}_{\overline{R}}$, since f is R-regular.

(1) We take an exact sequence
$$0 \to \overline{R} \xrightarrow{\psi} \mathrm{K}_{\overline{R}} \to D \to 0$$
of $\overline{R}$-modules so that D is an Ulrich $\overline{R}$-module of dimension $d-2$. Let $\xi \in \mathrm{K}_R$ such that $\psi(1) = \overline{\xi}$, where $\overline{\xi}$ denotes the image of ξ in $\mathrm{K}_{\overline{R}} = \mathrm{K}_R/f\mathrm{K}_R$. We now consider the exact sequence
$$R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$
of R-modules with $\varphi(1) = \xi$. Then, because $\psi = \overline{R} \otimes_R \varphi$, we get $D = C/fC$, whence $\dim_R C < d$, because $\dim_R D = d-2$. Consequently, the homomorphism φ must be injective, and hence C is a Cohen-Macaulay R-module of dimension $d-1$. Therefore, f is C-regular, so that C is an Ulrich R-module and $f \notin \mathfrak{m}^2$. Hence R is almost Gorenstein.

(2) Notice that f is a C-regular element, because f is superficial for C with respect to $\mathfrak{m}$ and $\dim_R C = d-1 > 0$. Therefore the exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ gives rise to the exact sequence of $\overline{R}$-modules
$$0 \to \overline{R} \to \mathrm{K}_{\overline{R}} \to C/fC \to 0$$
where C/fC is an Ulrich $\overline{R}$-module. Hence $\overline{R}$ is an almost Gorenstein local ring. $\square$

We apply Theorem 3.3 (1) to get the following.

Corollary 3.4. *Suppose that $d > 0$. If $R/(f)$ is an almost Gorenstein local ring for every non-zerodivisor $f \in \mathfrak{m}$, then R is a Gorenstein local ring.*

Let us note one example of almost Gorenstein local rings.

Example 3.5 (cf. [18]). Let $S = k[[X_1, X_2, \ldots, X_n, Y_1, Y_2, , \ldots, Y_n]]$ $(n \geq 2)$ be the formal power series ring over a field k and put

$$R = S/\mathrm{I}_2(\mathbb{M})$$

where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of S generated by 2×2 minors of the matrix

$$\mathbb{M} = \left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{smallmatrix}\right).$$

Then R is an almost Gorenstein local ring with $\dim R = n+1$ and $\mathrm{r}(R) = n-1$.

Proof. It is well-known that R is a Cohen-Macaulay normal local ring with $\dim R = n+1$ and $\mathrm{r}(R) = n-1$. The sequence $\{X_i - Y_{i-1}\}_{1 \leq i \leq n}$ (here $Y_0 = Y_n$ for convention) forms a regular sequence in R and

$$R/(X_i - Y_{i-1} \mid 1 \leq i \leq n)R \cong k[[X_1, X_2, \ldots, X_n]]/\mathrm{I}_2(\mathbb{N}) =: T$$

where

$$\mathbb{N} = \left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & \cdots & X_n & X_1 \end{smallmatrix}\right).$$

Then T is a Cohen-Macaulay local ring with $\dim T = 1$, such that $\mathfrak{n}^2 = x_1\mathfrak{n}$ and $\mathrm{K}_T \cong (x_1, x_2, \ldots, x_{n-1})$, where $\mathfrak{n}$ stands for the maximal ideal of T and x_i denotes the image of X_i in T. Hence T is an almost Gorenstein local ring, because $\mathfrak{n}(x_1, x_2, \ldots, x_{n-1}) \subseteq (x_1)$. Thus R is an almost Gorenstein local ring by Theorem 3.3 (1). $\square$

We are now interested in the question of how the almost Gorenstein property is inherited under flat local homomorphisms of Noetherian local rings. Let us begin with the following.

Theorem 3.6. *Let $(S, \mathfrak{n})$ be a Noetherian local ring and let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a regular local ring. Then R is an almost Gorenstein local ring if and only if so is S.*

Proof. Suppose that R is an almost Gorenstein local ring and consider an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $C = (0)$, then R is a Gorenstein ring, so is S. Suppose $C \neq (0)$. Then $S \otimes_R C$ is an Ulrich S-module, because C is an Ulrich R-module. Note that $\mathrm{K}_S \cong S \otimes_R \mathrm{K}_R$ as S-modules, since $S/\mathfrak{m}S$ is a Gorenstein local ring. Thus S is almost Gorenstein, thanks to the exact sequence of S-modules

$$0 \to S \to \mathrm{K}_S \to S \otimes_R C \to 0.$$

Suppose now that S is an almost Gorenstein local ring. Let $n = \dim S/\mathfrak{m}S$. We have to show that R is an almost Gorenstein local ring. Assume the contrary and choose a counterexample S so that $\dim S = n + d$ is as small as possible. Then S is not a Gorenstein ring, so that $\dim S = n + d > 0$. We take an exact sequence

$$0 \to S \to \mathrm{K}_S \to D \to 0$$

of S-modules with $\mu_S(D) = \mathrm{e}^0_{\mathfrak{n}}(D)$. Suppose $n > 0$. If $d > 0$, then choose an element $g \in \mathfrak{n}$ so that g is superficial for D with respect to $\mathfrak{n}$ and $\overline{g}$ is a part of a regular system of parameters of $S/\mathfrak{m}S$, where $\overline{g}$ denotes the image of g in $S/\mathfrak{m}S$. Then g is S-regular and the composite homomorphism $R \overset{\varphi}{\to} S \to S/gS$ is flat. Therefore the minimality of $n + d$ forces R to be an almost Gorenstein local ring, because S/gS is an almost Gorenstein local ring by Theorem 3.3 (2). Thus $d = 0$ and $\mathfrak{p} = \mathfrak{m}S$ is a minimal prime ideal of S. Hence the induced flat local homomorphism $R \overset{\varphi}{\to} S \to S_{\mathfrak{p}}$ shows that R is a Gorenstein ring, because $S_{\mathfrak{p}}$ is a Gorenstein ring. Thus $n = 0$ and $\mathfrak{n} = \mathfrak{m}S$.

Suppose now that $d \geq 2$. Then because $\mathfrak{n} = \mathfrak{m}S$, we may choose an element $f \in \mathfrak{m}$ so that f is R-regular and $\varphi(f)$ is superficial for D with respect to $\mathfrak{n}$. Then by Theorem 3.3 (2) S/fS is an almost Gorenstein local ring, while the homomorphism $R/fR \to S/fS$ is flat. Consequently, R/fR is an almost Gorenstein local ring, so that by Theorem 3.3 (1) R is an almost Gorenstein local ring.

Thus $d = 1$ and $\mathfrak{n} = \mathfrak{m}S$, so that R is an almost Gorenstein local ring by [7, Proposition 3.3], which is the required contradiction. $\square$

The following plays an important role when we consider the almost Gorenstein property of the Rees algebras.

Lemma 3.7. *Let R be an almost Gorenstein local ring and choose an exact sequence*

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $\varphi(1) \in \mathfrak{m}\mathrm{K}_R$, then R is a regular local ring. Therefore $\mu_R(C) = \mathrm{r}(R) - 1$, provided R is not regular.

Proof. Suppose $\varphi(1) \in \mathfrak{m}\mathrm{K}_R$. Then $C \neq (0)$ and therefore $d > 0$. Assume $d = 1$. Then $\mathrm{Q}(R)$ is a Gorenstein ring. Therefore we get an exact sequence

$$0 \to R \xrightarrow{\psi} I \to C \to 0$$

of R-modules with $\psi(1) \in \mathfrak{m}I$, where I ($\subsetneq R$) is an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. Let $a = \psi(1)$. Then $\mathfrak{m}I = (a)$, because $\mathfrak{m}C = (0)$ and $a \in \mathfrak{m}I$. Hence R is a discrete valuation ring. Let $d > 1$ and assume that our assertion holds true for $d-1$. Let $f \in \mathfrak{m}$ be a non-zerodivisor of R such that f is superficial for C with respect to $\mathfrak{m}$. We set $\overline{R} = R/(f)$ and $\overline{C} = C/fC$. Then by Theorem 3.3 (2) $\overline{R}$ is an almost Gorenstein local ring with the exact sequence

$$0 \to \overline{R} \xrightarrow{\overline{\varphi}} \mathrm{K}_{\overline{R}} \to \overline{C} \to 0$$

of $\overline{R}$-modules, where $\overline{\varphi} = \overline{R} \otimes_R \varphi$ and $\mathrm{K}_{\overline{R}} = \mathrm{K}_R/f\mathrm{K}_R$. Therefore, because $\overline{\varphi}(1) \in \mathfrak{m}\mathrm{K}_{\overline{R}}$, the induction argument on d shows $\overline{R}$ is regular and hence so is R.

The second assertion follows from the fact that

$$\mu_R(C) = \mu_R(\mathrm{K}_R) - 1 = \mathrm{r}(R) - 1$$

since $\varphi(1) \notin \mathfrak{m}\mathrm{K}_R$. $\square$

When R contains a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$, we have the following characterization for $A = R \ltimes \mathfrak{p}$ to be an almost Gorenstein local ring, which is a generalization of [7, Theorem 6.5].

Theorem 3.8. *Suppose that $d = \dim R > 0$. Let $\mathfrak{p} \in \mathrm{Spec} R$ and suppose that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. Then the following conditions are equivalent.*
(1) *$A = R \ltimes \mathfrak{p}$ is an almost Gorenstein local ring.*
(2) *R is an almost Gorenstein local ring.*

The following example extends [7, Example 6.10].

Example 3.9. Suppose that R is a Gorenstein local ring of positive dimension. Let $\mathfrak{p} \in \mathrm{Spec} R$ and assume that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. We set $A = R \ltimes \mathfrak{p}$. Then, thanks to Theorem 3.8, A is an almost Gorenstein local ring. Therefore because $\mathfrak{p} \times \mathfrak{p} \in \mathrm{Spec}\, A$ with $A/[\mathfrak{p} \times \mathfrak{p}] \cong R/\mathfrak{p}$, setting

$$R_n = \begin{cases} R & (n = 0) \\ R_{n-1} \ltimes \mathfrak{p}_{n-1} & (n > 0) \end{cases}, \qquad \mathfrak{p}_n = \begin{cases} \mathfrak{p} & (n = 0) \\ \mathfrak{p}_{n-1} \ltimes \mathfrak{p}_{n-1} & (n > 0) \end{cases},$$

we get an infinite family $\{R_n\}_{n \geq 0}$ of almost Gorenstein local rings. Note that R_n is not a Gorenstein ring, if $n \geq 2$ (see [7, Lemma 6.6]).

Example 3.10. Let k be an infinite field and $S = k[[X, Y, Z, U, V, W]]$ a formal power series ring over k. Set

$$A = k[[X, Y, Z, U, V, W]]/I$$

where

$$I = (X^3 - Z^2, Y^2 - ZX) + (U, V, W)^2 + (YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW).$$

Then it is routine to check that the isomorphism

$$A \cong k[[t^4, t^5, t^6]] \ltimes (t^4, t^5, t^6)$$

and hence A is an almost Gorenstein local ring.

Let us note a characterization of almost Gorenstein property in terms of canonical ideals, which is a generalization of [7, Theorem 3.11].

Theorem 3.11. *Suppose that $d = \dim R > 0$ and $\mathrm{Q}(R)$ is a Gorenstein ring. Let I ($\subsetneq R$) be an ideal of R such that $I \cong \mathrm{K}_R$. Then the following conditions are equivalent.*
(1) *R is an almost Gorenstein local ring.*
(2) *R contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$ and $\mathfrak{m}(I + Q) = \mathfrak{m}Q$.*

When this is the case, if $d \geq 2$ and R is not a Gorenstein ring, we have the following, where $J = I + Q$.
(*a*) $\mathrm{red}_Q(J) = 2$.

(b) $\ell_R(R/J^{n+1}) = \ell_R(R/Q)\cdot\binom{n+d}{d} - \mathrm{r}(R)\cdot\binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$ *for* $\forall n \geq 0$.
Hence $\mathrm{e}_1(J) = \mathrm{r}(R)$.

(c) *Let* $G = \mathrm{gr}_J(R)$. *Then* $f_2, f_3, \dots, f_d$ *is a super-regular sequence with respect to* J *and* $\mathrm{depth} G = d-1$.

Theorem 3.12 (S. Goto)**.** *Suppose that* R *is a non-Gorenstein almost Gorenstein local ring with* $\dim R \geq 1$. *Let* M *be a finitely generated* R*-module. If*

$$\mathrm{Ext}^i_R(M, R) = (0)$$

for all $i \gg 0$, *then* $\mathrm{pd}_R M < \infty$.

As a direct consequence of Theorem 3.12, we have the following.

Corollary 3.13. *Suppose that* R *is an almost Gorenstein local ring with* $\dim R \geq 1$. *If* R *is not a Gorenstein ring, then* R *is G-regular in the sense of* [17], *that is*

$$\mathrm{Gdim}_R M = \mathrm{pd}_R M$$

for every finitely generated R*-module* M.

4. Semi-Gorenstein local rings

In this section we maintain the notation as in Section 3. Let $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ denote a filtration of ideals of R such that $I_0 = R$, $I_1 \neq R$. We now consider the R-algebras

$$\mathcal{R} = \sum_{n\geq 0} I_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n\in\mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}], \quad \text{and} \quad G = \mathcal{R}'/t^{-1}\mathcal{R}'$$

associated to $\mathcal{F}$, where t is an indeterminate. Notice that $\mathcal{R}' = \mathcal{R}[t^{-1}]$ and that $G = \oplus_{n\geq 0} I_n/I_{n+1}$. Let N denote the graded maximal ideal of $\mathcal{R}'$.

Let us begin with the following.

Theorem 4.1. *Suppose that* $\mathcal{R}$ *is a Noetherian ring. If* G_N *is an almost Gorenstein local ring and* $\mathrm{r}(G_N) \leq 2$, *then* R *is an almost Gorenstein local ring.*

Proof. We may assume $\mathrm{r}(G_N) = 2$. Since $\mathcal{R}'_N$ is an almost Gorenstein local ring with $\mathrm{r}(\mathcal{R}'_N) = 2$, we have

$$0 \to \mathcal{R}'_N \to \mathrm{K}_{(\mathcal{R}'_N)} \to C \to 0$$

where C is isomorphic to a regular local ring of dimension d. Let $\mathfrak{p} = \mathfrak{m}R[t, t^{-1}]$ and set $P = \mathfrak{p} \cap \mathcal{R}'$. Then $P \subseteq N$, so that $R[t, t^{-1}]_\mathfrak{p}$ is an almost Gorenstein local ring, because

$$R[t, t^{-1}]_\mathfrak{p} = \mathcal{R}'_P = (\mathcal{R}'_N)_{P\mathcal{R}'_N}.$$

Hence R is an almost Gorenstein local ring, since $R \to R[t, t^{-1}] \to R[t, t^{-1}]_\mathfrak{p}$ is a flat homomorphism. $\square$

Example 4.2 (Barucci-Dobbs-Fontana)**.** Let $R = k[[x^4, x^6 + x^7, x^{10}]] \subseteq V$, where $V = k[[x]]$ denotes the formal power series ring over an infinite field k of $\mathrm{ch}\, k \neq 2$. Let $H = \{v(a) \mid 0 \neq a \in R\}$ be the value semigroup of R. We consider the filtration $\mathcal{F} = \{(xV)^n \cap R\}_{n\in\mathbb{Z}}$ of ideals of R. We then have the following.

(1) $H = \langle 4, 6, 11, 13 \rangle$.
(2) $G \cong k[x^4, x^6, x^{11}, x^{13}]$ $(\subseteq k[x])$ and G_N is an almost Gorenstein local ring with $\mathrm{r}(G_N) = 3$.
(3) R is not an almost Gorenstein local ring and $\mathrm{r}(R) = 2$.

Therefore $(\mathcal{R}'_N)_{P\mathcal{R}'_N}$ is not an almost Gorenstein local ring. Hence local rings $R_\mathfrak{p}$ ($\mathfrak{p} \in \mathrm{Spec}R$) of an almost Gorenstein local ring R are not necessarily almost Gorenstein in general. Now we deal with the special class of almost Gorenstein rings which preserves under localization.

Definition 4.3. We say that R is *a semi-Gorenstein local ring*, if R is an almost Gorenstein local ring which possesses an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

such that either $C = (0)$, or C is an Ulrich R-module and $C = \oplus_{i=1}^{\ell} C_i$ for some cyclic R-submodule C_i of C.

Hence every Gorenstein local ring is a semi-Gorenstein local ring and every one-dimensional almost Gorenstein local ring is semi-Gorenstein, since $\mathfrak{m}C = (0)$. We notice that in exact sequence of Definition 4.3, if $C \neq (0)$, then each C_i is a cyclic Ulrich R-module of dimension $d-1$, whence

$$C_i \cong R/\mathfrak{p}_i$$

for some prime ideal $\mathfrak{p}_i$ of R such that $R/\mathfrak{p}_i$ is a regular local ring of dimension $d-1$.

We note the following.

Proposition 4.4. *Let R be a semi-Gorenstein local ring. Then $R_\mathfrak{p}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \mathrm{Spec}R$.*

Proof. We may assume that R is not a Gorenstein ring. Choose an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules which satisfies the condition as in Definition 4.3. Hence $C = \oplus_{i=1}^{\ell} R/\mathfrak{p}_i$, where for each $1 \le i \le \ell$, $\mathfrak{p}_i \in \mathrm{Spec}R$ and $R/\mathfrak{p}_i$ is a regular local ring of dimension $d-1$. Let $\mathfrak{p} \in \mathrm{Spec}R$. Then since $\mathrm{K}_{R_\mathfrak{p}} = (\mathrm{K}_R)_\mathfrak{p}$, we get an exact sequence

$$0 \to R_\mathfrak{p} \to \mathrm{K}_{R_\mathfrak{p}} \to C_\mathfrak{p} \to 0$$

of $R_\mathfrak{p}$-modules, where $C_\mathfrak{p} = \oplus_{\mathfrak{p}_i \subseteq \mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$ is a direct sum of finite cyclic Ulrich $R_\mathfrak{p}$-modules $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$, so that by definition the local ring $R_\mathfrak{p}$ is semi-Gorenstein. $\square$

Let us now consider a characterization of semi-Gorenstein local rings in terms of their minimal free resolutions, which is a natural generalization of [7, Corollary 4.2].

Theorem 4.5. *Let $(S, \mathfrak{n})$ be a regular local ring and $\mathfrak{a} \subsetneq S$ an ideal of S with $n = \mathrm{ht}_S \mathfrak{a}$. Let $R = S/\mathfrak{a}$. Then the following conditions are equivalent.*

(1) *R is a semi-Gorenstein local ring but not a Gorenstein ring.*

(2) *R is Cohen-Macaulay, $n \geq 2$, $r = \mathrm{r}(R) \geq 2$, and R has a minimal S-free resolution of the form*:

$$0 \to F_n = S^r \overset{\mathbb{M}}{\to} F_{n-1} = S^q \to F_{n-2} \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

where

$$^t\mathbb{M} = \begin{pmatrix} y_{21}y_{22}\cdots y_{2\ell} & y_{31}y_{32}\cdots y_{3\ell} & \cdots & y_{r1}y_{r2}\cdots y_{r\ell} & z_1z_2\cdots z_m \\ x_{21}x_{22}\cdots x_{2\ell} & 0 & 0 & 0 & 0 \\ 0 & x_{31}x_{32}\cdots x_{3\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_{r1}x_{r2}\cdots x_{r\ell} & 0, \end{pmatrix},$$

$\ell = n+1$, $q \geq (r-1)\ell$, $m = q-(r-1)\ell$, and $x_{i1}, x_{i2}, \ldots, x_{i\ell}$ is a part of a regular system of parameters of S for every $2 \leq i \leq r$.

When this is the case, one has the equality

$$\mathfrak{a} = (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2 \left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix} \right),$$

where $\mathrm{I}_2(\mathbb{N})$ denotes the ideal of S generated by 2×2 minors of the submatrix $\mathbb{N} = \left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix} \right)$ of $\mathbb{M}$.

We explore one example.

Example 4.6. Let $V = k[[t]]$ be the formal power series ring over an infinite field k and set $R = k[[t^5, t^6, t^7, t^9]]$. Let $S = k[[X, Y, Z, W]]$ be the formal power series ring and let $\varphi : S \to R$ be the k-algebra map defined by

$$\varphi(X) = t^5, \ \varphi(Y) = t^6, \ \varphi(Z) = t^7, \ \text{ and } \ \varphi(W) = t^9.$$

Then R has a minimal S-free resolution of the form

$$0 \to S^2 \overset{\mathbb{M}}{\to} S^6 \to S^5 \to S \to R \to 0,$$

where

$$^t\mathbb{M} = \left(\begin{smallmatrix} W & X^2 & XY & YZ & Y^2-XZ & Z^2-XW \\ X & Y & Z & W & 0 & 0 \end{smallmatrix} \right).$$

Hence R is a semi-Gorenstein local ring with $\mathrm{r}(R) = 2$ and

$$\mathrm{Ker}\varphi = (Y^2 - XZ, Z^2 - XW) + \mathrm{I}_2 \left(\begin{smallmatrix} W & X^2 & XY & YZ \\ X & Y & Z & W \end{smallmatrix} \right).$$

5. Almost Gorenstein graded rings

Let us now discuss the graded ring. Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring and assume that R_0 is a local ring and there exists the graded canonical module K_R. Let $a = \mathrm{a}(R)$ be an a-invariant of R.

Definition 5.1. Then R is called *an almost Gorenstein graded ring*, if there exists an exact sequence
$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$
of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mathfrak{M}$ is the unique graded maximal ideal of R. Remember that $\mathrm{K}_R(-a)$ stands for the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Note that Gorenstein graded ring is by definition an almost Gorenstein graded ring. If R is an almost Gorenstein graded ring, then the local ring $R_{\mathfrak{M}}$ is an almost Gorenstein local ring. Unfortunately, the converse is not true in general.

Example 5.2. Let $U = k[s,t]$ be the polynomial ring over an infinite field k and look at the subring $R = k[s, s^3t, s^3t^2, s^3t^3] \subseteq U$. Let $S = k[X,Y,Z,W]$ be the weighted polynomial ring such that
$$\deg X = 1,\ \deg Y = 4,\ \deg Z = 5,\ \text{and}\ \deg W = 6.$$
Let $\psi : S \to R$ be the k-algebra map defined by
$$\psi(X) = s,\ \psi(Y) = s^3t,\ \psi(Z) = s^3t^2,\ \text{and}\ \psi(W) = s^3t^3.$$
Then $\operatorname{Ker}\psi = \mathrm{I}_2\left(\begin{smallmatrix} X^3 & Y & Z \\ Y & Z & W \end{smallmatrix}\right)$ and R has a graded minimal S-free resolution
$$0 \to S(-13) \oplus S(-14) \xrightarrow{\left(\begin{smallmatrix} X^3 & Y \\ Y & Z \\ Z & W \end{smallmatrix}\right)} S(-10) \oplus S(-9) \oplus S(-8) \xrightarrow{(\Delta_1\ \Delta_2\ \Delta_3)} S \xrightarrow{\psi} R \to 0$$
where $\Delta_1 = Z^2 - YW$, $\Delta_2 = X^3W - YZ$, and $\Delta_3 = Y^2 - X^3Z$. Therefore, because $\mathrm{K}_S \cong S(-16)$, we get
$$(\sharp) \qquad S(-6) \oplus S(-7) \oplus S(-8) \xrightarrow{\left(\begin{smallmatrix} X^3 & Y & Z \\ Y & Z & W \end{smallmatrix}\right)} S(-3) \oplus S(-2) \xrightarrow{\varepsilon} \mathrm{K}_R \to 0.$$
Hence $\mathrm{a}(R) = -2$. Let $\xi = \varepsilon(\binom{1}{0}) \in [\mathrm{K}_R]_3$ and we have
$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R(3) \to S/(Y,Z,W)(1) \to 0$$
where $\varphi(1) = \xi$. Hence R_M is a semi-Gorenstein local ring.

On the other hand, by $(\sharp)$ we get $[\mathrm{K}_R]_2 = k\eta \neq (0)$, where $\eta = \varepsilon(\binom{0}{1})$. Hence if R is an almost Gorenstein graded ring, we must have
$$\mu_R(\mathrm{K}_R/R\eta) = \mathrm{e}^0_M(\mathrm{K}_R/R\eta)$$
which is impossible, because $\mathrm{K}_R/R\eta \cong [S/(X^3, Y, Z)](-3)$.

We explore the almost Gorenstein property of the homogeneous ring.

Theorem 5.3. *Let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring with $d = \dim R \geq 1$. Suppose that k is an infinite field and R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein graded ring and level.*

(2) $\mathrm{Q}(R)$ *is a Gorenstein ring and* $\mathrm{a}(R) = 1 - d$.

Let us consider a few examples.

Example 5.4 (cf. [18]). Let $S = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\ ]$ $(2 \leq m \leq n)$ be the polynomial ring over an infinite field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \leq t \leq m$, $X = [X_{ij}]$. Then R is an almost Gorenstein graded ring if and only if either $m = n$, or $m \neq n$ and $t = m = 2$.

Example 5.5. Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ be a polynomial ring over an infinite field k. Let $n \geq 1$ be an integer. Then the following assertions are hold.

(1) $R^{(n)} = k[R_n]$ is an almost Gorenstein graded ring, if $d \leq 2$.
(2) Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if either $n \mid d$, or $d = 3$ and $n = 2$.

6. Two-dimensional rational singularities

Let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring of dimension $d \geq 0$, admitting the canonical module K_R. We assume that $R/\mathfrak{m}$ is infinite. Let $\mathrm{v}(R) = \mu_R(\mathfrak{m})$ and $\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R)$. We denote by

$$G = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

the associated graded ring of $\mathfrak{m}$ and put $\mathfrak{M} = G_+$. The purpose of this section is mainly to study the question of when G is an almost Gorenstein graded ring. Remember that $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$ if and only if $\mathfrak{m}^2 = Q\mathfrak{m}$ for some (and hence any) minimal reduction Q of $\mathfrak{m}$. When this is the case, G is a Cohen-Macaulay ring and $\mathrm{a}(G) = 1 - d$, provided R is not a regular local ring.

The answer for the above question is stated as follows.

Theorem 6.1. *The following assertions hold true.*

(1) *Suppose that R is an almost Gorenstein local ring with* $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$. *Then G is an almost Gorenstein level graded ring.*
(2) *Suppose that G is an almost Gorenstein level graded ring. Then R is an almost Gorenstein local ring.*

Proof. We only prove the assertion (1). We may assume that R is not a Gorenstein local ring. Hence $d > 0$ and $\mathrm{a}(G) = 1 - d$. We will show that G is an almost Gorenstein graded ring by induction on d. First we consider the case $d = 1$. Let $\overline{R}$ denote the integral closure of R in $\mathrm{Q}(R)$. Choose an R-submodule K of $\overline{R}$ so that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as an R-module. We have $\mathfrak{m}K \subseteq R$ by [7, Theorem 3.11] as R is an almost Gorenstein local ring. Hence $\mathfrak{m}K = \mathfrak{m}$, and $\mathfrak{m}^n K = \mathfrak{m}^n$ for all $n \geq 1$. Let $C = K/R$ and consider the $\mathfrak{m}$-adic filtrations of R, K, and C. We then have the exact sequence

$$(\sharp) \qquad 0 \to G \to \mathrm{gr}_{\mathfrak{m}}(K) \to \mathrm{gr}_{\mathfrak{m}}(C) \to 0$$

of graded G-modules induced from the canonical exact sequence

$$0 \to R \to K \to C \to 0$$

of filtered R-modules. Note that $\mathrm{gr}_{\mathfrak{m}}(C) = [\mathrm{gr}_{\mathfrak{m}}(C)]_0$. By the exact sequence ($\sharp$), G is an almost Gorenstein graded ring, because $\mathrm{gr}_{\mathfrak{m}}(K) \cong \mathrm{K}_G$ as a graded G-module.

Suppose that $d > 1$ and that our assertion holds true for $d-1$. Let

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

be an exact sequence of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. We take $a \in \mathfrak{m}$ so that a is a part of a minimal reduction of $\mathfrak{m}$ and a is superficial for C with respect to $\mathfrak{m}$. Let $f = \overline{a}$ ($\in \mathfrak{m}/\mathfrak{m}^2$) denote the image of a in $G = \mathrm{gr}_{\mathfrak{m}}(R)$. We then have

$$G/fG = \mathrm{gr}_{\mathfrak{m}}(R/(a))$$

and $\mathrm{v}(R/(a)) = \mathrm{e}(R/(a)) + d - 2$. By the induction argument, G/fG is an almost Gorenstein graded ring, because $R/(a)$ is an almost Gorenstein local ring. Choose an exact sequence

$$0 \to G/fG \to \mathrm{K}_{G/fG}(d-2) \to X \to 0$$

of graded G/fG-modules so that $\mu_{G/fG}(X) = \mathrm{e}^0_{[G/fG]_+}(X)$. Recall that $\mathrm{K}_{G/fG}(d-2) \cong \mathrm{K}_{G/f\mathrm{K}_G}(d-1)$ as a graded G-module and we get an exact sequence

$$0 \to G \to \mathrm{K}_G(d-1) \to Y \to 0$$

of graded G-modules such that $\mu_G(Y) = \mathrm{e}^0_{\mathfrak{M}}(Y)$. Consequently G is an almost Gorenstein graded ring. $\square$

In the case where R has a minimal multiplicity, the almost Gorenstein property of R is equivalent to the Gorenstein property of $\mathrm{Q}(G)$.

Corollary 6.2. *Suppose that* $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$. *Then the following are equivalent.*

(1) *R is an almost Gorenstein local ring,*
(2) *G is an almost Gorenstein graded ring,*
(3) $\mathrm{Q}(G)$ *is a Gorenstein ring.*

We say that $\mathfrak{m}$ is *a normal ideal*, if $\mathfrak{m}^n$ is an integrally closed ideal for every $n \geq 1$.

Corollary 6.3. *Suppose that* $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$ *and that R is a normal ring. If $\mathfrak{m}$ is a normal ideal, then R is an almost Gorenstein local ring.*

Proof. Let $\mathcal{R}' = \mathcal{R}'(\mathfrak{m}) = R[\mathfrak{m}t, t^{-1}]$ be the extended Rees algebra of $\mathfrak{m}$, where t is an indeterminate. Then $\mathcal{R}'$ is a normal ring, because R is a normal local ring and $\mathfrak{m}$ is a normal ideal. Hence the total ring of fractions of $G = \mathcal{R}'/t^{-1}\mathcal{R}'$ is a Gorenstein ring, so that R is almost Gorenstein by Corollary 6.2. $\square$

The following is a direct consequence of Corollary 6.3.

Corollary 6.4. *Every 2-dimensional rational singularity is an almost Gorenstein local ring.*

By the result of M. Auslander [1], every two-dimensional Cohen-Macaulay complete local ring R of finite Cohen-Macaulay representation type is a rational singularity, provided R contains a field of characteristic 0. Hence we get the following.

Corollary 6.5. *Every two-dimensional Cohen-Macaulay complete local ring R of finite Cohen-Macaulay representation type is an almost Gorenstein local ring, provided R contains a field of characteristic* 0*.*

7. Almost Gorenstein Rees algebras

In this section we study the problem of when the Rees algebras of ideals and modules over two-dimensional regular local rings are almost Gorenstein graded rings.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 2$ and let $I \subsetneq R$ be an $\mathfrak{m}$-primary ideal of R. Assume that I contains a parameter ideal $Q = (a, b)$ of R such that $I^2 = QI$. We set $J = Q : I$. Let

$$\mathcal{R} = R[It] \subseteq R[t] \quad \text{and} \quad T = R[Qt] \subseteq R[t],$$

where t stands for an indeterminate over R. Notice that the Rees algebra $\mathcal{R}$ of I is a Cohen-Macaulay ring with $\mathrm{a}(\mathcal{R}) = -1$ and

$$\mathcal{R} = T + T{\cdot}It$$

while the Rees algebra T of Q is a Gorenstein ring of dimension 3 and $\mathrm{a}(T) = -1$. Hence $\mathrm{K}_T(1) \cong T$ as a graded T-module, where K_T denotes the graded canonical module of T.

Let us begin with the following, which is a special case of [19, Theorem 2.7 (a)].

Lemma 7.1. $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ *as a graded $\mathcal{R}$-module.*

Proof. Since $\mathcal{R}$ is a module-finite extension of T, we get

$$\mathrm{K}_R(1) \cong \mathrm{Hom}_T(\mathcal{R}, \mathrm{K}_T)(1) \cong \mathrm{Hom}_T(\mathcal{R}, T) \cong T :_F \mathcal{R}$$

as graded $\mathcal{R}$-modules, where $F = \mathrm{Q}(T) = \mathrm{Q}(\mathcal{R})$ is the total ring of fractions. Therefore

$$T :_F \mathcal{R} = T :_T It$$

because $\mathcal{R} = T + T{\cdot}It$. Since $Q^n \cap [Q^{n+1} : I] = Q^n[Q : I]$ for every $n \geq 0$, we have

$$T :_T It = JT.$$

Hence $T :_F \mathcal{R} = JT$, so that $JT = J\mathcal{R}$. Thus $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ as a graded $\mathcal{R}$-module. $\square$

Corollary 7.2. *Suppose that $\mathcal{R}$ is a normal ring. Then $J = Q : I$ is integrally closed.*

Proof. Since $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$, $J\mathcal{R}$ is unmixed and of height one. Therefore $J\mathcal{R}$ is integrally closed in $\mathcal{R}$, whence J is integrally closed in R, because $\overline{J} \subseteq J\mathcal{R}$. $\square$

The following is the key in our argument.

Theorem 7.3. *The following conditions are equivalent.*

(1) *$\mathcal{R}$ is a strongly almost Gorenstein graded ring, namely there exists an exact sequence*
$$0 \to \mathcal{R} \to \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$
such that $MC = (\xi, \eta)C$ for some homogeneous elements $\xi, \eta \in M$.

(2) *There exist elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that*
$$IJ = gJ + Ih \text{ and } \mathfrak{m}J = fJ + \mathfrak{m}h$$

When this is the case, $\mathcal{R}$ is an almost Gorenstein graded ring.

Proof. (2) $\Rightarrow$ (1) Notice that $\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h$, since $IJ = gJ + Ih$ and $\mathfrak{m}J = fJ + \mathfrak{m}h$. Consider the exact sequence
$$\mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$
of graded $\mathcal{R}$-modules where $\varphi(1) = h$. We then have $\mathfrak{M}C = (f, gt)C$, so that $\dim_{\mathcal{R}_{\mathfrak{M}}} C_{\mathfrak{M}} \leq 2$. Hence by [15, Lemma 3.1] the homomorphism φ is injective and $\mathcal{R}$ is an almost Gorenstein graded ring.

(1) $\Rightarrow$ (2) We may assume that $\mathcal{R}$ is not a Gorenstein ring and consider the exact sequence
$$0 \to \mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$
of graded $\mathcal{R}$-modules with $C \neq (0)$ and $\mathfrak{M}C = (\xi, \eta)C$ for some homogeneous elements ξ, η of $\mathfrak{M}$. Hence $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring. We set $h = \varphi(1) \in J$, $m = \deg \xi$, and $n = \deg \eta$. Hence $C = J\mathcal{R}/\mathcal{R}h$. Remember that $h \notin \mathfrak{m}J$, since $\mathcal{R}_{\mathfrak{M}}$ is not a regular local ring. If $\min\{m, n\} > 0$, then $\mathfrak{M}C \subseteq \mathcal{R}_+C$, whence $\mathfrak{m}C_0 = (0)$. Therefore $\mathfrak{m}J \subseteq (h)$, so that we have $J = (h) = R$. Thus $\mathcal{R}h = J\mathcal{R}$ and $\mathcal{R}$ is a Gorenstein ring, which is impossible. Assume $m = 0$. If $n = 0$, then $\mathfrak{M}C = \mathfrak{m}C$ since $\xi, \eta \in \mathfrak{m}$, so that
$$C_1 \subseteq \mathcal{R}_+C_0 \subseteq \mathfrak{m}C$$
and therefore $C_1 = (0)$. Hence $IJ = Ih$ which shows (h) is a reduction of J, so that $(h) = R = J$. Therefore $\mathcal{R}$ is a Gorenstein ring, which is impossible. If $n \geq 2$, then because
$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq \xi{\cdot}J\mathcal{R} + \eta{\cdot}J\mathcal{R} + \mathcal{R}h,$$
we get $IJ \subseteq \xi IJ + Ih$, whence $IJ = Ih$. This is impossible as we have shown above. Hence $n = 1$. Let us write $\eta = gt$ with $g \in I$ and take $f = \xi$. We then have
$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h,$$
whence $\mathfrak{m}J \subseteq fJ + Rh$. Because $h \notin \mathfrak{m}J$, we get $\mathfrak{m}J \subseteq fJ + \mathfrak{m}h$, so that $\mathfrak{m}J = fJ + \mathfrak{m}h$, while $IJ = gJ + Ih$, because $IJ \subseteq fIJ + gJ + Ih$. This completes the proof. $\square$

We are now in a position to prove the following.

Theorem 7.4. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field and I an $\mathfrak{m}$-primary integrally closed ideal in R. Then the Rees algebra $\mathcal{R}$ of I is an almost Gorenstein graded ring.*

Proof. We choose a parameter ideal Q of R so that $Q \subseteq I$ and $I^2 = QI$, whence the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ is a Cohen-Macaulay ring. Because $\mathcal{R}$ is a normal ring, $J = Q : I$ is also an integrally closed ideal in R. Consequently, choosing three elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ so that f, h are a joint reduction of $\mathfrak{m}, J$ and g, h are a joint reduction of I, J, we readily get by the equalities

$$\mathfrak{m}J = fJ + \mathfrak{m}g \quad \text{and} \quad IJ = gJ + Ih$$

by the result of J. Verma. Hence $\mathcal{R} = \mathcal{R}(I)$ is an almost Gorenstein graded ring. $\square$

As a direct consequence we have the following.

Corollary 7.5. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every integer $\ell > 0$.*

Closing this article, let us explore the question of when the Rees algebras of socle ideals are almost Gorenstein graded rings. Let $(R, \mathfrak{m})$ be a regular local ring with $d = \dim R \geq 2$ possessing an infinite residue class field $R/\mathfrak{m}$ of R. Let Q be a parameter ideal of R such that $Q \neq \mathfrak{m}$. We set

$$I = Q : \mathfrak{m}$$

the *socle ideal of* Q.

With this notation, we have the following.

Theorem 7.6. *Suppose that $d \geq 3$. Then the Rees algebra $\mathcal{R}(I)$ of I is an almost Gorenstein graded ring if and only if either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

For each ideal I of R, we set

$$\mathrm{o}(I) = \sup\{n \geq 0 \mid I \subseteq \mathfrak{m}^n\}.$$

Theorem 7.7. *Suppose that $d = 2$. Then the Rees algebra $\mathcal{R}(I)$ of I is an almost Gorenstein graded ring if and only if $\mathrm{o}(Q) \leq 2$.*

References

[1] M. Auslander, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc., **293** (1986), no. 2, 511–531.

[2] V. Barucci and R. Fröberg, *One-dimensional almost Gorenstein rings*, J. Algebra, **188** (1997), no. 2, 418–442.

[3] J. P. Brennan, J. Herzog and B. Ulrich, *Maximally generated maximal Cohen-Macaulay modules*, Math. Scand., **61** (1987), no. 2, 181–203.

[4] S. Goto, *Integral closedness of complete intersection ideals*, J. Algebra, **108** (1987), 151–160.

[5] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong and W. V. Vasconcelos, *Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals*, J. London Math. Soc., **81** (2010) 679–695.

[6] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong and W. V. Vasconcelos, *The Chern numbers and Euler characteristics of modules*, Acta Math. Vietnam., **40** (2015), no. 1, 37–60.

[7] S. Goto, N. Matsuoka and T. T. Phuong, *Almost Gorenstein rings*, J. Algebra, **379** (2013), 355–381.
[8] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, *The almost Gorenstein Rees algebras over two-dimensional regular local rings*, J. Pure Appl. Algebra, **220** (2016), 3425–3436.
[9] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, *The almost Gorenstein Rees algebras of parameters*, J. Algebra, **452** (2016), 263-278.
[10] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, *On the almost Gorenstein property in the Rees algebras of contracted ideals*, Preprint 2016.
[11] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, *The almost Gorenstein Rees algebras of p_g-ideals, good ideals, and powers of the maximal ideals*, Preprint 2016.
[12] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*, Michigan Math. J., **57** (2008), 359-381.
[13] S. Goto, K. Nishida, and K. Ozeki, *The structure of Sally modules of rank one*, Math. Res. Lett., **15** (2008) 881–892.
[14] S. Goto and Y. Shimoda, *On the Rees algebras of Cohen-Macaulay local rings*, Commutative algebra (Fairfax, Va., 1979), 201–231, Lecture Notes in Pure and Appl. Math., 68, Dekker, New York, 1982.
[15] S. Goto, R. Takahashi and N. Taniguchi, *Almost Gorenstein rings – towards a theory of higher dimension*, J. Pure Appl. Algebra, **219** (2015), 2666–2712.
[16] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics, 238, *Springer-Verlag, Berlin-New York*, 1971.
[17] R. Takahashi, *On G-regular local rings*, Comm. Algebra, **36** (2008), no. 12, 4472–4491.
[18] N. Taniguchi, *On the almost Gorenstein property of determinantal rings*, Preprint 2016.
[19] B. Ulrich, *Ideals having the expected reduction number*, Amer. J. Math., **118** (1996), no. 1, 17–38.

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: taniguchi@meiji.ac.jp
URL: http://www.isc.meiji.ac.jp/~taniguchi/