

PRIMITIVE AUTOMORPHISMS OF A SIMPLE ABELIAN VARIETY

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ABSTRACT. We shall prove that an automorphism of a simple abelian variety is primitive if and only if it is of infinite order.

1. INTRODUCTION

This note provides a supplementary result (Theorem 1.1) of my talk at the sixty-first Algebra Symposium of Mathematical Society of Japan, held at Saga University on September 7–10, 2016. My talk there was based on my previous [Og16-2].

Throughout this note, the base field is assumed to be the complex number field \mathbb{C} . Let M be a smooth projective variety of dimension $m \geq 2$ and $f \in \text{Bir}(M)$.

f is said to be *imprimitive* if there are a smooth projective variety B with $0 < \dim B < m$ and a dominant rational map $\pi : M \dashrightarrow B$ with connected fibers such that π is f -equivariant, i.e., there is $f_B \in \text{Bir}(B)$ satisfying $\pi \circ f = f_B \circ \pi$. As π is just a rational dominant map, smoothness assumption of B is harmless by Hironaka resolution of singularities ([Hi64]). We say that f is *primitive* if it is not imprimitive.

The notion of primitivity is introduced by De-Qi Zhang [Zh09]. Note that if f is primitive, then $\text{ord}(f) = \infty$. Indeed, otherwise, the invariant field $\mathbb{C}(M)^{f^*}$ is of the same transcendental degree m as the rational function field $\mathbb{C}(M)$. Thus we have $\varphi \in \mathbb{C}(M)^f \setminus \mathbb{C}$ as $m \geq 1$. Then the Stein factorization of $\varphi : M \dashrightarrow \mathbb{P}^1$ is f -equivariant. f is then imprimitive as $m \geq 2$.

Assume that $f \in \text{Aut}(M)$. The *topological entropy* $h_{\text{top}}(f)$ of f is a fundamental quantity measuring the complexity of the orbit behaviour under f^n ($n \geq 0$). Let r_p be the spectral radius of $f^*|H^{p,p}(M)$. Then, by Gromov-Yomdin's theorem, $h_{\text{top}}(f)$ satisfies

$$0 \leq h_{\text{top}}(f) = \log \max_{0 \leq p \leq m} r_p(f)$$

In this note, it is harmless to regard this formula as the definition of $h_{\text{top}}(f)$ (See eg. [Og15] and references therein for details).

The aim of this note is to remark the following:

Theorem 1.1. *Let A be a simple abelian variety of dimension $m \geq 2$ and $f \in \text{Aut}(A)$. Then f is primitive if and only if $\text{ord}(f) = \infty$. In particular, the translation automorphism t_a ($a \in A$) defined by $x \mapsto x + a$ is primitive if a is a non-torsion point of A with fixed zero. Moreover, if in addition A is of CM type, then A admits a primitive automorphism of positive entropy, possibly after replacing A by an isogeny.*

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Here and hereafter, an abelian variety $A = \mathbb{C}^m/\Lambda$ is said to be *simple* if A has no abelian subvariety B such that $0 < \dim B < \dim A$. A simple abelian variety A is called of *CM type* if the endomorphism ring $E := \text{End}_{\text{group}}(A) \otimes \mathbb{Q}$ is a CM field with $[E : \mathbb{Q}] = 2 \dim A$. By definition, a field E is a *CM field* if E is a totally imaginary quadratic extension of a totally real number field K . Note that if an abelian variety B is isogenous to a simple abelian variety of CM type, then so is B with the same endomorphism ring as A . However, $\text{Aut}_{\text{group}}(A) \not\cong \text{Aut}_{\text{group}}(B)$ in general (even for elliptic curves of CM type).

The "only if" part of Theorem 1.1 is clear as already remarked. Theorem 1.1 is a generalization of our earlier work [Og16-2, Theorem 4.3]. The last statement of Theorem 1.1 gives an affirmative answer to a question asked by Gongyo at the symposium.

Our proof is a fairly geometric one based on works due to Amerik-Campana [AC13] and Bianco [Bi16] and is in some sense close to [Og16-3].

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2. PROOF OF THEOREM 1.1.

Let A be a simple abelian variety of dimension $m \geq 2$ and $f \in \text{Aut}(A)$ such that $\text{ord}(f) = \infty$. We first show that f is primitive.

The following two well-known propositions will be frequently used:

Proposition 2.1. *Let V be a subvariety of A such that $\dim V < m = \dim A$ and \tilde{V} is a Hironaka resolution of V . Then \tilde{V} is of general type.*

Proof. See [Ue75, Corollary 10.10]. □

Proposition 2.2. *Let M be a smooth projective variety of general type defined over a field k of characteristic 0. Then the birational automorphism group $\text{Bir}(M/k)$ of M over k is a finite group*

Proof. By the Lefschetz principle, we may reduce to [Ue75, Corollary 14.3]. □

Lemma 2.3. *Let P be a very general closed point of A . Then the $\langle f \rangle$ -orbit $\{f^n(P) \mid n \in \mathbb{Z}\}$ of P is Zariski dense in A .*

Proof. As P is very general, f^n is defined at P for all $n \in \mathbb{Z}$. By [AC13, Théorème 4.1], there is a smooth projective variety B and a dominant rational map $\rho : A \dashrightarrow B$ such that $\rho \circ f = \rho$ and $\rho^{-1}(\rho(P))$ is the Zariski closure of $\langle f \rangle$ -orbit of P . It suffices to show that $\dim B = 0$. In what follows, assume to the contrary that $\dim B > 0$, we derive a contradiction.

Let $\eta \in B$ be the generic point in the sense of scheme and A_η be the fiber over η . Then by Proposition 2.1 and specialization, a Hironaka resolution of each irreducible component of A_η is of general type over $\mathbb{C}(B)$. By $\rho \circ f = \rho$, f faithfully acts on A_η over $\mathbb{C}(B)$. Thus, by Proposition 2.2, $f^n = \text{id}$ on A_η for some positive integer n . Thus $f^n = \text{id}$ on A , as the generic point η_A of A is in A_η . This contradicts to $\text{ord} f = \infty$. □

The following general, useful proposition is due to Bianco:

Proposition 2.4. *Let X be a projective variety and $g \in \text{Bir}(X)$. Assume that $\pi : X \dashrightarrow B$ is a g -equivariant dominant rational map to a smooth projective variety B with $\dim B < \dim X$. Assume that a Hironaka resolution \tilde{X}_b of the fiber X_b is of general type for a general closed point $b \in B$. Then for any very general closed point $P \in X$, the $\langle g \rangle$ -orbit $\{g^n(P) | n \in \mathbb{Z}\}$ of P is never Zariski dense in X .*

Proof. See [Bi16, Section 4]. See also [Og16-3, Remark 2.6] for a minor clarification. \square

The next proposition completes the first part of Theorem 1.1:

Proposition 2.5. *Let A be a simple abelian variety of dimension ≥ 2 and f be an automorphism of A of infinite order. Then f is primitive.*

Proof. Let $\pi : A \dashrightarrow B$ be an f -equivariant dominant rational map to a smooth projective variety B with $\dim B < \dim A$ and with connected fibers. If $\dim B > 0$, then by Proposition 2.1, a Hironaka resolution \tilde{A}_b of the fiber A_b over $b \in B$ is of general type for general $b \in B$. Then, by Proposition 2.4, the $\langle f \rangle$ -orbit of a very general closed point $P \in A$ is not Zariski dense. This contradicts to Lemma 2.3. Thus $\dim B = 0$, i.e., f is primitive. \square

We shall show the last part of Theorem 1.1.

Let A be a simple abelian variety of CM type of dimension $m \geq 2$. We write $E := \text{End}_{\text{group}}(A) \otimes \mathbb{Q}$. Then by definition, E is a totally imaginary quadratic extension of a totally real number field K with $[K : \mathbb{Q}] = m \geq 2$. First we make A explicit up to isogeny. As E is a totally imaginary field with $[E : \mathbb{Q}] = 2m$, there are exactly $2m$ different complex embeddings $\varphi_i : E \rightarrow \mathbb{C}$ ($1 \leq i \leq 2m$) such that $\varphi_{2m-i} = \overline{\varphi_i}$. Here $\overline{}$ is the complex conjugate of \mathbb{C} . Note that there are exactly $2^m \cdot m!$ ways of numberings I of the embeddings here. Choosing one such numbering I , we consider the embedding:

$$\varphi_I := (\varphi_1, \varphi_2, \dots, \varphi_m) : E \rightarrow \mathbb{C}^m ; a \mapsto (\varphi_1(a), \varphi_2(a), \dots, \varphi_m(a)) .$$

Let O_E (resp. O_K) be the integral closure of \mathbb{Z} in E (resp. in K). Then

$$B_I := \mathbb{C}^m / \varphi_I(O_E)$$

is an abelian variety and A is isogenous to B_I for some numbering I (See eg. [Mi06, Chapter I, Section 3]).

From now, we shall prove that the abelian variety $B := B_I$ admits an automorphism of positive entropy.

Definition 2.6. Let $\overline{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} , $\overline{\mathbb{Z}}$ be the integral closure of \mathbb{Z} in $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Z}}^\times$ be the unit group of the ring $\overline{\mathbb{Z}}$. A real algebraic integer is an element of $\overline{\mathbb{Z}} \cap \mathbb{R}$. A real algebraic integer α is called a *Pisot number* if $\alpha > 1$ and $|\alpha'| < 1$ for all Galois conjugates $\alpha' \neq \alpha$ of α over \mathbb{Q} . A Pisot number α is called a *Pisot unit* if $\alpha \in \overline{\mathbb{Z}}^\times$.

Then, by [BDGPS92, Theorem 5.2.2], we have

Theorem 2.7. *For any real number field L , there is a Pisot unit $\alpha \in L$ such that $L = \mathbb{Q}(\alpha)$.*

As K is (totally) real, there is then a Pisot unit α such that $K = \mathbb{Q}(\alpha)$. Consider the linear automorphism of \mathbb{C}^m defined by:

$$\tilde{f}_\alpha : \mathbb{C}^d \rightarrow \mathbb{C}^d ; (z_1, z_2, \dots, z_m) \mapsto (\varphi_1(\alpha)z_1, \varphi_2(\alpha)z_2, \dots, \varphi_m(\alpha)z_m) .$$

As α is a unit in O_K (hence in O_E), so are $\varphi_i(\alpha)$ in $\varphi_i(O_E)$. Thus $\tilde{f}_\alpha(\varphi_I(O_E)) = \varphi_I(O_E)$ by the definition of φ_I . Hence \tilde{f}_α descends to an automorphism f_α of B . We set $f := f_\alpha$.

As K is totally real, regardless of I , we have

$$\{\varphi_i(\alpha) \mid 1 \leq i \leq m\} = \{\alpha := \alpha_1, \alpha_2, \dots, \alpha_m\}.$$

Here the right hand side is the set of all Galois conjugates of α over \mathbb{Q} . By the construction of f from \tilde{f}_α , the left hand side set also coincides with the set of eigenvalues of $f_*|H^0(B, \Omega_B^1)^*$, and therefore, coincides with the set of eigenvalues of $f^*|H^0(B, \Omega_B^1)$. As B is an abelian variety, we have

$$H^{1,1}(B) = H^0(B, \Omega_B^1) \otimes \overline{H^0(B, \Omega_B^1)}.$$

Here $\overline{H^0(B, \Omega_B^1)}$ is the complex conjugate of $H^0(B, \Omega_B^1) \subset H^1(B, \mathbb{Z}) \otimes \mathbb{C}$. As α is real, it follows that α^2 is an eigenvalue of the action of f on $H^{1,1}(B)$. Hence

$$h_{\text{top}}(f) \geq r_1(f) \geq \alpha^2 > 1.$$

Here the last inequality follows from the fact that $\alpha > 1$. Thus f is of positive entropy. In particular, $\text{ord}(f) = \infty$. Therefore, f is primitive as well by the first part of Theorem 1.1. This completes the proof of Theorem 1.1.

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