Derived category
and
Cohen-Macaulay representations

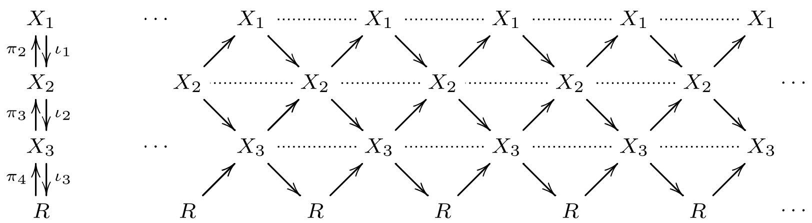
Osamu Iyama

Nagoya University

August 31, 2015

$R := k[x]/(x^n)$

- Indecomposable R-modules are
  $X_i = k[x]/(x^i)$ $(1 \le i \le n)$
- Irreducible morphisms are
  $\pi_i : X_i \twoheadrightarrow X_{i-1}$ and $\iota_{i-1} : X_{i-1} \overset{x}{\rightarrowtail} X_i$
- *Auslander-Reiten quiver* of mod R is



Gluing *Auslander-Reiten sequences*

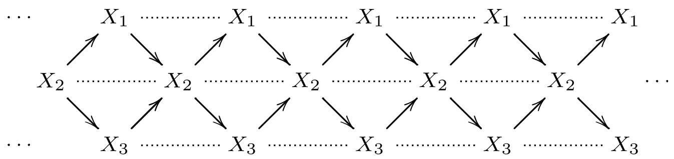
$$0 \to X_i \xrightarrow{(\pi_i\ \iota_i)} X_{i-1} \oplus X_{i+1} \xrightarrow{\binom{\iota_{i-1}}{-\pi_{i+1}}} X_i \to 0$$

Relations of mod R are

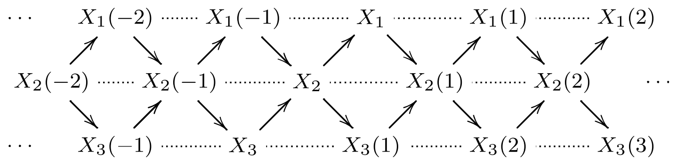
$(X_i \xrightarrow{\pi_i} X_{i-1} \xrightarrow{\iota_{i-1}} X_i) = (X_i \xrightarrow{\iota_i} X_{i+1} \xrightarrow{\pi_{i+1}} X_i)$ $(1 \leq i < n)$

$0 = (X_1 \xrightarrow{\iota_1} X_2 \xrightarrow{\pi_2} X_1)$ $\quad (R \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\iota_{n-1}} R) \neq 0$

- $\underline{\text{mod}}\, R := (\text{mod}\, R)/[R]$: *stable category*



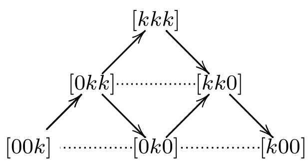
- $\underline{\text{mod}}^{\mathbb{Z}} R := (\text{mod}^{\mathbb{Z}} R)/[R]$: *stable category*



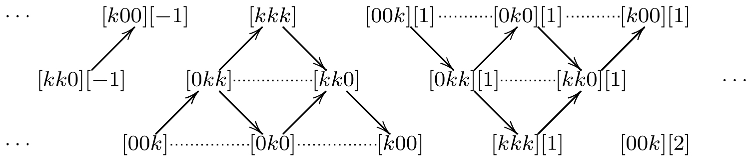
$\mathbb{A}_n \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$

$$k\mathbb{A}_n = \begin{bmatrix} k & k & \cdots & k \\ 0 & k & \cdots & k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}$$: path algebra over a field k

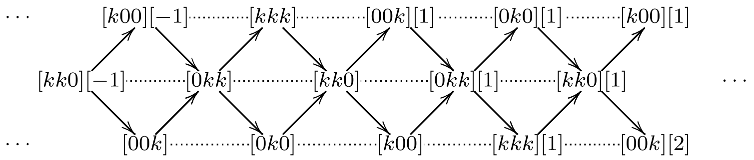
- Indecomposable $k\mathbb{A}_n$-modules are $[0 \cdots 0 \overset{i}{k} \cdots \overset{j}{k}\ 0 \cdots 0]$ $(1 \leq i \leq j \leq n)$
- Auslander-Reiten quiver of mod $k\mathbb{A}_3$ is



• Glueing



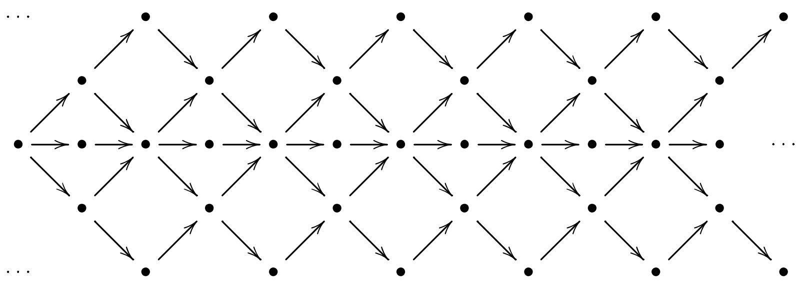
we obtain Auslander-Reiten quiver of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k\mathbb{A}_3)$



$\exists$ equivalence of categories
$\underline{\mathsf{mod}}^{\mathbb{Z}}(k[x]/(x^n)) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \mathbb{A}_{n-1})$

• [Gabriel, Happel] Q : *Dynkin quiver* ($\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$)
$\Longrightarrow$ Auslander-Reiten quiver of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kQ)$ is $\mathbb{Z}Q$

<u>Example</u> : For $Q = \mathbb{E}_6$, $\mathbb{Z}Q$ is



Cohen-Macaulay representation

$T = k[[T_1, \ldots, T_d]]$: formal power series ring
R : module-finite T-algebra
mod R : category of finitely generated R-modules

- No duality mod $R \simeq$ mod R^{op}
- mod R does not have enough injectives

CM $R := \{X \in \text{mod } R \mid X \text{ is a free } T\text{-module}\}$
: category of *Cohen-Macaulay* R-modules

- $\exists$ duality $\text{Hom}_T(-, T)$: CM $R \simeq$ CM R^{op}

- R is a *T-order* (i.e. $R \in$ CM R) $\Longrightarrow$
CM R has enough projectives and enough injectives

Commutative T-orders = Cohen-Macaulay rings with a Noetherian normalization T

R : *isolated singularity* $\Longleftrightarrow$ gl.dim $(T \otimes_R R_{\mathfrak{p}}) = \dim T_{\mathfrak{p}}$ for any non-maximal prime ideal $\mathfrak{p}$ of T

Theorem [Auslander]

R : T-order

- R is an isolated singularity $\Longrightarrow$
CM R has *Auslander-Reiten sequences*

R : *CM-finite* $\iff$ There are only finitely many indecomposable CM R-modules

- CM-finite $\Longrightarrow$ isolated singularity

R : CM-finite with dim $R \leq 2 \Longrightarrow$ The category CM R is mostly reconstructed from its Auslander-Reiten quiver

Example : Simple singularities $R = k[[x_0, \dots, x_d]]/(f)$

type	$f(x_0, \cdots, x_d)$
A_n	$x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2$
D_n	$x_0^{n-1} + x_0 x_1^2 + x_2^2 + \cdots + x_d^2$
E_6	$x_0^4 + x_1^3 + x_2^2 + \cdots + x_d^2$
E_7	$x_0^3 x_1 + x_1^3 + x_2^2 + \cdots + x_d^2$
E_8	$x_0^5 + x_1^3 + x_2^2 + \cdots + x_d^2$

Classification of CM-finite Cohen-Macaulay rings

Under reasonable assumptions

- dim 0 : $k[x]/(x^n)$
- dim 1 : Simple singularities and their overrings [Jacobinski, Drozd-Kirichenko-Roiter]
- dim 2 : Quotient singularities [Auslander, Esnault]
- Gorenstein case : Simple singularities [Knörrer, Buchweitz-Greuel-Schreyer]

$\exists$ further results for non-commutative case

$\underline{\mathsf{CM}}\, R := (\mathsf{CM}\, R)/[R]$: *Stable category*

Properties

Assume R is *Gorenstein*
(i.e. $\mathrm{Hom}_T(R, T)$ is a projective R-module)

- [Happel] $\underline{\mathsf{CM}}\, R$ is a *triangulated category*

- *Suspension functor* is given by cosyzygy
  $\Omega^{-1} : \underline{\mathsf{CM}}\, R \simeq \underline{\mathsf{CM}}\, R$
- *Triangles* are induced from short exact sequences
  $0 \to X \to Y \to Z \to 0$

- [Auslander] R : commutative $\Longrightarrow$ $\underline{\mathsf{CM}}\, R$ is
$(d-1)$-*Calabi-Yau* : $\underline{\mathrm{Hom}}_R(X, Y) \simeq D\mathrm{Ext}_R^{d-1}(Y, X)$

Tilting theory

Tilting theory reduces the study of a triangulated category to that of a derived category of a ring

$\mathcal{T}$: triangulated category

Definition

$U \in \mathcal{T}$: *tilting object* $\Longleftrightarrow$

- $\forall i \neq 0$ $\mathrm{Hom}_{\mathcal{T}}(U, U[i]) = 0$
- U generates $\mathcal{T}$ (as a thick subcategory)

Example : $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ has a tilting object

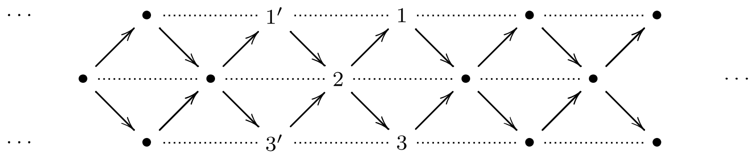
$$\Lambda = (\cdots \to 0 \to 0 \to \overset{0}{\Lambda} \to 0 \to 0 \to \cdots)$$

Tilting Theorem [Rickard, Keller]

$\mathcal{T}$: *algebraic* idempotent complete triangulated category
$U \in \mathcal{T}$: tilting object $\Longrightarrow \mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\mathrm{End}_{\mathcal{T}}(U))$

Example

$\mathcal{T}$



$$\mathrm{End}_{\mathcal{T}}(U_1) = k[\,1 \longrightarrow 2 \longleftarrow 3\,]$$
$$\mathrm{End}_{\mathcal{T}}(U_2) = k[\,1 \longrightarrow 2 \longrightarrow 3'\,]$$
$$\mathrm{End}_{\mathcal{T}}(U_3) = k[\,1' \longleftarrow 2 \longrightarrow 3'\,]$$

They have the same derived categories

Applications of Tilting theory

- Group theory
- Lie theory
- Algebraic geometry
- Mirror symmetry
- $\cdots$

Aim of this talk

Apply Tilting theory to CM representations

G : finitely generated abelian group

Assume that R is Gorenstein and *G-graded*

(i.e. $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subset R_{g+h}$)

- $\mathrm{CM}^G R$: category of G-graded CM R-modules
- $\underline{\mathrm{CM}}^G R := (\mathrm{CM}^G R)/[R]$: *stable category*

This is a triangulated category

Key observation

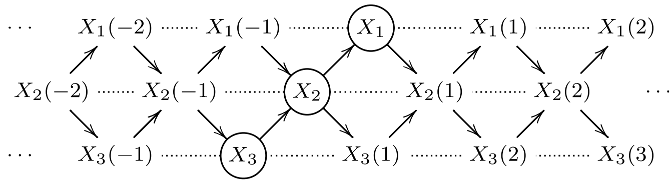
- $\operatorname{rank} G = 1 \Longrightarrow \underline{\mathrm{CM}}^G R$ is often triangle equivalent to the *derived category* of a finite dimensional algebra

  $\Longleftrightarrow \underline{\mathrm{CM}}^G R$ has a tilting object
- $\operatorname{rank} G = 0 \Longrightarrow \underline{\mathrm{CM}}^G R$ is often triangle equivalent to the *cluster category* of a finite dimensional algebra

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded Gorenstein k-algebra

- [Happel, Yamaura] $\dim R = 0$ and $\operatorname{gl.dim} R_0 < \infty \Longrightarrow$ $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ has a tilting object $T := \bigoplus_{i \geq 0} R(i)_{\geq 0}$

Example : $R = k[x]/(x^n) \Longrightarrow \underline{\mathrm{End}}_R^{\mathbb{Z}}(T) \simeq k\mathbb{A}_{n-1}$



- [Buchweitz-I.-Yamaura, in prepration]

  A result for $d = 1$

- [Geigle-Lenzing, Kajiura-Saito-Takahashi]
R is a simple singularity with $\dim R = 2 \Longrightarrow$
$\underline{\mathsf{CM}}^{\mathbb{Z}} R$ has a tilting object and $\underline{\mathsf{CM}}^{\mathbb{Z}} R \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)$

In this case, the *preprojective algebra* Π and an idempotent $e \in \Pi_0$ satisfy $e\Pi e = R$ and $\Pi_0/(e) = kQ$

- [Amiot-I.-Reiten]
A result replacing Π by a $\mathbb{Z}$-graded $(d+1)$-Calabi-Yau algebra with Gorenstein parameter 1

- [Ueda, Iyama-Takahashi, Mori-Ueyama]
Results for quotient singularities

Geigle-Lenzing complete intersection

$d = 1$ [Geigle-Lenzing, 1987]
$d \geq 0$ [Herschend-I.-Minamoto-Oppermann, 2014]

- k : field
- $d \geq 0$
- $C := k[T_0, \ldots, T_d]$: polynomial algebra
- $n \geq 0$
- $\ell_1, \ldots, \ell_n \in C$: linear forms
- $p_1, \ldots, p_n \geq 2$: integers (*weights*)

Definition

- $R := C[X_1, \ldots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$
- $\mathbb{L} := \langle \vec{c}, \vec{x}_1, \ldots, \vec{x}_n \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle$

- $\mathbb{L}$ is an abelian group of rank one
- R is an $\mathbb{L}$-graded k-algebra : $\deg T_j := \vec{c}$, $\deg X_i := \vec{x}_i$
- R is a complete intersection ring in dimension $d+1$

Definition

$(R, \mathbb{L})$: *Geigle-Lenzing (GL) complete intersection* $\Longleftrightarrow$ $\forall$ at most $d+1$ elements from $\ell_1, \ldots, \ell_n$ are linearly independent

- $\mathrm{Ext}^d_R(k, R(\vec{\omega})) \simeq k$ (i.e. R has an *a-invariant* $\vec{\omega}$) for $\vec{\omega} := (n-d-1)\vec{c} - \sum_{i=1}^n \vec{x}_i$
- *Auslander-Reiten duality*
  $\underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathbb{L}} R}(X, Y) \simeq D\mathrm{Ext}^d_{\mathrm{mod}^{\mathbb{L}} R}(Y, X(\vec{\omega}))$

- $n \leq d+1 \Longleftrightarrow R$ is regular
- $n = d+2 \Longleftrightarrow R$ is a hypersurface
  $R \simeq k[X_1, \ldots, X_{d+2}]/(\sum_{i=1}^n \alpha_i X_i^{p_i})$
- The case $d = 0$
  $R \simeq k[X_1, \cdots, X_n]/(X_i^{p_i} - \alpha_i X_1^{p_1})_{2 \leq i \leq n}$

The case $d = 1$

$(R, \mathbb{L})$ is the *weighted projective line* of Geigle-Lenzing
$R \simeq k[X_1, \cdots, X_n]/(X_i^{p_i} - \alpha_{i1} X_1^{p_1} - \alpha_{i2} X_2^{p_2})_{3 \leq i \leq n}$

- *domestic* $\Leftrightarrow$ (p, q), $(2, 2, p)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$

In this case the Veronese subring $R^{(\vec{\omega})}$ is a simple singularity in dimension 2 and $\mathrm{CM}^{\mathbb{L}} R \simeq \mathrm{CM}^{\mathbb{Z}} R^{(\vec{\omega})}$

Theorem [HIMO]

$\exists$ a finite dimensional k-algebra A^{CM} and
a triangle equivalence $\underline{\mathsf{CM}}^{\mathbb{L}}R \simeq \mathsf{D}^{\mathsf{b}}(\mathrm{mod}\, A^{\mathsf{CM}})$

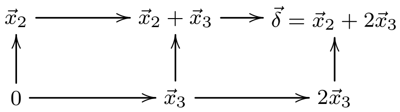
The case $n = d + 2$: [Kussin-Meltzer-Lenzing] ($d = 1$), [Futaki-Ueda], [Ballard-Favero-Katzarkov]

- $\mathbb{L}$ has a partial order :
  $\vec{x} \geq \vec{y} \iff \vec{x} - \vec{y} \in \langle \vec{c}, \vec{x}_1, \ldots, \vec{x}_n \rangle_{\mathsf{monoid}}$
- $\vec{\delta} := d\vec{c} + 2\vec{\omega} \in \mathbb{L}$
- $A^{\mathsf{CM}} := (R_{\vec{x}-\vec{y}})_{0 \leq \vec{x}, \vec{y} \leq \vec{\delta}}$: *CM-canonical algebra*
  k-algebra by product in R and matrix multiplication

Example : $n = d + 2 \Longrightarrow$
$\vec{\delta} = \sum_{i=1}^{n} (p_i - 2)\vec{x}_i$ and $A^{\mathsf{CM}} = \bigotimes_{i=1}^{n} k\mathbb{A}_{p_i - 1}$

Example : $d = 1$ with weights $(2, 3, 4)$
$R = k[X_1, X_2, X_3]/(\alpha_1 X_1^2 + \alpha_2 X_2^3 + \alpha_3 X_3^4)$

$A^{\mathsf{CM}} = k\mathbb{A}_2 \otimes k\mathbb{A}_3$

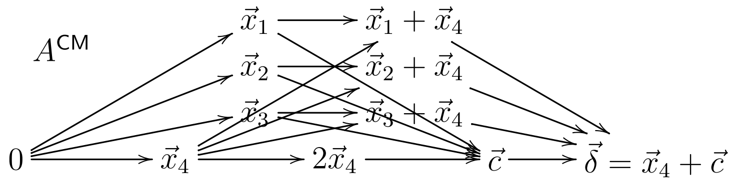


Corollary (Knörrer periodicity)

$(R, \mathbb{L})$, $(R', \mathbb{L}')$: GL hypersurfaces with weights $(p_1, \ldots, p_{d+2})$ and $(2, p_1, \ldots, p_{d+2})$ respectively $\Longrightarrow$
$\underline{\mathsf{CM}}^{\mathbb{L}}R \simeq \underline{\mathsf{CM}}^{\mathbb{L}'}R'$

Example : $d = 1$ with weights $(2,2,2,3)$

$$R = \frac{k[X_1, X_2, X_3, X_4]}{(X_3^2 - \alpha_{31}X_1^2 - \alpha_{32}X_2^2,\ X_4^3 - \alpha_{41}X_1^2 - \alpha_{42}X_2^2)}$$



Idea of proof of $\underline{\mathsf{CM}}^{\mathbb{L}}R \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, A^{\mathsf{CM}})$

- [Buchweitz, Orlov] $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{\mathbb{L}}R)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}^{\mathbb{L}}R) \simeq \underline{\mathsf{CM}}^{\mathbb{L}}R$
- $(-)^* := \mathsf{RHom}_R(-, R) : \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{\mathbb{L}}R) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{\mathbb{L}}R)$
- Step 1 :

Let $\mathcal{X} := \{X \in \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{\mathbb{L}_+}R) \mid X^* \in \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{-\mathbb{L}_+}R)\}$
By Orlov's semiorthogonal decomp., the composition
$\mathcal{X} \subset \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{\mathbb{L}}R) \to \underline{\mathsf{CM}}^{\mathbb{L}}R$ is an equivalence

- Step 2 : Show $\mathcal{X} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}^{[0,\vec{\delta}]}R)$ by regular sequence
- Step 3 : Show $\mathsf{mod}^{[0,\vec{\delta}]}R \simeq \mathsf{mod}\, A^{\mathsf{CM}}$ by Morita theory

□

$(R,\mathbb{L})$: *CM-finite* $\iff$ there are only finitely many indecomposable objects in $\mathsf{CM}^{\mathbb{L}}R$ up to degree shift

- [Geigle-Lenzing] For $d=1$, CM-finite $\iff$ domestic

Corollary (Classification) [HIMO]

$(R,\mathbb{L})$: GL complete intersection is CM-finite

$\iff$ • $n \leq d+1$ (i.e. R is regular) or

• $n = d+2$ and weights are $(2,\ldots,2,2,p)$, $(2,\ldots,2,3,3)$, $(2,\ldots,2,3,4)$, $(2,\ldots,2,3,5)$ (i.e. R is domestic up to Knörrer periodicity)

Higher Auslander-Reiten theory

Definition

A full subcategory $\mathcal{C} \subset \mathsf{CM}^{\mathbb{L}}R$ is *d-cluster tilting* $\iff$

$$\mathcal{C} = \{X \in \mathsf{CM}^{\mathbb{L}}R \mid \forall i \in [1,d-1]\ \mathrm{Ext}^i_{\mathrm{mod}^{\mathbb{L}}R}(\mathcal{C},X)=0\}$$
$$= \{X \in \mathsf{CM}^{\mathbb{L}}R \mid \forall i \in [1,d-1]\ \mathrm{Ext}^i_{\mathrm{mod}^{\mathbb{L}}R}(X,\mathcal{C})=0\}$$

and $\mathcal{C}$ is functorially finite

In this case

- $\mathcal{C} = \mathcal{C}(\vec{\omega})$ holds
- $\mathcal{C}$ has *d-Auslander-Reiten sequences* $0 \to X(\vec{\omega}) \to C_{d-1} \to \cdots \to C_0 \to X \to 0$
- $\mathcal{C}$ has *d-fundamental sequences* $0 \to R(\vec{\omega}) \to C_{d-1} \to \cdots \to C_0 \to R \to 0$

Definition

$(R, \mathbb{L})$: *d-CM-finite*
$\iff \exists\, \mathcal{C} \subset \mathsf{CM}^{\mathbb{L}} R$: d-cluster tilting subcategory s.t. $\mathcal{C}$ contains only finitely many isoclasses of indecomposable objects up to shift by $\mathbb{Z}\vec{\omega}$

$d = 1 \implies$ CM-finite=1-CM-finite

- Define a group homomorphism $\mathsf{d} : \mathbb{L} \to \mathbb{Z}$ by $\mathsf{d}(\vec{c}) = 1$, $\mathsf{d}(\vec{x}_i) = \frac{1}{p_i}$
- $\mathsf{d}(\vec{\omega}) = n - d - 1 - \sum_{i=1}^{n} \frac{1}{p_i}$
- $(R, \mathbb{L})$: *Fano* $\iff \mathsf{d}(\vec{\omega}) < 0$

Example : For $d = 1$, Fano $\iff$ domestic

Theorem (Criterion for d-CM-finiteness) [HIMO]

$(R, \mathbb{L})$: GL complete intersection
Then (a)⇒(b)+(c) holds

(a) $\exists$ finite dimensional k-algebra A s.t.
gl.dim $A \leq d$ and $\underline{\mathsf{CM}}^{\mathbb{L}} R \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, A)$

(b) $(R, \mathbb{L})$ is d-CM-finite

(c) $(R, \mathbb{L})$ is Fano

Example : These conditions are satisfied if

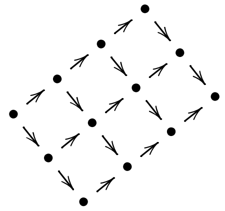
- $n \leq d + 1$ or
- $n = d + 2$ and weights are

$(2, 2, p_3, p_4, \ldots, p_n)$, $(2, 3, 3, p_4, \ldots, p_n)$,
$(2, 3, 4, p_4, \ldots, p_n)$, $(2, 3, 5, p_4, \ldots, p_n)$

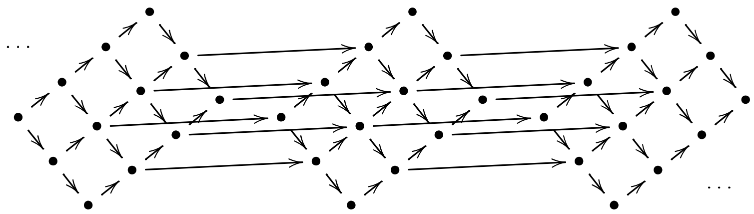
Example : $d = 2$ with weights $(2, 2, 4, 5)$

$R = k[X_1, X_2, X_3, X_4]/(\alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^4 + \alpha_4 X_4^5)$

$A^{\text{CM}} = k\mathbb{A}_3 \otimes k\mathbb{A}_4$



A 2-cluster tilting subcategory $\mathcal{C} \subset \text{CM}^{\mathbb{L}} R$:



Conjecture

All conditions (a), (b) and (c) are equivalent

References :

[GL] W. Geigle, H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles, 265–297, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

[HIMO] M. Herschend, O. Iyama, H. Minamoto, S. Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, arXiv:1409.0668.