SINGULARITIES OF MODULI OF STABLE SHEAVES ON SOME ELLIPTIC SURFACES

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Abstract. Let X be some type of elliptic surface X over $\mathbb{C}$ with $\kappa(X) = 1$, and $M(c)$ the coarse moduli scheme of rank-two stable sheaves with Chern classes $(c_1, c_2) = (0, c)$ on X. Then $M(c)$ allows only canonical singularities. By using it, we hope eventually to calculate the Kodaira dimension of $M(c)$.

1. Introduction

Let be $\mathcal{O}(1)$ an ample line bundle on a non-singular projective surface X over $\mathbb{C}$. A torsion-free sheaf E on X is $\mathcal{O}(1)$-*stable (resp. semistable)* if for any proper subsheaf F of E one has $\chi(F(n))/\mathrm{rk}(F) < \chi(E(n))/\mathrm{rk}(E)$ (resp. $\leq$) when $n \gg 0$. There exists the coarse moduli scheme $M(c)$ of $\mathcal{O}(1)$-stable rank-two sheaves with Chern classes $(c_1, c_2) = (0, c) \in \mathrm{Pic}(X) \times \mathbb{Z}$ by Gieseker-Maruyama. If c is odd, then $M(c)$ is projective over $\mathbb{C}$. By Donaldson and Zuo, if c is sufficiently large w.r.t. X and $\mathcal{O}(1)$, then $M(c)$ is normal, l.c.i., and of dimension $\mathrm{ext}^1(E, E)^0 - \mathrm{ext}^2(E, E)^0$ with $E \in M(c)$.

In this article, we shall consider the following question, and report the following theorem.

Question 1.1. (1) How is the birational property of $M(c)$, e.g. its Kodaira dimension $\kappa(M(c))$? (2) Does $M(c)$ allow only canonical singularities?
(See Definition 2.1 for definition of terms)

Theorem 1.2. *Let X be a minimal elliptic surface over $\mathbf{P}^1$ s.t. (i) $\chi(\mathcal{O}_X) = 1$, (ii) its singular fibers are either rational integral curve with one node (I_1) or multiple fiber with smooth reduction (nI_0), and (iii) X has just two multiple fibers with multiplicities $(2, m)$, where m is odd and $m \geq 3$. In particular, $\kappa(X)$ is 1. Let $\mathcal{O}(1)$ be c-suitable, that is, $\mathcal{O}(1)$ is so close to the fiber class $\mathfrak{f}$ of the elliptic fibration $X \to \mathbf{P}^1$, that $\mathcal{O}(1)$ and $\mathfrak{f}$ is not divided by any c-wall (*[1, Def. 2.1]*).*

Then $M(c)$ admits only canonical singularities.

For some history of Question 1.1, see Section 3. As explained there, this question is settled mainly in the one case where $p_g(X) \neq 0$ and one can use generically non-degenerate two-forms, or in the another case where moduli of sheaves is related to

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some more-clarified scheme, e.g. Hilbert scheme of points, but neither case fits the situation of Theorem 1.2. Expecting to calculate Kodaira dimension of $M(c)$ by using its original definition and Theorem 1.2, the author is now trying to estimate the dimension of pluricanonical maps of $M(c)$, and hopes to report it somewhere else. We end with notifying that such a plan worked well in the following.

Fact 1.3. [6, Y] *Let M be a moduli scheme of stable sheaves with fixed Chern classes on an Enriques surface or a hyper-elliptic surface. If its expected dimension is not less than 7, then M admits only canonical singularities. Moreover, if M is compact, then its Kodaira dimension is zero. Also the characteristic of singular points of M is obtained at* [6, Lem. 13(a)].

Notation. For a line bundle L, $\mathrm{Ext}^i(E, E\otimes L)^\circ$ denotes the kernel of trace map $\mathrm{Ext}^i(E, E\otimes L) \to H^i(L)$. Denote $\dim \mathrm{Ext}^i(E,F)$ and $\dim \mathrm{Ext}^i(E,E\otimes)^\circ$ by $\mathrm{ext}^i(E,F)$ and $\mathrm{ext}^i(E,E\otimes L)^\circ$ respectively.

2. Ideas in the proof of Theorem 1.2

Let us begin with recalling the definition of some terms.

Definition 2.1. (1) Given any variety V_0, define its *Kodaira dimension* $\kappa(V_0)$ to be $\max\{\dim \Phi_{mK_{\tilde{V}}} \mid m \in \mathbb{N}\}$, where $\tilde{V}$ is a desingularization of a completion of V_0. Kodaira dimension is birational invariant.
(2) A normal variety V is said to admit only *canonical singularities* when (i) K_V is $\mathbb{Q}$-Cartier, and (ii) if $\phi : \tilde{V} \to V$ is a desingularization with except divisors E_i, then

$$K_{\tilde{V}} = \phi^* K_V + \textstyle\sum_i a_i E_i \qquad (a_i \geq 0).$$

When V does so and V is complete, $\kappa(V)$ equals $\max\{\dim \Phi_{mK_{\tilde{V}}} \mid m \in \mathbb{N}\}$, so we need not consider its desingularization $\tilde{V}$ in calculating $\kappa(V)$.

Lemma 2.2. *Under assumptions in Theorem 1.2, any sheaf $E \in M$ satisfies that* $\mathrm{ext}^2(E,E)^\circ = \hom(E, E(K_X))^\circ \leq 1$.

The next fact results from deformation theory of sheaves and singularities theory.

Fact 2.3 ([6] Lem. 2.5.). *Let E be a stable sheaf on a projective surface.*
(1) If $\hom(E,E(K_X))^\circ = 0$*, then moduli M is non-singular at E.*
(2) Suppose $\hom(E, E(K_X))^\circ = 1$ *so* $\mathrm{Hom}(E, E(K_X))^\circ = \mathbb{C}\cdot f$*. Then $f : E \to E(K_X)$ define a map $H^1(f_-) : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^1(E, E(K_X))$ by $H^1(f_-)(\alpha) = f\circ\alpha - \alpha\circ f$. If $\mathrm{rk} H^1(f_-) \geq 3$, then M admits only canonical singularity at E.*

Thus it's important to estimate $\mathrm{rk}H^1(f_-)$. Let $k(\mathbf{P}^1)$ denote the function field of $\mathbf{P}^1$ and $\overline{k(\mathbf{P}^1)}$ its algebraic closure. We set $\eta = \eta(\mathbf{P}^1) = \mathrm{Spec}(k(\mathbf{P}^1)), \bar{\eta} = \mathrm{Spec}((\overline{k(\mathbf{P}^1)}))$, $X_\eta = X \times_{\mathbf{P}^1} \eta$, and $X_{\bar{\eta}} = X \times_{\mathbf{P}^1} \bar{\eta}$. $X_{\bar{\eta}}$ is a nonsingular elliptic curve over $\bar{\eta}$. Any sheaf F on X induces F_η on X_η, and $F_{\bar{\eta}}$ on $X_{\bar{\eta}}$. For $E \in M(c)$, $E_{\bar{\eta}}$ is degree-zero semi-stable vector bundle on $X_{\bar{\eta}}$ since $\mathcal{O}(1)$ is c-suitable, and so Atiyah's classification of vector bundles on an elliptic curve deduces the following.

Lemma 2.4. *For $E \in M(c)$, one of the following holds:*

(A) $E_{\bar{\eta}}$ is decomposable, that is, $E_{\bar{\eta}} \simeq \mathcal{O}_{X_{\bar{\eta}}}(F) \oplus \mathcal{O}_{X_{\bar{\eta}}}(-F)$ on $X_{\bar{\eta}}$. Moreover,

(A-1) $\mathcal{O}_{X_{\bar{\eta}}}(F)$ is not rational over $k(\mathbf{P}^1)$. Let $C \to \mathbf{P}^1$ be the double cover consisting of nonsingular curves which corresponds to the stabilizer subgroup of $\mathcal{O}_{X_{\bar{\eta}}}(F)$ in $\mathrm{Gal}(\overline{k(\mathbf{P}^1)}/k(\mathbf{P}^1))$. Then $\mathcal{O}_{X_{\bar{\eta}}}(F)$ is rational over $\eta' = \mathrm{Spec}(k(C))$.

(A-2) $\mathcal{O}_{X_{\bar{\eta}}}(F)$ is rational over $k(\mathbf{P}^1)$.

(B) $E_{\bar{\eta}}$ is indecomposable on $X_{\bar{\eta}}$.

Let E be a singular point of $M(c)$, and then there exists a traceless homomorphism $f : E \to E(K_X)$ by Fact 2.3 (1). We study E and f with Lemma 2.4 in mind, and get the following.

Proposition 2.5. *Under assumptions in Theorem 1.2, any singular point $E \in M(c)$ satisfies the following: In Lemma 2.4, only Case (A-1) occurs; any traceless homomorphism $f : E \to E(K_X)$ satisfies $\det f \neq 0$; the determinant $\det f \in \Gamma(2K_X)$ induces double covers $C \to \mathbf{P}^1$ and $\gamma : Y = X \times_{\mathbf{P}^1} C \to \mathbf{P}^1$, and decompositions of $\gamma^* E$ on Y*

$$0 \longrightarrow F_\pm \longrightarrow \gamma^* E \longrightarrow G_\pm \longrightarrow 0, \tag{1}$$

that extend decompositions of $E_{\bar{\eta}}$ on $X_{\bar{\eta}}$

$$0 \longrightarrow Ker(f \pm s) \longrightarrow E_{\bar{\eta}} \longrightarrow Im(f \pm s) \longrightarrow 0,$$

where $\pm s$ are eigenvalues of $f_{\bar{\eta}} : E_{\bar{\eta}} \to E(K_X)_{\bar{\eta}} \simeq E_{\bar{\eta}}$.

To estimate the rank of $H^1(f_-) : \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(E, E(K_X))$, we look into $RHom(F_\pm, G_\pm)$, and so on. Since only Case (A-1) occurs by Proposition 2.5, subsheaves $F_\pm \subset \gamma^* E$ do not descend to subsheaves of E. Consequently several cohomology groups coming from $RHom(F_\pm, G_\pm)$ etc. vanish, and thus we can obtain good estimation of $\mathrm{rk} H^1(f_-)$ from below. In such a way, we can prove Theorem 1.2.

3. Appendix: History of Question 1.1

Here we note some historical background of Question 1.1; refer to [4, Section 11] for more. When X is a minimal surface with $K_X = 0$, i.e. a $K3$ surface or a torus, moduli scheme M of rank-two stable sheaves is of Kodaira dimension zero by [4, Thm. 11.1.7.]. If X is a minimal surface of general type, the expected dimension of moduli scheme $\mathrm{ext}^1(E, E)^0 - \mathrm{ext}^2(E, E)^0$ is even, and $|K_X|$ contains a reduced curve, then M is of general type for $c_2 \gg 0$ by [5]. In these works, they utilize generically non-degenerate two-forms, a generalization of symplectic forms introduced by Mukai. When X is an Enriques surface or hyper-elliptic, see Fact 2.3.

Let X be an elliptic surface. When $c_1(E) \cdot \mathfrak{f}$ is odd, M is non-singular. If in addition X has just two multiple fibers, then moduli M is birationally equivalent to $\mathrm{Sym}^t(J^{e+1}(X))$ by [2, Thm. 3.14], where $c_1(E) \cdot \mathfrak{f} = 2e+1$. This work uses the fact that $E \in M$ is obtained by a sequence of elementary transforms of a special sheaf V_0 s.t. $V_0|_f$ is stable for *every* fiber f.

When $c_1 \cdot \mathfrak{f}$ is even and X is an elliptic surface over $\mathbf{P}^1$ with just two multiple fibers (plus some conditions), then M birationally fibers over some projective space whose fibers are isomorphic to finite union of Jacobian of some hyperelliptic curves by [1, Section 7]. They construct this fibration using the spectral cover induced by a stable sheaf (cf. [3, p. 229]). Some upper bound of $\kappa(M)$ is obtained there, but $\kappa(M)$ itself is still unknown.

Question 1.1 is unsolved yet in the following cases: (a) X is an Enriques surface, but moduli of stable sheaves is not compact. (b) X is of Kodaira dimension one, but $c_1(E) \cdot \mathfrak{f}$ is even. (c) X is of general type, but conditions in [5] do not hold; for example, the expected dimension of moduli is odd, or $p_g(X) = 0$. (d) Most of results above holds when $c_2 \gg 0$. How is the case where c_2 is not sufficiently large?

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