ULRICH IDEALS AND MODULES

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Contents

1. Introduction and definition

Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 0$. Let M be a finitely generated A-module. In [BHU] J. Brennan, J. Herzog, and B. Ulrich gave structure theorems of MGMCM (<u>**m**</u>aximally <u>**g**</u>enerated <u>**m**</u>aximal <u>**C**</u>ohen–<u>**M**</u>acaulay) modules that is, maximal Cohen–Macaulay A–modules M with $\mathrm{e}^0_{\mathfrak{m}}(M) = \mu_A(M)$, where $\mathrm{e}^0_{\mathfrak{m}}(M)$ (resp. $\mu_A(M)$) denotes the multiplicity of M with respect to $\mathfrak{m}$ (resp. the number of elements in a minimal system of generators of M). In [HK] these modules are simply called *Ulrich* modules.

The purpose of my talk is to study Ulrich modules, and ideals as well, with a slightly generalized definition. To state our definition, let I be an $\mathfrak{m}$–primary ideal in A and assume that I contains a parameter ideal Q of A as a reduction; hence $I^{n+1} = QI^n$ for all $n \gg 0$. Remember that the latter condition, that is the existence of reductions, is satisfied, when the residue class field $A/\mathfrak{m}$ of A is infinite.

Definition 1.1. Let M ($\neq (0)$) be a finitely generated A–module. Then we say that M is an Ulrich A–module with respect to I, if

(1) M is a Cohen-Macaulay A–module with $\dim_A M = d$,
(2) $\mathrm{e}^0_I(M) = \ell_A(M/IM)$, and
(3) M/IM is A/I–free,

where $\mathrm{e}^0_I(M)$ denotes the multiplicity of M with respect to I and $\ell_A(*)$ denotes the length.

This talk is based on a work [GOTWY] jointly with R. Takahashi, K. Ozeki, K.-i. Watanabe, and K.-i. Yoshida
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Let me give a few comments about Definition 1.1. Suppose that M is a maximal Cohen–Macaulay A-module. Then

$$\mathrm{e}_I^0(M) = \mathrm{e}_Q^0(M) = \ell_A(M/QM) \geq \ell_A(M/IM) \geq \ell_A(M/\mathfrak{m}M) = \mu_A(M).$$

Hence condition (2) is equivalent to saying that $QM = IM$. If $I = \mathfrak{m}$, then condition (3) is automatically satisfied, and in general we have $\mathrm{e}_\mathfrak{m}^0(M) \geq \mu_A(M)$, and $\mathrm{e}_\mathfrak{m}^0(M) = \mu_A(M)$ if and only if M is a **MGMCM** module in the sense of [BHU].

Definition 1.2. Our ideal I is called an Ulrich ideal of A, if

(1) $I \supsetneq Q$,
(2) $I^2 = QI$, and
(3) I/I^2 is A/I–free.

Here we notice that condition (2) equipped with (1) is equivalent to saying that the associated graded ring

$$\mathrm{gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$$

of I is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$, whence Definition 1.2 is independent of the choice of reductions Q, and the blowing-up of $\mathrm{Spec}A$ with center $\mathrm{V}(I)$ enjoys nice properties. When $I = \mathfrak{m}$, condition (3) is automatically satisfied and condition (2) equipped with (1) is equivalent to saying that A is not a regular local ring, but possesses *maximal embedding dimension* in the sense of J. Sally, i.e, the following equality

$$\mathrm{v}(A) = \mathrm{e}(A) + \dim A - 1$$

holds true, where $\mathrm{v}(A)$ and $\mathrm{e}(A)$ denote, respectively, the embedding dimension of A and the multiplicity of A with respect to $\mathfrak{m}$.

In my talk we shall discuss several basic properties of Ulrich modules and ideals, and the relation between them as well. In Section 2 we will summarize some auxiliary results on Ulrich ideals for the later use.

The main result in Section 3 is the following. Let $\mathrm{Syz}_A^i(A/I)$ denote the i–th syzygy module of A/I in a minimal A-free resolution.

Theorem 1.3 (cf. [BHU])**.** *The following conditions are equivalent.*

(1) *I is an Ulrich ideal of A.*
(2) *$\mathrm{Syz}_A^i(A/I)$ is an Ulrich A–module with respect to I for all $i \geq d$.*
(3) *There exists an exact sequence*

$$0 \to X \to F \to Y \to 0$$

of finitely generated A–modules such that
(a) *F is a finitely generated free A–modules,*
(b) *$X \subseteq \mathfrak{m}F$, and*

(c) *both X and Y are Ulrich A–modules with respect to I.*

If $d > 0$, we can add the following.

(4) $\mu_A(I) > d$, *I/I^2 is A/I–free, and* $\mathrm{Syz}_A^i(A/I)$ *is an Ulrich A–module with respect to I for some $i \geq d$.*

In Section 4 we will give a structure theorem of minimal free resolutions of Ulrich ideals and some applications as well. We shall discuss in Section 5 Ulrich ideals in numerical semi-group rings.

2. Preliminary steps

Let me begin with the following.

Example 2.1. Suppose that R is a Cohen–Macaulay local ring with maximal ideal $\mathfrak{n}$ and $\dim R = d$. Let $F = R^n$ for $n > 0$ and $A = R \ltimes F$ the idealization of R over F. Let $\mathfrak{q}$ be a parameter ideal in R and put $I = \mathfrak{q} \times F$ and $Q = qA$. Then A is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} = \mathfrak{n} \times F$ and $\dim A = d$, I is an $\mathfrak{m}$–primary ideal of A which contains the parameter ideal Q of A as a reduction. We furthermore have that I is an Ulrich ideal of A. Therefore A contains infinitely many Ulrich ideals, if $d = \dim R > 0$.

Question 2.2. I don't know whether those ideals $I = \mathfrak{q} \times F$ are all the Ulrich ideals in $A = R \ltimes F$.

Example 2.3. We have the following.

(1) In the ring $A = k[[X, Y, Z]]/(Z^2 - XY)$, the maximal ideal $\mathfrak{m} = (x, y, z)$ is a unique Ulrich ideal and $\mathfrak{p} = (z, x)$ is a unique indecomposable Ulrich A–module with respect to $\mathfrak{m}$.

(2) The ring $A = k[[t^3, t^5]] \cong k[[X, Y]]/(X^5 - Y^3)$ contains no Ulrich ideals.

We note here a proof of assertion (2). See Example 4.8 for the proof of assertion (1).

Proof of assertion (2). Let $A = k[[t^3, t^5]]$ and $V = k[[t]]$. Assume that A contains an Ulrich ideal, say I, and let $Q = (a)$ be a reduction of I. We put $B = \frac{I}{a} := \{\frac{x}{a} \mid x \in I\} \subseteq V$. Then $B = A[\frac{I}{a}]$ and B is a Gorenstein local ring with $\mu_A(B) = 2$, because

$$I = aB \text{ and } I \cong \mathrm{Hom}_A(B, A).$$

Thus $B \neq V$, since $\mu_A(V) = 3$. We have $t^7 \in B$, because $A : \mathfrak{m} = A + kt^7$ (remember that A is a Gorenstein ring) and $A \subsetneq B$. Hence $t^5 V \subseteq B$. Let $\mathfrak{c} = B :_{\mathrm{Q}(B)} V$ and write $\mathfrak{c} = t^n V$ with $n \geq 1$. Then, since B is a Gorenstein local ring,

$$n = \ell_B(V/\mathfrak{c}) = 2\ell_B(B/\mathfrak{c}).$$

Hence $n \leq 4$, because $t^5 V \subseteq B$. Thus $n = 2$ or $n = 4$. If $t^4 \in B$, then $t^3 V \subseteq B$, whence $n = 2$. Consequently, $k[[t^2, t^3]] \subseteq B \subsetneq V$, and $B = k[[t^2, t^3]]$, since $\ell_{k[[t^2,t^3]]}(V/k[[t^2,t^3]]) = 1$. This is impossible, because $\mu_A(k[[t^2,t^3]]) = 3$. Thus A contains no Ulrich ideals. $\square$

To provide examples of Ulrich modules, we need more preliminaries. For the moment, assume that $d > 0$. Let $a \in Q \backslash \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, and $\overline{Q} = Q/(a)$. We then have the following. Remember that $\mathrm{Syz}_A^i(A/I)$ denotes the i–th syzygy of A/I in a minimal A-free resolution.

Lemma 2.4 (W. V. Vasconcelos). *Suppose I/I^2 is A/I–free. Then*

$$\mathrm{Syz}_A^i(A/I)/a{\cdot}\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^{i}(\overline{A}/\overline{I})$$

for all $i \geq d$.

Proof. We have only to show $I/aI \cong A/I \oplus I/(a)$. Let $I = (a) + (x_1, x_2, \cdots, x_n)$ with $n = \mu_A(I) - 1$. Then $I/aI = A\overline{a} + \sum_{i=1}^n A\overline{x_i}$, where $\overline{a}$ and $\overline{x_i}$ denote the images of a and x_i in I/aI, respectively. Assume that $c\overline{a} + \sum_{i=1}^n c_i\overline{x_i} = 0$ with $c, c_i \in A$. Then $ca + \sum_{i=1}^n c_i x_i \in aI \subseteq I^2$. Since $\{\overline{a}, \overline{x_i} \in I/I^2\}_{1 \leq i \leq n}$ forms a free A/I-basis of I/I^2, we have $c, c_i \in I$ for $1 \leq i \leq n$. Therefore $c\overline{a} = \sum_{i=1}^n c_i\overline{x_i} = 0$, whence $I/aI \cong A/I \oplus I/(a)$. $\square$

Let me note the following.

Lemma 2.5. *Let I be an Ulrich ideal in a Cohen-Macaulay local ring A with $d = \dim A > 0$. Let $a \in Q \setminus \mathfrak{m}Q$, where $Q = (a_1, a_2, \ldots, a_d)$ is a reduction of I. Then $I/(a)$ is an Ulrich ideal of $A/(a)$.*

Proof. We set $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, and $\overline{Q} = Q/(a)$. Then $\overline{I} \supsetneq \overline{Q}$ and $\overline{I}^2 = \overline{Q}\,\overline{I}$. Let us consider the exact sequence

$$0 \to [(a) + I^2]/I^2 \to I/I^2 \to \overline{I}/\overline{I}^2 \to 0$$

of A-modules. We then have

$$I/I^2 \cong A/I \oplus \overline{I}/\overline{I}^2,$$

since I/I^2 is A/I-free and $\overline{a}$ which is the image of a in I/I^2 forms a part of A/I-free basis of I/I^2. Thus $\overline{I}/\overline{I}^2$ is also $\overline{A}/\overline{I}$-free, so that $\overline{I}$ is an Ulrich ideal of $\overline{A}$. $\square$

We note the following. To prove it, we just remember that in the exact sequence

$$0 \to Q/I^2 \to I/I^2 \to I/Q \to 0,$$

the A/I-module $Q/I^2 = Q/QI$ is free and is generated by a part of a minimal basis of I/I^2.

Proposition 2.6. *Suppose that A is a Cohen-Macaulay local ring and assume that $I^2 = QI$. Then the following conditions are equivalent.*

(1) *I/I^2 is A/I-free.*

(2) *I/Q is A/I-free.*

When this is the case, $I = Q :_A I$, if $Q \subsetneq I$; hence I is a good ideal of A in the sense of [GIW]*, if A is a Gorenstein ring.*

The following result shows the number of generators of Ulrich ideals I of A is bounded by the Cohen-Macaulay type $\mathrm{r}(A)$ and the dimension of A.

Proposition 2.7. *Suppose that A is a Cohen-Macaulay local ring and let I be an Ulrich ideal of A. Then we have the following, where $\mathrm{r}(A)$ denotes the Cohen-Macaulay type of A.*

(1) $\mathrm{r}(A) \geq \mu_A(I) - d$.

(2) *$\mu_A(I) = d + 1$ and $I/Q \cong A/I$, if A is a Gorenstein ring.*

Proof. (1) Let $n = \mu_A(I)$ $(> d)$. Then by Proposition 2.6, $I/Q \cong (A/I)^{n-d}$, so that $I = Q :_A I$. Hence $\mathrm{r}(A) = (n-d){\cdot}\mathrm{r}(A/I) \geq n - d > 0$, where $\mathrm{r}(A/I)$ denotes the Cohen-Macaulay type of A/I.

(2) As $\mathrm{r}(A) = 1$, we have $n - d = 1$ by assertion (1), whence $I/Q \cong A/I$. $\square$

3. Proof of Theorem 1.3; relation between Ulrich ideals and modules

The heart of the proof of the implication (3) $\Rightarrow$ (1) in Theorem 1.3 is the following.

Proposition 3.1. *Suppose that A is a Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal in A and assume that I contains a parameter ideal Q of A as a reduction. Assume that there exists an exact sequence*

$$0 \to X \to F \to Y \to 0$$

of finitely generated A-modules such that

(i) *F is a finitely generated free A-module,*

(ii) *$X \neq (0)$ and $X \subseteq \mathfrak{m}F$, and*

(iii) *Y is an Ulrich A-module with respect to I.*

Then the following conditions are equivalent.

(1) *X is an Ulrich A-module with respect to I.*

(2) *$I^2 \subseteq Q$ and I/Q is A/I-free.*

When this is the case, the following assertions hold true.

(a) *$I^2 = QI$ and I/I^2 is A/I-free, if the residue class field $A/\mathfrak{m}$ of A is infinite. Hence I is an Ulrich ideal of A.*

(b) $\mu_A(X) = \mu_A(Y){\cdot}\mathrm{rank}_{A/I}(I/Q)$.

Proof. We consider the exact sequence

$$(\sharp) \quad 0 \to X \to F \to Y \to 0$$

of A-modules. Because $X \neq (0)$ and F and Y are Cohen-Macaulay A-modules with $\dim_A F = \dim_A Y = d$, X is a Cohen-Macaulay A-module with $\dim_A X = d$. We set $m = \mathrm{rank}_A F$; hence $m = \mu_A(Y)$, because $X \subseteq \mathfrak{m}F$. Tensoring exact sequence ($\sharp$) by A/Q, we get an exact sequence

$$0 \to X/QX \to F/QF \to Y/QY \to 0$$

of A-modules, where $Y/QY = Y/IY \cong (A/I)^{\oplus m}$, because Y is an Ulrich A-modules with respect to I and $m = \mu_A(Y)$. Therefore, since $F/QF = (A/Q)^{\oplus m}$, we have $X/QX \cong (I/Q)^{\oplus m}$.

(1) $\Rightarrow$ (2) Since $IX = QX$ and $X/QX \cong (I/Q)^{\oplus m}$, we have $I{\cdot}(I/Q)^{\oplus m} = (0)$, whence $I^2 \subseteq Q$. Because $X/IX = X/QX \cong (I/Q)^{\oplus m}$ is A/I-free, the A/I-module I/Q is also free.

(2) $\Rightarrow$ (1) and (b) Since $I^2 \subseteq Q$ and $X/QX \cong (I/Q)^{\oplus m}$, we have $I{\cdot}(X/QX) = (0)$, whence $IX = QX$. Let $r = \mathrm{rank}_{A/I} I/Q$. Then

$$X/IX = X/QX \cong (I/Q)^{\oplus m} \cong (A/I)^{\oplus mr},$$

because $I/Q \cong (A/I)^{\oplus r}$. Thus X is an Ulrich A-module with respect to I and $\mu_A(X) = \mu_A(Y){\cdot}\mathrm{rank}_{A/I}(I/Q)$.

(a) Let $n = \mu_A(I)$ and write $I = (x_1, x_2, \cdots, x_n)$. Then since the residue class field $A/\mathfrak{m}$ of A is infinite, we may choose a minimal basis $\{x_i\}_{1 \le i \le n}$ of I so that the ideal $(x_{i_1}, x_{i_2}, \cdots, x_{i_d})$ is a reduction of I for any set $1 \le i_1 < i_2 < \cdots < i_d \le n$ of integers. We now fix a subset $\Lambda = \{i_1, i_2, \cdots, i_d\}$ of $\{1, 2, \cdots, n\}$ and put $Q = (x_{i_1}, x_{i_2}, \cdots, x_{i_d})$. We now consider the epimorphism

$$(A/I)^{\oplus n} \xrightarrow{\varphi} I/I^2 \to 0$$

of A/I-modules such that $\mu_A(\mathbf{e}_i) = \overline{x_i}$ for all $1 \le i \le n$, where $\{\mathbf{e}_i\}_{1 \le i \le n}$ is the standard basis of A/I-free module $(A/I)^{\oplus n}$ and $\overline{x_i}$ denotes the image of x_i in I/I^2. Assume that $\sum_{i=1}^n c_i \overline{x_i} = 0$ with $c_i \in A$. Then since

$$\sum_{i=0}^{n} c_i x_i \in I^2 \subseteq Q = (x_i \mid i \in \Lambda),$$

we have $\sum_{1 \le i \le n, i \notin \Lambda} c_i x_i \in Q$. Therefore because $\{\overline{x_i} \in I/Q\}_{1 \le i \le n, i \notin \Lambda}$ forms a A/I-free basis of I/Q, we get $c_i \in I$ for all $1 \le i \le n$ whenever $i \notin \Lambda$. After changing $\Lambda = \{i_1, i_2, \cdots, i_d\}$, we have $c_i \in I$ for all $1 \le i \le n$, whence $I/I^2 \cong (A/I)^{\oplus n}$.

We now show that $I^2 = QI$. Since $I^2 \subseteq Q$, it is enough to check that $Q \cap I^2 \subseteq QI$. Let $x \in Q \cap I^2$ and write $x = \sum_{1 \le j \le d} d_{i_j} x_{i_j}$ with $d_{i_j} \in A$. Then, because $\{\overline{x_{i_j}}\}_{1 \le j \le d}$ forms a part of A/I-free basis of I/I^2, we have $d_{i_j} \in I$ for all $1 \le j \le d$. Hence $x = \sum_{1 \le j \le d} d_{i_j} x_{i_j} \in QI$, so that $I^2 = QI$. Thus I is an Ulrich ideal of A. $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. (1) $\Rightarrow$ (2) We proceed by induction on d. Let $n = \mu_A(I)$ and $X_i = \mathrm{Syz}^i_A(A/I)$ for all $i \geq 1$. If $d = 0$, then we have $I^2 = (0)$ and $I \cong (A/I)^{\oplus n}$. Therefore $X_i \cong (A/I)^{\oplus n^i}$ for all $i \geq 1$. Thus X_i is an Ulrich A-module with respect to I for all $i \geq 0$. Assume that $d > 0$ and that our assertion holds true for $d-1$. Let $a \in Q \setminus \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, $\overline{Q} = Q/(a)$, and $\overline{X_i} = X_i/aX_i$ for $i \geq 1$. Then by Lemma 2.5 $\overline{I}$ is an Ulrich ideal of $\overline{A}$. Hence the hypothesis of induction on d guarantees that $\mathrm{Syz}^i_{\overline{A}}(\overline{A}/\overline{I})$ is an Ulrich $\overline{A}$-module with respect to $\overline{I}$ for all $i \geq d-1$, while we get by Lemma 2.4 an isomorphism

$$\overline{X_i} \cong \mathrm{Syz}^{i-1}_{\overline{A}}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}^{i}_{\overline{A}}(\overline{A}/\overline{I})$$

of A-modules, whence $\overline{X_i} \neq (0)$, $\overline{I}\,\overline{X_i} = \overline{Q}\,\overline{X_i}$, and $\overline{X_i}/\overline{I}\,\overline{X_i}$ is $\overline{A}/\overline{I}$-free for all $i \geq d$. Therefore $X_i \neq (0)$, $IX_i = QX_i$ and X_i/IX_i is A/I-free for all $i \geq d$, so that X_i is an Ulrich A-module with respect to I for all $i \geq d$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) By Proposition 3.1 we get the implication, because the residue class field $A/\mathfrak{m}$ of A is infinite.

(2) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (1) Let $a \in Q \setminus \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, $\overline{Q} = Q/(a)$, and $\overline{X_i} = X_i/aX_i$. We look at the isomorphism

$$\overline{X_i} \cong \mathrm{Syz}^{i-1}_{\overline{A}}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}^{i}_{\overline{A}}(\overline{A}/\overline{I})$$

obtained by Lemma 2.4, and set $Z = \mathrm{Syz}^{i-1}_{\overline{A}}(\overline{A}/\overline{I})$, $Z' = \mathrm{Syz}^{i}_{\overline{A}}(\overline{A}/\overline{I})$. Then $\overline{X_i}$ is an Ulrich $\overline{A}$-module with respect to $\overline{I}$ and $Z \neq (0)$. If $Z' = (0)$, then $\overline{X_i} \cong Z$ is $\overline{A}$-free. Then, since $\overline{I}\,\overline{X_i} = \overline{Q}\,\overline{X_i}$, we have $I = Q$, which is impossible; thus $Z' \neq (0)$. We now consider the exact sequence

$$0 \to Z' \to F_{i-1}/aF_{i-1} \to Z \to 0$$

of $\overline{A}$-modules. Because Z and Z' are Ulrich $\overline{A}$-modules with respect to $\overline{I}$, we have $\overline{I}^2 \subseteq \overline{Q}$ by Proposition 3.1, whence $I^2 \subseteq Q = (a_1, a_2, \cdots, a_d)$. On the other hand, since I/I^2 is A/I-free and $\{\overline{a_i}\}_{1 \leq i \leq d}$ forms a part of A/I-free basis of I/I^2 where $\overline{a_i}$ denotes the image of a_i in I/I^2, we get $Q \cap I^2 = QI$. Thus $I^2 = QI$, whence I is an Ulrich ideal of A. $\square$

4. Structure of minimal free resolutions of Ulrich ideals

In this section let me consider minimal free resolutions of Ulrich ideals. We fix the following notation. Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 0$. Let I be an Ulrich ideal of A and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal of A which is a reduction of I. Let

$$\mathrm{F}_\bullet : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to A/I \to 0$$

be a minimal A-free resolution of A/I. For $i \geq 0$ let $\beta_i = \beta_A^i(A/I)$ be the i-th betti number of A/I. Let $n = \beta_1 = \mu_A(I)$, the number of generators of I. We then have the following.

Theorem 4.1. *The following assertions hold true.*

(1)

$$\beta_i = \begin{cases} (n-d)^{i-d}(n-d+1)^d & (i \geq d), \\ \binom{d}{i} + (n-d)\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0) \end{cases}$$

for $i \geq 0$.

(2) $A/I \otimes_A \partial_i = 0$ *for all* $i \geq 1$.

(3) $\beta_i = \binom{d}{i} + (n-d)\beta_{i-1}$ *for all* $i \geq 1$.

Proof. We proceed by induction on d. Let $X_i = \mathrm{Syz}_A^i(A/I)$ for $i \geq 1$. If $d = 0$, then $I^2 = (0)$ and $I \cong (A/I)^{\oplus n}$. Hence $X_i \cong (A/I)^{\oplus n^i}$ for all $i \geq 1$. Therefore $\beta_i = n^i$ and $A/I \otimes_A \partial_i = 0$. Assume that $d > 0$ and that our assertion holds true for $d-1$. Let $a \in Q \setminus \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, and $\overline{X_i} = X_i/aX_i$ for $i \geq 1$. Then by Lemma 2.5 $\overline{I}$ is an Ulrich ideal of $\overline{A}$. By Lemma 2.4 we have an isomorphism

$$\overline{X_i} \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^{i}(\overline{A}/\overline{I})$$

of $\overline{A}$-modules for all $i \geq 1$. Hence $\beta_i = \overline{\beta_{i-1}} + \overline{\beta_i}$ for all $i \geq 1$, where $\overline{\beta_i} = \beta_{\overline{A}}^i(\overline{A}/\overline{I})$ denotes the i-th betti number of $\overline{A}/\overline{I}$. We set $\overline{n} = \mu_{\overline{A}}(\overline{I}) = n-1$ and $\overline{d} = \dim \overline{A} = d-1$.

(1) Suppose that $i \geq d$. Then by the hypothesis of induction on d we get

$$\overline{\beta_j} = (\overline{n} - \overline{d})^{j-\overline{d}} {\cdot} (\overline{n} - \overline{d} + 1)^{\overline{d}}$$

for $j \geq d-1$. Hence

$$\begin{aligned} \beta_i &= \overline{\beta_{i-1}} + \overline{\beta_i} \\ &= (\overline{n}-\overline{d})^{i-1-\overline{d}} {\cdot} (\overline{n}-\overline{d}+1)^{\overline{d}} + (\overline{n}-\overline{d})^{i-\overline{d}} {\cdot} (\overline{n}-\overline{d}+1)^{\overline{d}} \\ &= (n-d)^{i-d} {\cdot} (n-d+1)^{d-1} + (n-d)^{i-d+1} {\cdot} (n-d+1)^{d-1} \\ &= (n-d)^{i-d} {\cdot} (n-d+1)^{d-1} {\cdot} \{1 + (n-d)\} \\ &= (n-d)^{i-d} {\cdot} (n-d+1)^{d}. \end{aligned}$$

Suppose now that $1 \leq i \leq d$. Since $\beta_1 = n = \binom{d}{1} + (n-d)\beta_0$, our assertion holds true for the case where $i = 1$. If $2 \leq i \leq d-1$, then by the hypothesis of induction on d, we have

$$\overline{\beta_j} = \binom{\overline{d}}{j} + (\overline{n} - \overline{d})\overline{\beta_{i-1}}$$

for all $1 \le j \le d-1$. Therefore

$$\begin{aligned}
\beta_i &= \overline{\beta_{i-1}} + \overline{\beta_i} \\
&= \left\{ \binom{\overline{d}}{i-1} + (\overline{n}-\overline{d})\overline{\beta_{i-2}} \right\} + \left\{ \binom{\overline{d}}{i} + (\overline{n}-\overline{d})\overline{\beta_{i-1}} \right\} \\
&= \binom{d-1}{i-1} + (n-d)\overline{\beta_{i-2}} + \binom{d-1}{i} + (n-d)\overline{\beta_{i-1}} \\
&= \binom{d-1}{i-1} + \left\{ \binom{d}{i} - \binom{d-1}{i-1} \right\} + (n-d)\{\overline{\beta_{i-2}} + \overline{\beta_{i-1}}\} \\
&= \binom{d}{i} + (n-d)\beta_{i-1}.
\end{aligned}$$

If $i = d \ge 2$, then by the hypothesis of induction on d we have

$$\overline{\beta_{d-1}} = \binom{d-1}{\overline{d}} + (\overline{n}-\overline{d})\overline{\beta_{d-2}}$$

and

$$\begin{aligned}
\overline{\beta_d} &= (\overline{n}-\overline{d})^{d-\overline{d}} \cdot (\overline{n}-\overline{d}+1)^{\overline{d}} \\
&= (\overline{n}-\overline{d}) \cdot (\overline{n}-\overline{d}+1)^{\overline{d}} \\
&= (\overline{n}-\overline{d})\overline{\beta_{\overline{d}}} = (\overline{n}-\overline{d})\overline{\beta_{d-1}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\beta_d &= \overline{\beta_{d-1}} + \overline{\beta_d} \\
&= \left\{ \binom{d-1}{\overline{d}} + (\overline{n}-\overline{d})\overline{\beta_{d-2}} \right\} + (\overline{n}-\overline{d})\overline{\beta_{d-1}} \\
&= 1 + (n-d)\overline{\beta_{d-2}} + (n-d)\overline{\beta_{d-1}},
\end{aligned}$$

while

$$\binom{d}{d} + (n-d)\beta_{d-1} = 1 + (n-d)\{\overline{\beta_{d-2}} + \overline{\beta_{d-1}}\}.$$

Thus $\beta_d = \binom{d}{d} + (n-d)\beta_{d-1}$. Hence we get assertion (1).

(2) We have nothing to prove for the case where $i = 1$. Suppose that $i \ge 2$. Then by Lemma 2.4 we have an isomorphism

$$\overline{X_i} \cong \mathrm{Syz}^{i-1}_{\overline{A}}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}^{i}_{\overline{A}}(\overline{A}/\overline{I})$$

of $\overline{A}$-modules. Hence by the hypothesis of induction on d we get $A/I \otimes_A \partial_i = 0$ for all $i \ge 2$.

(3) We have only to consider the case where $i > d$. We get $\beta_i = (n-d)^{i-d}{\cdot}(n-d+1)^d$ for all $i \geq d$ by assertion (1), while

$$\begin{aligned}\binom{d}{i} + (n-d)\beta_{i-1} &= (n-d){\cdot}\{(n-d)^{i-1-d}{\cdot}(n-d+1)^d\} \\ &= (n-d)^{i-d}{\cdot}(n-d+1)^d.\end{aligned}$$

Hence $\beta_i = \binom{d}{i} + (n-d)\beta_{i-1}$ for all $i \geq 1$, which proves assertion (3). $\square$

Let $K_\bullet = K_\bullet(a_1, a_2, \cdots, a_d; A)$ denote the Koszul complex with differential maps $\partial_i^K : K_i \to K_{i-1}$. Then because $\beta_i = \binom{d}{i} + (n-i)\beta_{i-1}$ for all $i > 0$ by Theorem 4.1 (3), in the exact sequence

$$0 \to Q \to I \to I/Q \to 0$$

of A-modules a minimal A-free resolution of I is obtained by those of Q and I/Q, so that we have the following.

Proposition 4.2. *$F_i \cong K_i \oplus F_{i-1}^{\oplus(n-d)}$ for all $1 \leq i \leq d$ and $F_i \cong F_{i-1}^{\oplus(n-d)}$ for all $i \geq d+1$.*

Corollary 4.3. *Suppose that $d > 0$.*

(1) $\mathrm{Syz}_A^{i+1}(A/I) \cong [\mathrm{Syz}_A^i(A/I)]^{\oplus(n-d)}$ *for all $i \geq d$.*

(2) $F_{d+i} = F_d$ *and* $\partial_{d+i+1} = \partial_{d+1}$ *for all $i \geq 1$, if A is a Gorenstein local ring.*

Proof. Let $X_i = \mathrm{Syz}_A^i(A/I)$ for $i \geq 1$.

(1) Thsi is clear.

(2) Since A is a Gorenstein ring, we have $n - d = 1$ by Proposition 2.7, so that assertion (1) shows $F_{i+1} \cong F_i$ for all $i \geq d$. We now look at the following commutative diagram

$$\begin{array}{ccccc} F_{d+2} & \xrightarrow{\partial_{d+2}} & F_{d+1} & \xrightarrow{\partial_{d+1}} & F_d \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow \\ F_{d+1} & \xrightarrow{\partial_{d+1}} & F_d & \xrightarrow{\partial_d} & F_{d-1} \end{array}$$

of A-modules with isomorphisms $\alpha : F_{d+2} \to F_{d+1}$ and $\beta : F_{d+1} \to F_d$. It is standard to check that the following sequence

$$\cdots \to F_{d+1} \overset{\beta^{-1}\partial_{d+1}}{\to} F_{d+1} \to \cdots \to F_{d+1} \overset{\beta^{-1}\partial_{d+1}}{\to} F_{d+1} \overset{\partial_d\beta}{\to} F_{d-1} \overset{\partial_{d-1}}{\to} F_{d-2} \to \cdots$$

is also exact, which completes the proof of Corollary 4.3. $\square$

The following Theorem 4.4 plays a crucial role in the analysis of the problem of when the set $\mathcal{X}_A$ of Ulrich ideals in A is finite.

Theorem 4.4. $\mathrm{I}_1(\partial_i) = I$ *for all $i \geq 1$, where $\mathrm{I}_1(\partial_i)$ denotes the ideal of A generated by the entries of the matrix ∂_i.*

Proof. Let me begin with the following.

Claim 1. $\mathrm{I}_1(\partial_i) + Q = I$ *for all* $i \geq 1$.

Proof of Claim 1. We proceed by induction on d. We have nothing to prove when $d = 0$. Assume that $d > 0$ and that our assertion holds true for $d - 1$. Let $a = a_1 \in Q \setminus \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, and $\overline{Q} = Q/(a)$. Then $\overline{I}$ is an Ulrich ideal of $\overline{A}$. Let $X_i = \mathrm{Syz}_A^i(A/I)$ and put $\overline{X_i} = X_i/aX_i$ for all $i \geq 1$. Then by Lemma 2.4

$$\overline{X_i} \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^{i}(\overline{A}/\overline{I})$$

for all $i \geq 2$. Therefore the hypothesis of induction on d shows that $\mathrm{I}_1(\overline{\partial_i}) + \overline{Q} = \overline{I}$ for all $i \geq 2$, whence $\mathrm{I}_1(\partial_i) + Q = I$. Notice that $\mathrm{I}_1(\partial_1) = I$ clearly. Thus $\mathrm{I}_1(\partial_i) + Q = I$ for all $i \geq 1$ as claimed. $\square$

By Claim 1 we have only to show that $I_1(\partial_i) \supseteq Q$ for all $i \geq 1$. Suppose $2 \leq i \leq d$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i & \xrightarrow{\iota_i} & K_i \oplus F_{i-1}^{\oplus(n-d)} & \xrightarrow{p_i} & F_{i-1}^{\oplus(n-d)} & \longrightarrow & 0 \\
 & & {\scriptstyle \partial_i^K}\downarrow & & {\scriptstyle \partial_i}\downarrow & & {\scriptstyle \partial_{i-1}^{\oplus(n-d)}}\downarrow & & \\
0 & \longrightarrow & K_{i-1} & \xrightarrow{\iota_{i-1}} & K_{i-1} \oplus F_{i-2}^{\oplus(n-d)} & \xrightarrow{p_{i-1}} & F_{i-2}^{\oplus(n-d)} & \longrightarrow & 0
\end{array}$$

of A-modules, where $\iota_i(x) = (x, 0)$ and $p_i(x, y) = y$ with $x \in K_i$ and $y \in F_{i-1}^{\oplus(n-d)}$. Then, since $\partial_i \circ \iota_i = \iota_{i-1} \circ \partial_i^K$ and $p_{i-1} \circ \partial_i = \partial_{i-1}^{\oplus(n-d)} \circ p_i$, we have

$$\partial_i = \begin{pmatrix} \partial_i^K & * \\ 0 & \partial_{i-1}^{\oplus(n-d)} \end{pmatrix}.$$

Therefore $\mathrm{I}_1(\partial_i) \supseteq \mathrm{I}_1(\partial_i^K) = Q$. Thus $\mathrm{I}_1(\partial_i) = I$ for $2 \leq i \leq d$. Suppose that $i = d + 1$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & (0) \oplus F_d^{\oplus(n-d)} & \xrightarrow{p_{d+1}} & F_d^{\oplus(n-d)} & \longrightarrow & 0 \\
 & & & & {\scriptstyle \partial_{d+1}}\downarrow & & {\scriptstyle \partial_d^{n-d}}\downarrow & & \\
0 & \longrightarrow & K_d & \xrightarrow{\iota_d} & K_d \oplus F_{d-1}^{n-d} & \xrightarrow{p_d} & F_{d-1}^{n-d} & \longrightarrow & 0
\end{array}$$

of A-modules. Then $\partial_{d+1} = \begin{pmatrix} * \\ \partial_d^{n-d} \end{pmatrix}$, because $\partial_d^{n-d} \circ p_{d+1} = p_d \circ \partial_{d+1}$. Hence $\mathrm{I}_1(\partial_{d+1}) \supseteq \mathrm{I}_1(\partial_d) = I$, so that $\mathrm{I}_1(\partial_{d+1}) = I$. Thus by Corollary 4.3, $\mathrm{I}_1(\partial_{i+1}) = \mathrm{I}_1(\partial_i)$ for all $i \geq d+1$. Hence $\mathrm{I}_1(\partial_{i+1}) = \mathrm{I}_1(\partial_{d+1}) = I$ for $i \geq d$, which completes the proof of Theorem 4.4. $\square$

We are now in a position is to study the finiteness problem of Ulrich ideals. Let

$$\mathcal{X}_A = \{I \mid I \text{ is an Ulrich ideal of } A\}.$$

We are interested in the following question.

Question 4.5. When is $\mathcal{X}_A$ a finite set?

Let me begin with the following, which readily follows from Theorem 4.4.

Corollary 4.6. *Let I and J be Ulrich ideals of A. Then $I = J$ if and only if $\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_A^i(A/J)$ for some $i \geq 0$.*

Let me settle Problem 4.5 affirmatively in the following case.

Theorem 4.7. *Suppose that A is of finite CM-representation type. Then $\mathcal{X}_A$ is a finite set.*

Proof. We set $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$ be the set of isomorphism classes of $\mathrm{Syz}_A^d(A/I)$. Let $I \in \mathcal{X}_A$ with $n = \mu_A(I)$. Remember that

$$\mu_A(\mathrm{Syz}_A^d(A/I)) = (n-d+1)^d \leq (\mathrm{r}(A)+1)^d < \infty$$

by Theorem 4.1, since $n - d \leq \mathrm{r}(A)$ by Lemma 2.7. Therefore the set $\mathcal{Y}_A$ is finite, since A is of finite CM-representation type, so that by Corollary 4.6 $\mathcal{X}_A$ is also a finite set, because $\mathcal{X}_A \cong \mathcal{Y}_A$. $\square$

Let me explore one example.

Example 4.8. Let $A = k[[X,Y,Z]]/(Z^2 - XY)$. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Proof. The indecomposable maximal Cohen-Macaulay A-modules are A and $\mathfrak{p} = (z,x)$. We get $\mathfrak{m} \in \mathcal{X}_A$, since $\mathfrak{m}^2 = (x,y)\mathfrak{m}$. Let $I \in \mathcal{X}_A$. Then $\mu_A(I) = 3$. Let $X = \mathrm{Syz}_A^2(A/I)$ and consider the exact sequence

$$0 \to \mathrm{Syz}_A^2(A/I) \to A^3 \to A \to A/I \to 0$$

of A-modules. We then have

$$\mathrm{Syz}_A^2(A/I) \cong \mathfrak{p} \bigoplus \mathfrak{p},$$

because $\mu_A(X) = 4$ and $\mathrm{rank}_A X = 2$. Hence $I = \mathfrak{m}$ by Corollary 4.6. $\square$

There are many one-dimensional Cohen-Macaulay local rings of finite CM-representation type. Let me collect a few results.

Example 4.9. The following assertions hold true.

(1) $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$.
(2) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.
(3) $\mathcal{X}_{k[[X,Y]]/(Y(X^2-Y^{2a+1}))} = \{(x, y^{2a+1}), (x^2, y)\}$, where $a \geq 1$.
(4) $\mathcal{X}_{k[[X,Y]]/(Y(Y^2-X^3))} = \{(x^3, y)\}$.
(5) $\mathcal{X}_{k[[X,Y]]/(X^2-Y^{2a})} = \{(x^2, y), (x - y^a, y(x+y^a)), (x+y^a, y(x-y^a))\}$, where $a \geq 1$ and $\mathrm{ch}\, k \neq 2$.

5. Ulrich ideals in numerical semi-group rings

It seems interesting to ask how many Ulrich ideals are contained in a given Cohen-Macaulay local ring. We look at the numerical semigroup ring

$$A = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \subseteq k[[t]] = \overline{A},$$

where $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \dots, a_\ell) = 1$. Let

$$\mathcal{X}_A^g = \{\text{Ulrich ideals } I \text{ in } A \text{ such that } I = (\text{powers of } t)\}.$$

We then have the following.

Theorem 5.1. *The set $\mathcal{X}_A^g$ is finite.*

Proof. We have $I/Q \cong (A/I)^{\oplus(n-d)}$ and $\frac{I}{a} = A[\frac{I}{a}]$, where $Q = (a)$ and $\frac{I}{a} = a^{-1}I$. Hence

$$A/I \subseteq I/Q \cong A[\frac{I}{a}]/A \subseteq \overline{A}/A.$$

Therefore $A : \overline{A} = t^c{\cdot}k[[t]] \subseteq I$ for every Ulrich ideal I of A, where $c \geq 0$ denotes the conductor of the numerical semigroup

$$\langle a_1, a_2, \dots, a_\ell \rangle = \{\sum_{i=1}^{\ell} c_i a_i \mid 0 \leq c_i \in \mathbb{Z}\}.$$

Hence $\mathcal{X}_A^g$ is a finite set. $\square$

Although the sets $\mathcal{X}_A^g$ and $\mathcal{X}_A$ might be different, it is expected, first of all, to find what the set $\mathcal{X}_A^g$ is. Let me close this paper with a few (not complete) results.

Example 5.2. The following assertions hold true.

(1) $\mathcal{X}_{k[[t^3,t^5,t^7]]}^g = \{\mathfrak{m}\}$.
(2) $\mathcal{X}_{k[[t^4,t^5,t^6]]}^g = \{(t^4, t^6)\}$.
(3) $\mathcal{X}_{k[[t^a,t^{a+1},\cdots,t^{2a-2}]]}^g = \emptyset$, if $a \geq 5$.
(4) Let $1 < a < b \in \mathbb{Z}$ such that $\mathrm{GCD}(a, b) = 1$. Then $\mathcal{X}_{k[[t^a,t^b]]}^g \neq \emptyset$ if and only if a or b is even. (Compare with Example 5.3 (2).)
(5) Let $A = k[[t^4, t^6, t^{4a-1}]]$ $(a \geq 2)$. Then $\sharp\mathcal{X}_A^g = 2a - 2$.

Example 5.3 (with N. Taniguchi)**.** The following assertions hold true.

(1) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.
(2) $\mathcal{X}_{k[[t^3,t^7]]} = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$.
(3) $\mathcal{X}_{k[[t^{2a+i} \mid 1 \leq i \leq 2a]]} = \emptyset$ for $\forall a \geq 2$.
(4) $\mathcal{X}_{k[[t^{2a+i} \mid 0 \leq i \leq 2a-2]]} = \emptyset$ for $\forall a \geq 3$.
(5) $\sharp(\mathcal{X}_{k[[X,Y]]/(Y^n)}) = \infty$ for $\forall n \geq 2$.

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