APPLICATIONS OF MATHER DISCREPANCY

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ABSTRACT. In the talk at the Algebra Symposium, I showed the answers for the Mather verions of two of Shokurov's conjectures. In this article, I survey applications of jet schemes (including the Mather versions of Shokurov's conjectures) to birational geometry with the exposition of Mather discrepancies in the background.

1. INTRODUCTION

This is a survey article on Mather discrepancies. The readers interested in this topic are invited to read [5], [13], [4]. Varieties mean irreducible reduced schemes of finite type over an algebraically closed field k of characteristic zero. Let (X, B) be a pair consisting of a normal $\mathbb{Q}$-Gorenstein variety X over an algebraically closed field k of characteristic zero and an effective $\mathbb{R}$-Cartier divisor B on X. The minimal log discrepancy $\mathrm{mld}(W, X, B)$ with the center at a closed subset $w \subset X$ is defined for a pair and plays an important role in birational geometry. This is defined by using the discrepancy divisor $K_{Y/X} = K_Y - \varphi^* K_X$, where $\varphi : Y \to X$ is a log resolution of (X, B). On the other hand we can also define Mather minimal log discrepancy $\widehat{\mathrm{mld}}(W; X, \mathcal{J}_X B)$ with respect to the Jacobian ideal $\mathcal{J}_X$ of X by using Mather discrepancy $\widehat{K}_{Y/X}$ and the Jacobian ideal instead of usual discrepancy $K_{Y/X}$. These notion was introduced in [13] and [4]. Here we note that we need not to assume the $\mathbb{Q}$-Gorenstein condition on X or even X can be non-normal. The Mather minimal log discrepancy coincides with the ordinary one if (X, x) is locally a complete intersection. We expect that this "minimal log discrepancy" also plays an important role in algebraic geometry, since we observe that it has sometimes better properties than the usual minimal log discrepancy ([13], [4]).

2. PRELIMINARIES ON MATHER DISCREPANCY AND ARC SPACES

Let X be a $\mathbb{Q}$-Gorenstein variety of index r and $f : Y \to X$ a resolution of the singularities of X. Then the (usual) discrepancy divisor $K_{Y/X}$ is the unique $\mathbb{Q}$-divisor supported on the exceptional locus of f such that $rK_{Y/X}$ is linearly equivalent with $rK_Y - f^*(rK_X)$. Note that the usual discrepancy is defined only for a $\mathbb{Q}$-Gorenstein variety X, and the following Mather discrepancy is defined for every variety, even for non-normal variety.

Definition 2.1 ([5])**.** Let X be a variety of dimension d and $f : Y \to X$ a resolution of the singularities factoring through the Nash blow up. Then, the image of the canonical homomorphism

$$f^* \wedge^d \Omega_X \to \wedge^d \Omega_Y$$

is an invertible sheaf of the form $J \wedge^n \Omega_Y$, where J is the invertible ideal sheaf of $\mathcal{O}_Y$ that defines an effective divisor supported on the exceptional locus of f. This divisor is called the *Mather discrepancy divisor* and denoted by $\widehat{K}_{Y/X}$. For every prime divisor E on Y, we define

$$\widehat{k}_E := \mathrm{ord}_E(\widehat{K}_{Y/X}).$$

More generally, if v is a divisorial valuation over X, then we can assume without loss of generality that $v = q\,\mathrm{val}_E$ for a prime divisor E on some Y and a positive integer q, and define

$$\widehat{k}_v := q \cdot \widehat{k}_E.$$

For a $\mathbb{Q}$-Gorenstein variety X, we define $k_E := \mathrm{ord}_E(K_{Y/X})$ for a resolution $f : Y \to X$ and define also k_v in the similar way.

2.2. Let X be an d-dimensional $\mathbb{Q}$-Gorenstein variety of index r. We write the image of the homomorphism

$$(\wedge^n \Omega_X)^{\otimes r} \to \mathcal{O}_X(rK_X) = \omega_X^{[r]}$$

by $I_r \otimes \mathcal{O}_X(rK_X)$, where I_r is an ideal of $\mathcal{O}_X$. Let $f : Y \to X$ be a resolution factoring through the Nash blow up. Then, the relation of usual discrepancy and the Mather discrepancy is as follows:

$$f^*(I_r) \otimes \mathcal{O}_Y(r\widehat{K}_{Y/X}) = \mathcal{O}_Y(rK_{Y/X}).$$

In particular $\widehat{K}_{Y/X} \geq K_{Y/X}$. Let $\mathcal{J}_X$ be the Jacobian ideal of X and let $J_r = (\overline{\mathcal{J}_X^r} : I_r)$, then $J_r \cdot I_r$ and $\mathcal{J}_X^r$ have the same integral closure ([7, Corollary 9.4]). If X is locally a complete intersection, then $I_1 = \mathcal{J}_X$

Definition 2.3. Let X be a scheme of finite type over k and $K \supset k$ a field extension. For $m \in \mathbb{N}$, a k-morphism $\mathrm{Spec} K[t]/(t^{m+1}) \to X$ is called an m-jet of X and a k-morphism $\mathrm{Spec} K[[t]] \to X$ is called an *arc* of X.

2.4. We denote the space of m-jets of X by X_m and the space of arcs by X_∞. For terminologies and the basic properties of these spaces, we refer the paper [11].

Definition 2.5. Let X be a variety over k. We say an arc $\alpha : \mathrm{Spec} K[[t]] \to X$ is *thin* if α factors through a proper closed subset of X. An arc which is not thin is called a *fat arc*.

An irreducible closed subset C in X_∞ is called a *thin set* if the generic point of C is thin. An irreducible closed subset in X_∞ which is not thin is called a *fat set*.

One typical example of a fat set is a maximal divisorial set which is introduced in [12].

Definition 2.6. For a divisorial valuation v over a variety X, define the maximal divisorial set corresponding to v as follows:

$$C_X(v) := \overline{\{\alpha \in X_\infty \mid \mathrm{ord}_\alpha = v\}},$$

where $\overline{\{\ \ \}}$ is the Zariski closure in X_∞.

Proposition 2.7 ([12])**.** *Let $v = q\,\mathrm{val}_E$ be a divisorial valuation over a variety X. Let $f : Y \to X$ be a good resolution of the singularities of X such that the prime divisor E appears on Y. Here, a good resolution means a resolution whose exceptional locus is a simple normal crossing divisor. Then,*

$$C_X(v) = \overline{f_\infty(\mathrm{Cont}^q(E))}.$$

In particular, $C_X(v)$ is irreducible.

Definition 2.8 ([6])**.** For an ideal sheaf $\mathfrak{a}$ on a variety X, we define

$$\mathrm{Cont}^e(\mathfrak{a}) = \{\alpha \in X_\infty \mid \mathrm{ord}_\alpha(\mathfrak{a}) = e\}$$

and

$$\mathrm{Cont}^{\geq e}(\mathfrak{a}) = \{\alpha \in X_\infty \mid \mathrm{ord}_\alpha(\mathfrak{a}) \geq e\}.$$

These subset are called *contact loci* of an ideal $\mathfrak{a}$. The subset $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ is closed and $\mathrm{Cont}^m(\mathfrak{a})$ is locally closed. Both are cylinders. Here, a cylinder means the pull back $\psi_m^{-1}(S)$ of a constructible set $S \subset X_m$, where $\psi_m : X_\infty \to X_m$ is the canonical projection. We can define in the obvious way also subsets $\mathrm{Cont}^e(\mathfrak{a})_m$ (if $e \leq m$) and $\mathrm{Cont}^{\geq e}(\mathfrak{a})_m$ (if $e \leq m+1$) in X_m.

Proposition 2.9 ([5])**.** *Let X be an affine variety, and let $\mathfrak{a}_i \subset \mathcal{O}_X$ $(i = 1, \ldots, r)$ be non-zero ideals. Then, for $e_1, \ldots, e_r \in \mathbb{N}$, every fat irreducible component of the intersection $\mathrm{Cont}^{\geq e_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{\geq e_r}(\mathfrak{a}_r)$ is a maximal divisorial set.*

Note that [5, Proposition 2.12] is formulated for the case $r = 1$. But its proof works also for $r > 1$.

2.10. As X is a variety over k, the arc space X_∞ is irreducible by Kolchin's result. Therefore $\psi_m(X_\infty)$ is an irreducible constructible subset in X_m of dimension $(m+1)n$, where $n = \dim X$. Let $C \subset X_\infty$ be a cylinder $\psi_p^{-1}(A)$ contained in $\mathrm{Cont}^e(\mathcal{J}_X)$. Then, codimension of C is defined as follows:

$$\mathrm{codim}(C, X_\infty) := (m+1)n - \dim \psi_m(C)$$

for $m \geq \max\{p, e\}$.

For an arbitrary cylinder C, the codimension is defined as follows:

$$\mathrm{codim}(C, X_\infty) := \min\{\mathrm{codim}(C \cap \mathrm{Cont}^e(\mathcal{J}_X)) \mid e \in \mathbb{N}\}.$$

We sometimes write $\mathrm{codim}(C)$ for $\mathrm{codim}(C, X_\infty)$, when there is no possible confusion. Note that these are well defined by the following lemma. (For details, see [7, Section 5].)

3. Invariants based on Mather discrepancy

First we start this section with the well known invariants.

Definition 3.1. Let $(X, \mathfrak{a})$ be a pair consisting of $\mathbb{Q}$-Gorenstein variety X and a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_X$. The *log-canonical threshold* of $(X, \mathfrak{a})$ is defined as follows:

$$\mathrm{lct}(X, \mathfrak{a}) = \sup\{c \mid k_E - c \cdot \mathrm{ord}_E(\mathfrak{a}) + 1 \geq 0, E \text{ divisor over X}\}.$$

Let W be a closed subset of X. The *minimal log-discrepancy* of $(X, \mathfrak{a})$ along W is defined as follows:
If $\dim X \geq 2$,

$$\mathrm{mld}(W; X, \mathfrak{a}) = \inf\{k_E - \mathrm{ord}_E(\mathfrak{a}) + 1 \mid E \text{ divisor over } X \text{ with center in } W\}.$$

When $\dim X = 1$ we use the same definition of minimal log discrepancy, unless the infimum is negative, in which case we make the convention that $\mathrm{mld}(W, X, \mathfrak{a}) = -\infty$.

Remark 3.2. (i) The log-canonical threshold is also presented as

$$\mathrm{lct}(X, \mathfrak{a}) = \max\{c \mid k_{E_i} - c \cdot \mathrm{ord}_{E_i}(\mathfrak{a}) + 1 \geq 0,\ E_i : \text{exceptional prime divisor on } Y\}$$

for a fixed log-resolution $f : Y \to X$ of $(X, \mathfrak{a})$.

(ii) If $\mathrm{mld}(W; X, \mathfrak{a}) < 0$, then $\mathrm{mld}(W; X, \mathfrak{a}) = -\infty$. This is known when $\dim X \geq 2$, while it follows from the definition when $\dim X = 1$.

Now we will define the invariants modified from these invariants.

Definition 3.3. Let $(X,\mathfrak{a})$ be a pair consisting of an arbitrary variety X and a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_X$. The *Mather log-canonical threshold* of $(X,\mathfrak{a})$ is defined as follows:

$$\widehat{\mathrm{lct}}(X,\mathfrak{a}) = \sup\{c \mid \widehat{k}_E - c\cdot \mathrm{ord}_E(\mathfrak{a}) + 1 \geq 0, E \text{ divisor over X}\}.$$

Let W be a closed subset of X. The *Mather minimal log-discrepancy* of $(X,\mathfrak{a})$ along W is defined as follows:
If $\dim X \geq 2$,

$$\widehat{\mathrm{mld}}(W;X,\mathfrak{a}) = \inf\{\,\widehat{k}_E - \mathrm{ord}_E(\mathfrak{a}) + 1 \mid E \text{ divisor over } X \text{ with center in } W\}.$$

When $\dim X = 1$ we use the same definition of Mather minimal log discrepancy, unless the infimum is negative, in which case we make the convention that $\widehat{\mathrm{mld}}(W,X,\mathfrak{a}) = -\infty$.

Remark 3.4. (i) The Mather log-canonical threshold is represented as

$$\widehat{\mathrm{lct}}(X,\mathfrak{a}) = \max\{c \mid \widehat{k}_{E_i} - c\cdot\mathrm{ord}_{E_i}(\mathfrak{a}) + 1 \geq 0,\ E_i : \text{exceptional prime divisor on } Y\}$$

for a fixed log-resolution $f : Y \to X$ of $(X,\mathfrak{a})$ factoring through the Nash blow up, because for a sequence $Y' \xrightarrow{g} Y \xrightarrow{f} X$ of such log resolutions of $(X,\mathfrak{a})$, we have $\widehat{K}_{Y'/X} = K_{Y'/Y} + g^*\widehat{K}_{Y/X}$ with $K_{Y'/Y} \geq 0$.

(ii) If $\widehat{\mathrm{mld}}(W;X,\mathfrak{a}) < 0$, then $\widehat{\mathrm{mld}}(W;X,\mathfrak{a}) = -\infty$. This is proved by using the previous formula of $\widehat{K}_{Y'/X}$ when $\dim X \geq 2$, while it follows from the definition when $\dim X = 1$.

Proposition 3.5. *Let X be an arbitrary variety and $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}_X$. Then,*

$$\widehat{\mathrm{lct}}(X,\mathfrak{a}) = \min_{m\in\mathbb{N}} \frac{\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a}))}{m}.$$

As a corollary, we obtain the formula of lct for non-singular case.

Corollary 3.6 ([6])**.** *Let $(X,\mathfrak{a})$ be a pair consisting of a non-singular variety X and an ideal $\mathfrak{a} \subset \mathcal{O}_X$. Let Z be the subscheme defined by $\mathfrak{a}$. Then the log-canonical threshold is obtained as follows:*

$$\mathrm{lct}(X,\mathfrak{a}) = \min_{m\in\mathbb{N}} \frac{\mathrm{codim}(Z_{m-1}, X_{m-1})}{m}.$$

This follows immediately from the theorem, since the equality $\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) = \mathrm{codim}(Z_{m-1}, X_{m-1})$ holds for non-singular X.

The next is the formula for the Mather minimal log-discrepancy in terms of the arc space.

Proposition 3.7. *Let $(X, \mathfrak{a})$ be a pair consisting of an arbitrary variety X and a non-zero ideal $\mathfrak{a} \subset \mathcal{O}_X$. Let W be a proper closed subset of X and I_W be the (reduced) ideal of W. Then,*

$$(1) \qquad \widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \mathrm{Cont}^{\geq 1}(I_W)) - m\}.$$

We also have

$$(2) \qquad \widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq 1}(I_W)) - m\}.$$

Remark 3.8. Our formula can be easily extended for the combination of ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_r$ instead of one ideal $\mathfrak{a}$. I.e., we have

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{e_1}\mathfrak{a}_2^{e_2} \cdots \mathfrak{a}_r^{e_r}) =$$

$$\inf_{m_i \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^{m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{m_r}(\mathfrak{a}_r) \cap \mathrm{Cont}^{\geq 1}(I_W)) - \sum_i m_i e_i\},$$

where e_i's are positive real numbers. Here, any of $\mathrm{Cont}^{m_i}(\mathfrak{a}_i)$'s can be replaced by $\mathrm{Cont}^{\geq m_i}(\mathfrak{a}_i)$.

Proposition 3.9 (Inversion of Adjunction [13])**.** *Let X be an arbitrary varity, A a non-singular variety containing X as a closed subvariety of codimension c and W a proper closed subset of X. Let $\widetilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \widetilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is non-zero. Denote the ideal of X in A by I_X. Then,*

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\mathrm{mld}}(W; A, \widetilde{\mathfrak{a}} I_X^c).$$

Corollary 3.10 ([8], [7], [15])**.** *Let X be a normal closed subvariety in a non-singular variety A of codimension c and let W be a proper closed subset of X. Assume that X is $\mathbb{Q}$-Gorenstein variety of index r. Let $\widetilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \widetilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is non-zero. Then,*

$$\mathrm{mld}(W; X, \mathfrak{a} J_r^{1/r}) = \mathrm{mld}(W; A, \widetilde{\mathfrak{a}} I_X^c),$$

where I_X is the defining ideal of X in A and J_r is as in 2.2.

Corollary 3.11 (Adjunction formula)**.** *Let X be a closed subvariety of a variety X' of codimension c and let W be a proper closed subset of X. Let $\mathfrak{a}' \subset \mathcal{O}_{X'}$ be an ideal such that its image $\mathfrak{a} := \mathfrak{a}'\mathcal{O}_X \subset \mathcal{O}_X$ is non-zero. Let $I_{X/X'}$ be the defining ideal of X in X'. Then,*

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) \geq \widehat{\mathrm{mld}}(W; X', \mathfrak{a}'\mathcal{J}_{X'} I_{X/X'}^c).$$

Corollary 3.12. *Let $(X, \mathfrak{a})$ be a pair consisting of an arbitrary variety X and a non-zero ideal $\mathfrak{a} \subset \mathcal{O}_X$, then the function $x \mapsto \widehat{\mathrm{mld}}(x; X, \mathfrak{a}\mathcal{J}_X)$, ($x \in X$ closed point) is lower semicontinuous.*

Corollary 3.13. *Let X be a variety of dimension d. Then, for every closed point $x \in X$, the following inequality holds:*

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) \leq d,$$

where the equality holds if and only if (X, x) is non-singular.

In [19] Shokurov posed the following conjecture:

Conjecture 3.14. *Let X be a $\mathbb{Q}$-Gorenstein variety of dimension d. Then, for every closed point $x \in X$, the following inequality holds:*

$$\mathrm{mld}(x; X, \mathcal{O}_X) \leq d,$$

where the equality holds if and only if (X, x) is non-singular.

Our Corollary 3.13 is the answer to a modified version of this conjecture. In particular, if (X, x) is a complete intersection, then the affirmative answer to this conjecture follows from our corollary, because $\mathrm{mld}(x; X, \mathcal{O}_X) = \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X)$. This is already observed by Florin Ambro (private communication to the author) who published its special case in [2].

4. Mather version of Shokurov's Conjectures

As a refinement of Conjecture 3.14, Shokurov posed a conjecture as follows :

Conjecture 4.1 ([20], Conjecture 2)**.** *For a $\mathbb{Q}$-Gorenstein variety X and an $\mathbb{R}$-Cartier divisor B we have the inequality*

$$\mathrm{mld}(x; X, B) \leq \dim X,$$

where the equality holds if and only if (X, x) is nonsingular and $B = 0$ around x.

As a refinement of Corollary 3.13, we obtain the following (Mather version) answer to Conjecture 4.1.

Proposition 4.2 ([13],Corollary 3.15;[4],Corollary 4.15)**.** *For an arbitrary variety X and an effective $\mathbb{R}$-Cartier divisor B on X, we have the inequality*

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) \leq \dim X,$$

where the equality holds if and only if (X, x) is nonsingular and $B = 0$ around x.

Shokurov also posed a conjecture (which is not published) as follows:

Conjecture 4.3. *The inequality*

$$\dim X - 1 < \mathrm{mld}(x; X, B)$$

holds if and only if (X, x) *is nonsingular and* $\mathrm{mult}_x B < 1$. *In this case the minimal log discrepancy is computed by the exceptional divisor of the first blowup at* x.

The implication of "if" part of Conjecture 4.3 for 2-dimensional case was proved by Vyacheslav Shokurov in his unpublished paper and for 3-dimensional case was proved by Florin Ambro [1], however this conjecture is not yet proved in general. The main result of this paper is the following, which contains the affirmative answer to the Mather version of Conjecture 4.3.

Theorem 4.4 (—-, A. Reguera)**.** *A pair* (X, B) *consisting of an arbitrary variety* X *and an effective* $\mathbb{R}$*-Cartier divisor* B *on* X *satisfies*

$$\dim X - 1 \leq \widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X B)$$

if and only if either

(i) $B = 0$, $X \simeq C \times \mathbb{A}^r_k$ $(r \geq 0)$, *where* C *is a plane curve with an ordinary node at* p *and* $x = (p, \underline{0})$ *or*

(ii) $\dim X \geq 2$, $B = 0$ *and* (X, x) *is a compound Du Val singularity or*

(iii) (X, x) *is non-sigular and* $0 \leq \mathrm{mult}_x B \leq 1$.

In the cases (i) *and* (ii), *we have* $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X) = \dim X - 1$ *and in the case* (iii) *we have* $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X B) = \mathrm{mld}(x; X, B) = \dim X - \mathrm{mult}_x B$ *and the minimal log discrepancy is computed by the exceptional divisor of the first blowup at* x.

As a corollary, we obtain the "if" part of Conjecture 4.3 for usual mld:

Corollary 4.5. *The inequality*

$$\dim X - 1 < \mathrm{mld}(x; X, B)$$

holds if (X, x) *is nonsingular and mult*$_x B < 1$. *In this case the minimal log discrepancy is computed by the exceptional divisor of the first blowup at* x.

Let us show the outline of the proof of the theorem.

Definition 4.6. For $d \geq 1$, we say that a d-dimensional variety X has a *top singularity* at x_0, or that (X, x_0) is a top singularity, if $\widehat{\mathrm{mld}}(x_0; X, \mathfrak{J}_X) = d - 1$.

Definition 4.7. Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ with $d \geq 2$. A singularity (X, x_0) is called a *compound Du Val singularity* or cDV singularity if it is Du Val in case $d = 2$, if its general hyperplane section is Du Val singularity in case $d = 3$ and if its general hyperplane section is cDV singularity in case $d > 3$.

In the definition of cDV singularities, we assume the generality of hyperplane sections, but it is not necessary if one assume that the point is originally singular. The following is well known:

Lemma 4.8. *Assume that (X, x_0) is a germ of a d-dimensional hypersurface singularity of multiplicity 2 with $d \geq 3$, then (X, x_0) is a compound Du Val singularity if and only if there exist $(d-2)$ hyperplanes $H_1, \ldots, H_{d-2}$ such that $(X \cap H_1 \cap \cdots \cap H_{d-2}, x_0)$ is a Du Val singularity.*

Remark 4.9. We also have an analytic description of the previous lemma as follows: Let (X, x_0) be a germ of d-dimensional variety and $\widehat{R}$ be the M-adic completion of $R = \mathcal{O}_{X,x_0}$. Then (X, x_0) is a compound Du Val singularity if and only if there exists $g_1, \ldots, g_{d-2} \in \widehat{R}$ such that $\mathrm{Spec}\widehat{R}/(g_1, \ldots, g_{d-2})$ is a Du Val singularity.

Lemma 4.10. *Let (X, x_0) be a germ of a d-dimensional variety at a closed point x_0 and let $X' \subset X$ be a $(d-c)$-dimensional subvariety which is defined locally as the zero locus of c elements of $\mathcal{O}_X$. Let x_0 be a closed point in X'. If (X', x_0) is a top singularity, then (X, x_0) is a top singularity.*

Lemma 4.11. *If X has a top singularity at x_0, then X is locally at x_0 a hypersurface of multiplicity 2.*

Corollary 4.12. *A singularity (X, x_0) is a top singularity if and only if*

$$\dim X_m^0 = md + 1 \qquad \textit{for every } m \geq 1. \tag{3}$$

Example 4.13. Next we give an example of a top singularity of dimension $d = 1$: Let X be a plane curve with an ordinary node, *i.e.*, locally it is defined by $x_1 x_2 = 0$ in $\mathbb{A}_k^2$, and let us consider its germ $(X, \underline{0})$ at $\underline{0}$. For $m \geq 0$, we have

$$X_m^0 = \mathrm{Spec}\ k[\underline{X}_1, \ldots, \underline{X}_m] \ / \ (\{ \sum_{1 \leq i \leq n-1} X_{1,i} X_{2,n-i} \}_{1 \leq n \leq m}).$$

It follows that X_m^0 has m irreducible components, given by

$$X_{1,1} = X_{1,2} = \ldots = X_{1,r_1} = X_{2,1} = \ldots = X_{2,r_2} = 0$$

$$\text{for } r_1, r_2 \geq 0,\ r_1 + r_2 = m - 1.$$

Thus, each irreducible component has dimension $2m-(m-1) = m+1$, and hence $\dim X_m^0 = m + 1$ for $m \geq 0$. Therefore $(X, \underline{0})$ is a top singularity.

Definition 4.14. Let $X = C \times \mathbb{A}_k^{d-1}$ $(d \geq 1)$, where C is a plane curve with an ordinary node at p and $x_0 = (p, \underline{0})$. Then, the singularity (X, x_0) is called a *compound ordinary node* singularity (sometimes we call it a cON singularity).

Proposition 4.15. *A compound ordinary node singularity is a top singularity.*

Proof: For $d = 1$, the singularity $(X, x_0) = (C, p)$ is a top singularity by the previous example. Let $d \geq 2$ and $y_1, \ldots, y_{d-1}$ be a coordinate system of $\mathbb{A}_k^{d-1}$. The hyperplane cuts of (X, x_0) by $y_1 = \cdots = y_{d-1} = 0$ is (C, p). Then, by Lemma 4.10, (X, x_0) is a top singularity. $\square$

Remark 4.16. For $d \geq 2$, recall that, if $X \subset \mathbb{A}_k^{d+1}$ is a normal hypersurface and x_0 a closed point of X, then

$$\begin{aligned}\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) &= \mathrm{mld}(x_0; X, \mathcal{O}_X) = \\ &= \inf\{k_E + 1 \mid \ \nu_E \text{ divisorial valuation centered at } x_0\}.\end{aligned}$$

Note that, if $\pi : Y \to X$ and $\pi' : Y' \to X$ are two desingularizations of X, and Y' dominates Y, let $\rho : Y' \to Y$ be such that $\pi' = \rho \circ \pi$, then we have $K_{Y'/X} = K_{Y'/Y} + \rho^*(K_{Y/X})$ and $K_{Y'/Y}$ is effective. Therefore, given a normal hypersurface $X \subset \mathbb{A}_k^{d+1}$ of dimension $d \geq 2$, in order to prove that (X, x_0) is a top singularity, it suffices to show that there exists a desingularization $\pi : Y \to X$ such that

$$(4)\quad \inf\{k_E + 1 \mid \ E \text{ prime divisor on } Y \text{ such that } \pi(E) = x_0\} = d - 1.$$

Example 4.17. The equality (4) is satisfied for the minimal desingularizations of all rational double points of dimension 2 (also called Du Val singularities), since they are canonical singularities of dimension $d = 2$. The following is a list of rational double points, for each of them, the completion $\widehat{\mathcal{O}_{X,\underline{0}}}$ of the local ring $\mathcal{O}_{X,\underline{0}}$ of its germ at $\underline{0}$ is described as a quotient of the ring of series $k[[x_1, x_2, x_3]]$. More precisely, for each of the types of the rational double points in the left hand side, there exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and satisfying the equation in the right hand side (recall that char $k = 0$):

$$\begin{array}{ll}
\mathbf{A}_n (n \geq 1): & \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^{n+1} = 0 \\
\mathbf{D}_n (n \geq 4): & \mathbf{x}_1^2 + \mathbf{x}_2^2\mathbf{x}_3 + \mathbf{x}_3^{n-1} = 0 \\
\mathbf{E}_6: & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_3^4 = 0 \\
\mathbf{E}_7: & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_2\mathbf{x}_3^3 = 0 \\
\mathbf{E}_8: & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_3^5 = 0
\end{array}$$

Proposition 4.18. *A compound Du Val singularity is a top singularity.*

Proof: Let (X, x_0) be a compound Du Val singularity of dimension $d \geq 3$. Then, a successive $(d-2)$ hyperplane cuts produces a Du Val singularity. As in the previous example, Du Val singularities are top singularities. By Lemma 4.10, we obtain that (X, x_0) is a top singularity. □

4.19. We will see that the cDV and cON singularities are all of the top singularities. Recall that, given $f(x_1, \ldots, x_{d+1}) \in k[x_1, \ldots, x_{d+1}]$ (resp. $f \in k[[x_1, \ldots, x_{d+1}]]$), if in$f$ denotes the initial form of f in the graded ring $k[x_1, \ldots, x_{d+1}]$ (resp. $k[[x_1, \ldots, x_{d+1}]]$), with the usual graduation, then the smallest possible dimension τ of a linear subspace V_0 of $V = kx_1 + \ldots + kx_{d+1}$ such that in$f$ lies in the subalgebra $k[V_0]$ of $k[x_1, \ldots, x_{d+1}]$ is an invariant of the germ $(X, \underline{0})$ at $\underline{0}$ of the hypersurface $X \subset \mathbb{A}_k^{d+1}$ defined by $f(x_1, \ldots, x_{d+1}) = 0$ (resp. of Spec $k[[x_1, \ldots, x_{d+1}]]/(f)$) ([9], chap. III). We denote it by $\tau(X, \underline{0})$ (resp. by $\tau(f)$). Given a germ (X, x_0) of hypersurface in $\mathbb{A}_k^{d+1}$ at a closed point x_0, the τ-invariant $\tau(X, x_0)$ is defined as the τ-invariant of the germ of hypersurface obtained after a translation of x_0 to $\underline{0}$.

Lemma 4.20. *Let $(X, \underline{0})$ be the germ at $\underline{0}$ of an hypersurface $X \subset \mathbb{A}_k^{d+1}$ of multiplicity 2. Then, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1} \in \widehat{\mathcal{O}_{X,\underline{0}}}$ generating its maximal ideal and such that*

$$\mathbf{x}_1^2 + \ldots + \mathbf{x}_\tau^2 + g(\mathbf{x}_{\tau+1}, \ldots, \mathbf{x}_{d+1}) = 0$$

with $\tau = \tau(X, \underline{0})$, $g(x_{\tau+1}, \ldots, x_{d+1}) \in k[[x_{\tau+1}, \ldots, x_{d+1}]]$ and either $g = 0$ or mult $g \geq 3$.

Proposition 4.21. *Let (X, x_0) be a germ of a hypersurface of multiplicity 2 and $\tau(X, x_0) > 1$. Then (X, x_0) is either a cON singularity or a cDV singularity, therefore it is a top singularity.*

Proof: As in Lemma 4.20, let $\mathbf{x}_1, \dots, \mathbf{x}_{d+1} \in \widehat{\mathcal{O}_{X,\underline{0}}}$ generate its maximal ideal and satisfy

$$\mathbf{x}_1^2 + \dots + \mathbf{x}_\tau^2 + g(\mathbf{x}_{\tau+1}, \dots, \mathbf{x}_{d+1}) = 0.$$

with $\tau = \tau(X, \underline{0})$, $g(x_{\tau+1}, \dots, x_{d+1}) \in k[[x_{\tau+1}, \dots, x_{d+1}]]$ and either $g = 0$ or mult $g \geq 3$. If $\tau = 2$ and $g = 0$, then (X, x_0) is a cON singularity. If $\tau \geq 3$, , let X_0 be the intersection of X with the hyperplanes $x_i = 0$ for $4 \leq i \leq d+1$. Then X_0 is defined in $\mathbb{A}_k^3$ by

$$x_1^2 \ + \ x_2^2 \ + \ x_3^2 = 0$$

hence it has an $\mathbf{A}_1$-singularity at $\underline{0}$, therefore (X, x_0) is a cDV singularity. If $\tau = 2$ and $g \neq 0$. Therefore, there exists $\underline{\lambda} = (\lambda_4, \dots, \lambda_{d+1}) \in \mathbb{A}_k^{d-1}$ such that $g(x_3, \lambda_4 x_3, \dots, \lambda_{d+1} x_3)$ is nonzero and moreover, its multiplicity is $m = \mathrm{mult}_{\underline{0}}\ g(x_3, \dots, x_{d+1})$. Hence, the intersection X_0 of X with the hyperplanes $x_i = \lambda_i x_3$, $4 \leq i \leq d+1$, is defined in $\mathbb{A}_k^3$ by

$$x_1^2 \ + \ x_2^2 \ + \ u\ x_3^m = 0$$

where u is a unit in $k[x_3]$, hence $(X_0, \underline{0})$ is an $\mathbf{A}_{m-1}$-singularity (see Example 4.17), thus it is a cDV singularity. $\square$

4.22. Let $(X, \underline{0})$ be a germ of a hypersurface $X \subseteq \mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, \underline{0}) = 1$. Let $\mathbf{x}_1, \dots, \mathbf{x}_{d+1}$ be generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and such that

$$\mathbf{x}_1^2 + g(\mathbf{x}_2, \dots, \mathbf{x}_{d+1}) = 0, \tag{5}$$

where $g(x_2, \dots, x_{d+1}) \in k[[x_2, \dots, x_{d+1}]]$ and, since X is reduced, $g \neq 0$ and mult $g \geq 3$ (Lemma 4.20). Let us consider the germ at $\underline{0}$ of the hypersurface
$g(x_2, \dots, x_{d+1}) = 0$ in Spec $k[[x_1, \dots, x_{d+1}]]$. Although this germ depends on the choice of $x_1, \dots, x_{d+1}$, its multiplicity $m_2 :=$ mult g, and its τ-invariant at $\underline{0}$, let it be τ_2, only depend on $(X, \underline{0})$ (this follows from [10].). Given a germ (X, x_0) of hypersurface in $\mathbb{A}_k^{d+1}$ at a closed point x_0, we define $m_2(X, x_0)$ and $\tau_2(X, x_0)$ to be the invariants defined as before, after a translation of x_0 to $\underline{0}$.

Lemma 4.23. *Let $(X, \underline{0})$ be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, \underline{0}) = 1$. If $(X, \underline{0})$ is a top singularity then $d \geq 2$ and $m_2(X, \underline{0}) = 3$.*

Proposition 4.24. *Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, x_0) = 1$. If $m_2(X, x_0) = 3$ and $\tau_2(X, x_0) > 1$, then (X, x_0) is a cDV singularity, therefore it is a top singularity.*

4.25. Let $(X, \underline{0})$ be a germ of a hypersurface $X \subseteq \mathbb{A}_k^{d+1}$ of multiplicity 2, $\tau(X, \underline{0}) = 1$, $m_2(X, \underline{0}) = 3$ and $\tau_2(X, \underline{0}) = 1$. Then there exist $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and such that

$$\mathbf{x}_1^2 + \mathbf{x}_2^3 + g_3(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1})\, \mathbf{x}_2 + g_4(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1}) = 0, \tag{6}$$

where $g_i \in k[[x_3, \ldots, x_{d+1}]]$ and $\mathrm{mult}_{\underline{0}}\ g_i \geq i$, for $i = 3, 4$. In fact, there exist $x_1, \ldots, x_{d+1}$ whose classes in $\widehat{\mathcal{O}_{X,\underline{0}}}$ generate the maximal ideal, and $g \in k[[x_2, \ldots, x_{d+1}]]$ such that (5) holds, and moreover, since mult $g = m_2(X, x_0) = 3$ and $\tau_2(X, x_0) = 1$, by Weierstrass' preparation theorem and after a Tschirnhausen transformation, we may suppose that

$$g(x_2, \ldots, x_{d+1}) = u\ \left(x_2^3 + g_3(x_3, \ldots, x_{d+1})\ x_2 + g_4(x_3, \ldots, x_{d+1})\right)$$

where u is a unit in $k[[x_2, \ldots, x_{d+1}]]$ and $g_i \in k[[x_3, \ldots, x_{d+1}]]$ is such that mult $g_i \geq i$, for $i = 3, 4$. Replacing x_1 by vx_1 where v is a unit in $k[[x_2, \ldots, x_{d+1}]]$ such that $v^2 = u$, and considering the equality induced on the classes $\mathbf{x}_i$ of x_i in $\widehat{\mathcal{O}_{X,\underline{0}}}$, we obtain (6).

Proposition 4.26. *Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, x_0) = 1$, $m_2(X, x_0) = 3$ and $\tau_2(X, x_0) = 1$. Then the following are equivalent:*

(i) *(X, x_0) is a top singularity,*

(ii) *there exist $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that*

$$\mathbf{x}_1^2 + \mathbf{x}_2^3 + g_3(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1})\ \mathbf{x}_2 + g_4(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1}) = 0$$

where $g_i \in k[[x_3, \ldots, x_{d+1}]]$, mult $g_i \geq i$, *for $i = 3, 4$ and either* mult $g_3 = 3$ *or* $4 \leq$ mult $g_4 \leq 5$,

(iii) *(X, x_0) is a cDV singularity.*

The following summarizes the discussions of characterization of a top singularity (Proposition 4.21, Lemma 4.23, Proposition 4.24, Proposition 4.26).

Theorem 4.27. *A germ of a variety (X, x_0) of dimension d is a top singularity if and only if either*

(i) *there exist minimal system of generators $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d+1}$ of the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that $\mathbf{x}_1\mathbf{x}_2 = 0$, or*

(ii) *$d \geq 2$ and there exist minimal system of generators $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ of the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that one of the following holds:*

(a) $\mathbf{x}_1^2 + \ldots + \mathbf{x}_\tau^2 + g(\mathbf{x}_{\tau+1}, \ldots, \mathbf{x}_{d+1}) = 0$ *where* $\tau \geq 2$, $g(x_{\tau+1}, \ldots, x_{d+1}) \in k[[x_{\tau+1}, \ldots, x_{d+1}]]$ *and* mult $g \geq 3$.

(b) $\mathbf{x}_1^2 + \mathbf{x}_2^3 + p(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1})\, \mathbf{x}_2 + q(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1}) = 0$ *where* $p(x_3, \ldots, x_{d+1}),\ \ q(x_3, \ldots, x_{d+1}) \in k[[x_3, \ldots, x_{d+1}]]$, mult $p \geq 2$, mult $q \geq 3$, *and either* $2 \leq$ mult $p \leq 3$ *or* $3 \leq$ mult $q \leq 5$.

Theorem 4.28. *A germ of a variety* (X, x_0) *is a top singularity if and only if either*

(i) $X \simeq C \times \mathbb{A}_k^r$ $(r \geq 0)$*, where* C *is a plane curve with an ordinary node at* p *and* $x_0 = (p, \underline{0})$ *or*

(ii) $\dim X \geq 2$ *and* (X, x_0) *is a compound Du Val singularity*

Proof: The conditions (i) and (ii) imply that (X, x_0) is a top singularity by Proposition 4.15 and Proposition 4.18. The converse follows from the fact that a top singularity is a hypersurface double point and, under the classification of the defining equation according the invariants τ, m, a class which is of top singularities always satisfy the condition either (i) or (ii) (Proposition 4.21, Lemma 4.23, Proposition 4.24, Proposition 4.26).

5. A proof of the main theorem

Proof of Theorem 4.4: Let $d = \dim X$ and let (X, B) satisfy the condition $d - 1 \leq \widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X B)$ at a closed point $x \in X$. If (X, x) is singular, then by Proposition 4.2 we have $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X) \leq d - 1$ since $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X)$ is an integer by the definition. If $B \neq 0$ in a neighborhood of x, then

$$\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X B) < \widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X) \leq d - 1,$$

in which case (X, B) does not satisfy the condition of the theorem. Therefore if (X, x) is singular, then $B = 0$ and $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X) = d - 1$, *i.e.,* (X, x) is a top singularity. A top singularity is characterized in the Theorem 4.28 as in (i) and (ii).

Hence it is sufficient to characterize a pair (X, B) such that X is nonsingular and $\widehat{\mathrm{mld}}(x; X, \mathfrak{J}_X B) = \mathrm{mld}(x; X, B) \geq d - 1$ in terms of (iii).

If $d = \dim X = 1$, then it is obvious since $\mathrm{mld}(x; X, B) = 1 - \mathrm{mult}_x B$.

Assume $d = \dim X \geq 2$ and (X, B) satisfies the inequality $\mathrm{mld}(x; X, B) \geq d - 1$, then for the exceptional divisor E_1 of the blowup $\varphi_1 : X_1 \to X$

of X at x should have the log discrepancy

$$k_{E_1} - \mathrm{ord}_{E_1}\varphi_1^*B + 1 \geq d - 1.$$

As $k_E = d-1$ and $\mathrm{ord}_{E_1}\varphi_1^*B = \mathrm{mult}_x B$, this implies $\mathrm{mult}_x B \leq 1$.

Conversely we assume (iii) that $\mathrm{mult}_x B \leq 1$. Under this condition we check the log discrepancy of every prime divisor over X with the center at x.

Let E be a prime divisor over X with the center at x, let $y \in E$ be the generic point and let E appear in a resolution $f_0 : Y \to X$. Then, by Zariski's result (see, for example [16], VI, 1.3), we have a sequence of varieties $X_0, X_1, \ldots, X_n$ and rational maps as follows:

> $X_0 = X$, $f_0 = f$.
> If $f_i : Y \dashrightarrow X_i$ is already defined, then let $Z_i \subset X_i$ be the closure of $p_i = f_i(y)$. Let $X_{i+1} = B_{Z_i}X_i$ and $f_{i+1} : Y \dashrightarrow X_{i+1}$ be the induced map.
> Then the final birational map $f_n : Y \dashrightarrow X_n$ is isomorphic at y, *i.e.,* E appears on X_n.

Here, $B_{Z_i}X_i$ is the blowup of X_i with the center Z_i. Let $\varphi_i : X_i \to X_{i-1}$ be the blowup morphism and $E_i \subset X_i$ the exceptional divisor dominating Z_i. Note that the first blowup $\varphi_1 : X_1 \to X_0 = X$ is the blowup at the closed point x since the center of E on X is x and on the other hand f_n is isomorphic at the generic points of E_n and E. We also note that X_i and E_i are nonsingular at p_i for every $i = 1, \ldots n$. Indeed, this is proved inductively. As X_1 is the blowup at a closed point $x = p_0$, X_1 and E_1 are nonsingular at every point. Suppose $i \geq 2$ and X_{i-1} and E_{i-1} are nonsingular at p_{i-1}, then X_i is the blowup with the nonsingular center, when one restricts the morphism on a neighborhood of p_{i-1}. As p_i is on the pull back of this neighborhood, X_i and E_i are nonsingular at p_i.

Let $B^{(i)}$ be the strict transform of B on X_i, then by [9] II sec. 5, Theorem 3 (p.233) we have:

$$\mathrm{mult}_{p_i}B^{(i)} \leq \mathrm{mult}_{p_{i-1}}B^{(i-1)} \text{ for every } i = 1, \ldots, n \tag{7}$$

Let $a(E_i, X, B)$ be the discrepancy of (X, B) at the divisor E_i, *i.e.,*

$$a(E_i; X, B) = \mathrm{ord}_{E_i}(K_{X_i/X} - \Phi_i^*(B)),$$

where $\Phi_i : X_i \to X$ is the composite $\varphi_1 \circ \cdots \circ \varphi_i$. Note that the log discrepancy of (X, B) at the divisor E_i is $a(E_i; X, B) + 1$.

Claim. For every $i = 1, \ldots, n$

$$a(E_i, X, B) \geq 0 \text{ and } a(E_i, X, B) \geq a(E_{i-1}, X, B).$$

By abuse of notation, we denote the strict transform of $E_{i-1} \subset X_{i-1}$ on X_j $(j \geq i)$ by the same symbol E_{i-1}. Then, we have

$$\varphi_i^*(E_{i-1}) = E_{i-1} + E_i \tag{8}$$

by the nonsingularity of X_{i-1} and E_{i-1} at p_{i-1} guaranteed in the discussion above.

Now we prove the claim by induction on i. First for $i = 1$, by substituting $K_{X_1/X} = (d-1)E_1$ and $\varphi_1^*(B) = (\mathrm{mult}_x B)E_1 + B^{(1)}$ into

$$a(E_1; X, B) = \mathrm{ord}_{E_1}(K_{X_1/X} - \varphi_1^*(B)),$$

we obtain

$$a(E_1; X, B) = (d-1) - \mathrm{mult}_x B \geq d - 2,$$

which is of course nonnegative by our assumption $d \geq 2$.

Let $i \geq 2$ and assume that $a(E_j; X, B) \geq 0$ for all $j \leq i-1$ by induction hypothesis. Then,

$$\begin{aligned}
a(E_i; X, B) &= \mathrm{ord}_{E_i}(K_{X_i/X} - \Phi_i^*(B)) \\
&= \mathrm{ord}_{E_i}\left(K_{X_i/X_{i-1}} + \varphi_i^*\left(K_{X_{i-1}/X} - \Phi_{i-1}^*(B)\right)\right) \\
&= \mathrm{ord}_{E_i}\left(K_{X_i/X_{i-1}} + \varphi_i^*\Big(\sum_{j \leq i-1} a(E_j, X, B)E_j - B^{(i-1)}\Big)\right) \\
&\geq \mathrm{ord}_{E_i}(K_{X_i/X_{i-1}}) + a(E_{i-1}; X, B) - \mathrm{mult}_{p_{i-1}} B^{(i-1)}.
\end{aligned}$$

Here, we used (8) and the hypothesis of the induction. We may assume that $\mathrm{codim}\overline{\{p_{i-1}\}} \geq 2$, because if $\mathrm{codim}\overline{\{p_{i-1}\}} = 1$, then f_{i-1} is already isomorphic at the generic point $y \in E$. (We may assume that n is taken to be minimal.) As $\mathrm{ord}_{E_i}(K_{X_i/X_{i-1}}) = \mathrm{codim}\overline{\{p_{i-1}\}} - 1 \geq 1$ and $\mathrm{mult}_{p_{i-1}} B^{(i-1)} \leq \mathrm{mult}_x B \leq 1$ by (7), we obtain

$$a(E_i, X, B) \geq a(E_{i-1}, X, B) \geq 0$$

as claimed.

By this, we have the log discrepancy at E is $a(E; X, B) + 1 = a(E_n; X, B) + 1 \geq a(E_1; X, B) + 1 = d - \mathrm{mult}_x B \geq d - 1$. Therefore, the inequality $\dim X - 1 \leq \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) = \mathrm{mld}(x; X, B)$ holds and the minimal log discrepancy $d - \mathrm{mult}_x B$ and is computed by the exceptional divisor of the first blowup at x. $\square$

As a corollary of the theorem we have the "if" part of Conjecture 4.3:

Corollary 5.1. *The inequality*

$$\dim X - 1 < \mathrm{mld}(x; X, B)$$

holds if (X, x) is nonsingular and $mult_x B < 1$. In this case the minimal log discrepancy is computed by the exceptional divisor of the first blowup at x.

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