MODULI OF ABELIAN VARIETIES AND THEIR COMPACTIFICATIONS

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Abstract. What are the moduli space of elliptic curves and its compactification? In order to explain the issues involved, we discuss the case of planar cubic curves in detail. A basic idea for compactification is the GIT-stability of Mumford.

In arbitrary dimension, GIT-stability canonically compactifies the moduli space of abelian varieties over $\mathbf{Z}[\zeta_N, 1/N]$ for some large $N \geq 3$. In the smallest possible case, dimension one and $N = 3$, the problem is reduced to the study of planar cubic curves, more specifically Hesse cubic curves. Every GIT-stable cubic curve is isomorphic to a Hesse cubic curve.

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1. Hesse cubic curves

The major part of this note is taken from [Nakamura04]. Please visit [Nakamura04] if you are interested in, since it could be still be a good introduction to the subject as well as a survey on it. We also would like to invite the readers to visit a pdf file of our homepage

http://www.math.sci.hokudai.ac.jp/˜nakamura/Okayama201108.pdf

Let $\mathbf{P}^2_k$ be the projective plane over an algebraically closed field k of characteristic different from 3. We could think of the base field k as the field $\mathbf{C}$ of complex numbers.

A Hesse cubic curve is by definition a cubic curve on the plane $\mathbf{P}^2_k$ defined by the following equation:

$$C(\mu) \ : \ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \tag{1}$$

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for some $\mu \in k$, or $\mu = \infty$ (in which case we understand that $C(\infty)$ is the curve defined by $x_0x_1x_2 = 0$). Let ζ_3 is a primitive cube root of unity. For $\mu \neq \infty, 1, \zeta_3, \zeta_3^2$, the curve $C(\mu)$ is a nonsingular elliptic curve, while for $\mu = \infty, 1, \zeta_3, \zeta_3^2$ $C(\mu)$ consists of three nonsingular rational curves, pairwise intersecting at a distinct point so that the three irreducible components of $C(\mu)$ form a cycle.

When $k = \mathbf{C}$, if $C(\mu)$ is nonsingular, it is topologically a real 2-torus. Otherwise, it is a cycle of 3 rational curves, (or equivalently a union of three lines in $\mathbf{P}^2_k$ in general position), topologically a cycle of three real 2-spheres, which looks like a rosary of three beads. When μ approaches ∞ or ζ_3^k, then a real 2-torus is pinched locally at three distinct meridians into a cycle of three 2-spheres.

This class of curves was studied by Hesse in the middle of 19th century. His paper published in 1849 (see [Hesse]) is summarized as follows.

Theorem 1.1. (1) *Any nonsingular cubic curve can be converted into one of the Hesse cubic curves (1) under the action of* $\mathrm{SL}(3, k)$, *namely it is isomorphic to one of* $C(\mu)$ *for* $\mu \neq \infty, \mu^3 \neq 1$.

(2) *Every Hesse cubic curve* $C(\mu)$ *has nine inflection points , independent of* μ: $[1 : -\beta : 0], [0 : 1 : -\beta], [-\beta : 0 : 1]$ *where* $\beta^3 = 1$.

(3) $C(\mu)$ *is transformed isomorphically onto* $C(\mu')$ *under* $\mathrm{SL}(3, k)$ *with each of nine inflection points fixed if and only if* $\mu = \mu'$.

The first and third assertions of the theorem show that any isomorphism class of nonsingular cubic curves is represented by $\mu \in k$ with $\mu^3 \neq 1$. In other words,

$$k \setminus \{1, \zeta_3, \zeta_3^2\} = \text{the moduli space of nonsingular cubic curves with ordered nine inflection points}.$$

We mean by a moduli space the space naturally representing the isomorphism classes of some geometric objects as above. In this case the noncompact moduli space

$$k \setminus \{1, \zeta_3, \zeta_3^2\} = \mathrm{Spec}\ k[\mu, \frac{1}{\mu^3 - 1}]$$

can be compactified into $\mathbf{P}^1_k$ as the theorem of Hesse shows, where the exceptional values $\mu = 1, \zeta_3, \zeta_3^2$ and $\mu = \infty$ correspond to a singular Hesse cubic curve, that is a union of three lines in general position. It is remarkable that the compactified moduli space is also the moduli space of (isomorphism classes of) certain geometric objects, in this case Hesse cubic curves possibly singular.

In what follows we mean by "compactification of a moduli space" roughly what we saw above.

In this article, via diverse modern interpretations of the theorem of Hesse, we will present an analogy to it and construct for each large symplectic finite abelian group K a compactification $SQ_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$ of the moduli space of abelian varieties where ζ_N is a primitive N-th root of unity for N suitably chosen.

It is important to recognize that the problem of compactifying a moduli space is not the problem of finding all limits of some geometric objects. For example, the compactification of the moduli space of nonsingular Hesse cubic curves is $\mathbf{P}^1_k$. As this outcome suggests, one could say that the problem of compactification is to single out an important class or a relatively narrow class of limits only so that the class of limits may form a complete (or compact) algebraic variety set-theoretically.

This problem of compactifying a moduli space is algebro-geometrically quite interesting, whatever the objects to consider may be chosen. It was natural to ask whether one could construct compactifications of the moduli spaces at least for curves, abelian varieties and K3 surfaces because they are basic objects in algebraic geometry. As is well known, we have the Deligne-Mumford compactification for curves [DM69], whereas there are quite a lot of compactifications of the moduli of abelian varieties ([AMRT75], [FC90]).

Nevertheless we will construct *one and only one* new compactification $SQ_{g,K}$ of the moduli of abelian varieties. This compactification is natural enough because, as we will see below, there are three natural approaches to the compactification problem including GIT-stability and representation theory of Heisenberg groups, each of which leads us to the same compactification $SQ_{g,K}$.

Let us explain what is a modern interpretation of the theorem of Hesse. Let us look first at the curve

$$C(\infty)\ :\ x_0x_1x_2 = 0. \tag{2}$$

Let $C(\infty)^0 := C(\infty) \setminus \{\text{singular points}\}$. Then $C(\infty)^0$ has a group scheme structure, and as group schemes

$$C(\infty)^0 = \mathbf{G}_m \times (\mathbf{Z}/3\mathbf{Z}) \tag{3}$$

where $\mathbf{G}_m$ is the multiplicative group, so $\mathbf{G}_m = \mathbf{C}^*$ if $k = \mathbf{C}$. The family of Hesse cubic curves (1) is therefore an analogue of the so-called Tate curve over a complete discrete valuation ring (or the unit disc). Moreover theta functions on cubic curves, namely nonsingular elliptic curves or one-dimensional abelian varieties over $\mathbf{C}$, are Fourier series, and similarly functions on the group variety $\mathbf{G}_m$ has also Fourier series expansions. In this sense $\mathbf{G}_m$ is one of the nice limits of cubic curves. This viewpoint of degeneration of elliptic curves into nice group varieties was also very fruitful in higher dimension as is shown by the work of Grothendieck, Raynaud, Mumford, Faltings and Chai. This is the first modern interpretation of the theorem of Hesse, though this should now be considered classical.

From a different point of view, a finite group $G(3)$ of order 27 called the Heisenberg group acts on the Hesse cubic curves linearly. Let x_0, x_1, x_2 be the homogeneous coordinates of the plane $\mathbf{P}^2_k$ and V the vector space spanned by x_i ($i = 0, 1, 2$). The Heisenberg group $G(3)$ is a subgroup of $\mathrm{GL}(V)$ generated by the following two linear transformations σ and τ of V:

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i = 0, 1, 2 \mod 3) \tag{4}$$

which are subject to the relation

$$\sigma^3 = \tau^3 = \mathrm{id}_V, \quad \sigma\tau = (\zeta_3 \cdot \mathrm{id}_V)\tau\sigma. \tag{5}$$

Let us regard $\mathbf{P}_k^2$ as the space of row 3-vectors. Then σ and τ induce automorphisms of $\mathbf{P}_k^2$ by

$$\begin{aligned}\bar{\sigma} : [x_0, x_1, x_2] &\mapsto [\sigma(x_0), \sigma(x_1), \sigma(x_2)] = [x_0, \zeta_3 x_1, \zeta_3^2 x_2],\\ \bar{\tau} : [x_0, x_1, x_2] &\mapsto [\tau(x_0), \tau(x_1), \tau(x_2)] = [x_1, x_2, x_0].\end{aligned} \tag{6}$$

These restrict to automorphisms of Hesse cubic curves, which are the translations of the elliptic curves by their 3-torsion points. The vector space V is by (4) a three-dimensional representation of $G(3)$. This is often called the Schrödinger representation of $G(3)$, which is easily shown to be irreducible. Therefore by the famous lemma of Schur about irreducible representations, the basis x_0, x_1 and x_2, which are transformed by σ and τ as above, are uniquely determined up to constant multiples. This property enables us to identify x_0, x_1 and x_2 with a natural basis of theta functions over $\mathbf{C}$. In this sense theta functions usually defined over $\mathbf{C}$ have natural counterparts in positive characteristic. This is the second modern interpretation of the work of Hesse.

The third important interpretation of it is based on the GIT-stability of Mumford, which we omit in this report. We will see that the above three interpretations are essentially the same and that Hesse cubic curves and their equations are derived naturally from any of the interpretations. Thus each of the three interpretations is a route to one and the same compactification $\mathbf{P}_k^1$. Each interpretation provides us with a natural approach to the problem of compactifying the moduli of abelian varieties in higher dimension.

2. Satake compactification and toroidal compactification

A compactification as a complex analytic space of the moduli space A_g of principally polarized abelian varieties was constructed by Satake in the 1950's, now known as the Satake compactification of A_g. Later as an application of the theory of torus embeddings, quite a lot of compactifications of complex spaces similar to A_g were constructed by Mumford et al. [AMRT75], which we call toroidal compactifications. Thereafter, toroidal compactifications were algebrized by Faltings and Chai [FC90] into compactifications of A_g or its analogues as a scheme over $\mathbf{Z}$. A particular case of toroidal compactification, referred to as the Voronoi compactification of A_g, was discussed in [Namikawa76] in connection with proper degeneration of abelian varieties. This is very relevant to the subject of the present article.

Toroidal compactification is sufficiently general, and it seems that there is no other class of algebro-geometrically natural compactifications. However there is a feature missing in this compactification. As we explained for Hesse cubic curves in §1, we would like to demand a given compactification of the moduli space to be again the moduli space of *compact geometric objects of the same dimension.* All of the compactifications mentioned above do not meet the demand, though they are moduli spaces of noncompact objects by [FC90]. On the other hand it is still an open problem whether the Voronoi compactification in [Namikawa76] is a moduli space.

Therefore the first problem in this direction would be the existence of a compactification of A_g or its analogues which meets the above demand. Since the uniqueness of the compactification is not true for toroidal compactifications, there might exist many less important compactifications which meet the above demand. Thus a more important problem would be to single out a natural significant compactification and study its structure in detail.

The major purpose of my talk at Okayama was to report that there does exist actually an algebro-geometrically natural compactification $SQ_{g,K}$ defined over $\mathbf{Z}[\zeta_N, 1/N]$, where $N = \sqrt{|K|}$. It is immediate from its construction that it is projective. The precise relationship of $SQ_{g,K}$ with the Voronoi compactification will be understood in a not-so-far future since this is basically not a difficult problem.

We expect that the results will be extended over $\mathbf{Z}$ or $\mathbf{Z}[\zeta_N]$ in a way analogous to the Drinfeld compactification of modular curves [KM85]. This is now in progress, and we will be able to report it somewhere else in the near future. Though we reported a bit on it in Okayama, we omit it here because it is not final.

3. The space of closed orbits

3.1. **Example.** Now let us consider what the principle for singling out a nice compactification ought to be. One of the principles is suggested by GIT of Mumford, the geometric invariant theory [MFK94]. Let us look at the following example. Let $\mathbf{C}^2$ be the complex plane, (x, y) its coordinates. Let us consider the action of $\mathbf{C}^*$ on $\mathbf{C}^2$:

$$(\alpha, x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbf{C}^*) \tag{7}$$

What is (or ought to be) the quotient space of $\mathbf{C}^2$ by the action of $\mathbf{C}^*$? Let us decompose $\mathbf{C}^2$ into orbits first. Since the function xy is constant on any orbit, we see that there are four kinds of orbits:

$$\begin{gathered} O(a,1) = \{(x,y) \in \mathbf{C}^2; xy = a\} \quad (a \neq 0), \\ O(0,1) = \{(0,y) \in \mathbf{C}^2; y \neq 0\}, \\ O(1,0) = \{(x,0) \in \mathbf{C}^2; x \neq 0\}, \\ O(0,0) = \{(0,0)\} \end{gathered} \tag{8}$$

where there are the closure relations of orbits

$$\overline{O(1,0)} \supset O(0,0), \ \overline{O(0,1)} \supset O(0,0).$$

It is tempting to define the quotient to be the orbit space, namely the set of all orbits, but its natural topology is not Hausdorff. In fact, if it is Hausdorff, then we see

$$O(1,0) = \lim_{x \to 0} O(1,x) = \lim_{x \to 0} O(x,1) = O(0,1) \tag{9}$$

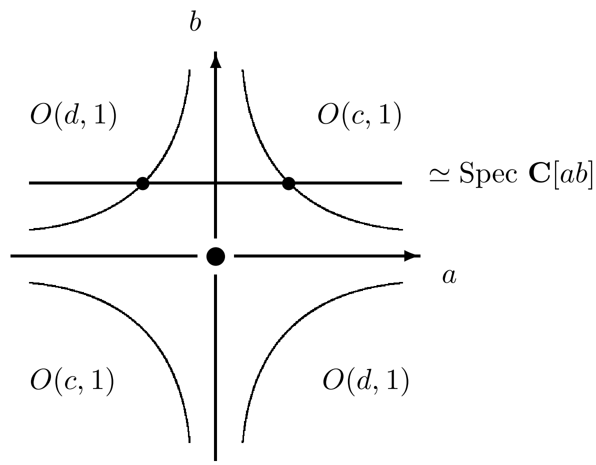
because $O(a, 1) = O(1, a)$ $(a \neq 0)$. Hence the natural topology of the orbit space is not Hausdorff. In order to avoid this, we need to turn to the ring of invariants. By (7) the ring of invariants by the $\mathbf{C}^*$-action is a polynomial ring generated by xy. Thus we define the desirable quotient space to be

$$\mathbf{C}^2 /\!/ \mathbf{C}^* = \{t; t \in \mathbf{C}\} \simeq \operatorname{Spec} \mathbf{C}[t]. \tag{10}$$

where $t = xy$. Since $t = xy$ is a function on $\mathbf{C}^2$, there is a natural morphism from $\mathbf{C}^2$ onto $\mathbf{C}^2 /\!/ \mathbf{C}^*$. The three orbits $O(1,0)$, $O(0,1)$ and $O(0,0)$ are projected to the origin $t = 0$ of Spec $\mathbf{C}[t]$. Among the three orbits, $O(0,0)$ is the unique closed orbit, while $O(0,1)$ and $O(1,0)$ are not closed. Hence one could think that the origin $t = 0$ is represented by the unique closed orbit $O(0,0)$. This is a very common phenomenon for orbits spaces. Now we make an important remark to summarize the above:

Theorem 3.2. *The quotient space* $\mathbf{C}^2//\mathbf{C}^*$ *is set-theoretically the space of closed orbits.*

To be more precise, the theorem asserts the following: For any $a \in \mathbf{C}$ in the right hand side of (10), there is a unique closed orbit $O(a,1)$ or $O(0,0)$ respectively if $a \neq 0$ or $a = 0$.



The same is true in general. There is a notion of a semistable point, which we will define soon after stating the following theorems.

Theorem 3.3. (Seshadri-Mumford) *Let X be a projective scheme over a closed field k, G a reductive algebraic k-group acting on X. Then there exists an open subscheme X_{ss} of X consisting of all semistable points in X, and a quotient of X_{ss} by G in a certain reasonable sense.*

To be more precise, there exist a projective k-scheme Y and a G-invariant morphism π from X_{ss} onto Y such that

(1) *For any k-scheme Z on which G acts, and for any G-equivariant morphism $\phi : Z \to X$ there exists a unique morphism $\bar{\phi} : Z \to Y$ such that $\bar{\phi} = \pi\phi$,*
(2) *For given points a and b of X_{ss}*
$$\pi(a) = \pi(b) \text{ if and only if } \overline{O(a)} \cap \overline{O(b)} \neq \emptyset$$
where the closure is taken in X_{ss},
(3) *$Y(k)$ is regarded as the set of G-orbits closed in X_{ss}.*

We denote the (categorical) quotient Y by $X_{ss}//G$.

A reductive group in Theorem 3.3 is by definition an algebraic group whose maximal solvable normal subgroup is an algebraic torus; for example $\mathrm{SL}(n)$ and $\mathbf{G}_m$ are reductive.

We restate Theorem 3.3 in a much simpler form, though the statement of it is not precise.

Theorem 3.4. *Let X be a projective variety, G a reductive group acting on X. Then $X_{ss}//G$ is projective and it is identified with the set of G-orbits closed in X_{ss}.*

Let R be the graded ring of all G-invariant homogeneous polynomials on X. Then the (categorical) quotient Y of X_{ss} by G is defined to be

$$Y = \mathrm{Proj}\,(R).$$

The most important point to emphasize is the fact that

$$Y = \text{the space of orbits closed in } X_{ss}.$$

Now we give the definition of the term "semistable" in Theorem 3.3.

Definition 3.5. We keep the same notation as in Theorem 3.3. Let $p \in X$.

(1) the point p is said to be *semistable* if there exists a G-invariant homogeneous polynomial F on X such that $F(p) \neq 0$,
(2) the point p is said to be *Kempf-stable* if the orbit $O(p)$ is closed in X_{ss},
(3) the point p is said to be *properly-stable* if p is Kempf-stable and the stabilizer subgroup of p in G is finite.

We denote by X_{ps} or X_{ss} the set of all properly-stable points or the set of all semistable points respectively. Very often in the recent literatures "properly-stable" is only referred to as "stable", however in the present article we will use "properly-stable" for it in order to strictly distinguish it from "Kempf-stable".

Thus Theorem 3.3 tells us what the quotient space Y ought to be. What is the subset of X lying over Y? It is X_{ss}, namely the subset of X consisting of all points where at least a G-invariant homogeneous polynomial does not vanish. Any homogeneous polynomial now is not a function on X, instead the quotient of a pair of G-invariant homogeneous polynomials of equal degree is a function on X. Therefore X_{ss} is the subset of X where G-invariant functions are defined possibly by choosing a suitable denominator. Therefore

$$\begin{aligned} X \setminus X_{ss} &= \text{the common zero locus of all } G\text{-invariant} \\ &\qquad \text{homogeneous polynomials on } X \\ &= \text{the subset of } X \text{ where no } G\text{-invariant} \\ &\qquad \text{functions are defined } (0/0\ !). \end{aligned}$$

However since it is a very difficult task to determine the ring of all G-invariant homogeneous polynomials on X, so is it to determine X_{ss}. The geometric invariant theory is the theory in which Mumford intended to determine semistability or the subset X_{ss} without knowing explicitly the ring of all G-invariant homogeneous polynomials, instead by studying the geometric structure of X and the G-action on it. In lucky situations this is really the case, for instance, semistable vector bundles on a variety (Takemoto, Maruyama and Mumford), stable curves (Gieseker and Mumford) and in addition abelian varieties (Kempf) and their natural limits PSQASes as we will see in Theorem 7.4.

3.6. **Comparison of notions.** In order to apply Theorem 3.3 to moduli problems we compare various notions in GIT and moduli theories as follows:

$$\begin{aligned}
X &= \text{the set of geometric objects},\\
G &= \text{the group of isomorphisms of objects in } X,\\
//G &= \text{mod } G \text{ plus some extra relation in Theorem 3.3}\\
&\quad\ \text{called the orbit closure relation},\\
X_{ps} &= \text{the set of properly-stable geometric objects}\\
&= \text{the set of generic geometric objects},\\
X_{ss} &= \text{the set of semistable geometric objects}\\
&= X_{ps} \cup \text{moderately degenerating limits of objects in } X_{ps},\\
X_{ps}//G &= \text{the moduli space of generic geometric objects},\\
X_{ss}//G &= \text{compactification of the moduli space}.
\end{aligned}$$

We note that if $a, b \in X_{ps}$,

$$\begin{aligned}
\pi(a) = \pi(b) &\iff \overline{O(a)} \cap \overline{O(b)} \neq \emptyset\\
&\iff O(a) \cap O(b) \neq \emptyset\\
&\iff O(a) = O(b)\\
&\iff a \text{ and } b \text{ are isomorphic}.
\end{aligned}$$

Each point of X_{ps} gives a closed orbit and the moduli space $X_{ps}//G$ is an ordinary orbit space X_{ps}/G. Moreover it is compactified by $X_{ss}//G$. This is currently one of the most powerful principles for compactifying moduli spaces.

The first approximation to our moduli is $Y^0 := X_{ps}/G$. Therefore the first candidate for a compactification of our moduli could be $X_{ss}//G$. However in many cases X_{ss} is too big to determine explicitly. There are too many orbits in $X_{ss}//G$ which are unnecessary in understanding the space $X_{ss}//G$ itself. In this sense it is more practical to restrict our attention to Kempf-stable points, though the set of Kempf-stable points in X_{ss} is not even an algebraic subscheme of X_{ss} in general.

4. GIT-stability and stable critical points

4.1. **What is implied by GIT-stability?** The Morse theory is well known as a method of studying the topology of a differentiable manifold by a Morse function. In studying the topology it is important to know the critical exponents of the Morse function at critical points. When the critical exponent at a critical point is maximal, namely the Hessian of the Morse function is positive definite, the critical point is called a stable critical point. This is the case where the Morse function takes a local minimum at the point. If one considers the function as a sort of energy function in physics, the critical point corresponds to a stable point (or a stable physical state) where the energy attains its local minimum. To our knowledge, the term *stable* is used in this sense in most cases. However it may seem that GIT-stability has nothing to do with it, at least from the definition. Nevertheless as the following theorem of Kempf and Ness shows, GIT-stability does have to do with a stable critical point.

Let V a finite-dimensional complex vector space, G a reductive algebraic group acting on V. Let K be a maximal compact subgroup of G and $\| \ \|$ be a K-invariant Hermitian norm on V. For instance if $G = \mathrm{SL}(2, \mathbf{C})$, then $K = \mathrm{SU}(2)$. If one takes the example in 3.1, then $V = \mathbf{C}^2$, $G = \mathbf{C}^*$, $K = S^1 = \{w \in \mathbf{C}^*; \|w\| = 1\}$ and the K-invariant Hermitian norm on V is given by

$$\|(a,b)\| = |a|^2 + |b|^2,$$
$$\|g_\lambda \cdot (a,b)\| = |\lambda a|^2 + |\lambda^{-1} b|^2$$

where $g_\lambda = \mathrm{diag}(\lambda, \lambda^{-1})$.

Definition 4.2. Let $v \in V$, $v \neq 0$.

(1) the vector v is said to be *semistable* if there exists a G-invariant homegeneous polynomial F on V such that $F(v) \neq 0$,
(2) the vector v is said to be *Kempf-stable* if the orbit $O(v)$ is closed in V,
(3) the vector v is said to be *properly-stable* if p is Kempf-stable and the stabilizer subgroup of v in G is finite.

If v is Kempf-stable, then v is semistable. Let $\pi : V \setminus \{0\} \to \mathbf{P}(V)$ be the natural surjection. Now we compare the above with the previous one in Definition 3.5. Then v is semistable (resp. Kempf-stable, properly-stable) if and only if $\pi(v)$ is semistable (resp. Kempf-stable, properly-stable).

For $v \neq 0$, we define $p_v(g) := \|g \cdot v\|$ on G. Then p_v is a function on an orbit $O(v)$, which is invariant by the action of K from the left. Then the following theorem is known.

Theorem 4.3. (Kempf-Ness)

(1) *p_v obtains its minimum on $O(v)$ at any critical point of p_v.*
(2) *The second order derivation of p_v at the minimum is "positive".*
(3) *v is Kempf-stable if and only if p_v obtains a minimum on $O(v)$.*

Thus summarizing the above, we see

$$v \text{ is Kempf-stable} \iff p_v \text{ has a stable critical point on } O(v).$$

In this sense, we can justify the term "stable" or "stability" in GIT.

In this connection we would like to add a few words about the history of coining the term "stability". The first edition of GIT was published in 1965, and then was followed by Deligne-Mumford's paper on stable curves in 1969 and the paper of Kempf and Ness in 1979. In the first edition of GIT the following theorem has been proved:

Theorem. Any nonsingular hypersurface of $\mathbf{P}^n$ is properly-stable.

This suggests that Mumford had an idea of compactifying the moduli space of nonsingular curves by stability at latest in 1965, though the notion of stable curves has maybe not been established yet because the article of Deligne and Mumford appeared in 1969. The term stability might come from the stable reduction theorem of abelian varieties (due to Grothendieck) and the stable curves that were probably being born at that time. On the other hand the theorem of Kempf and Ness is not so difficult to prove, though its discovery would not be so easy. Taking all of these into consideration, we suspect that Mumford was unaware of the connection of GIT-stability with Morse-stability like

Theorem 4.3 when he first used the term stability in GIT. It might have been a mere accident. But it was a very excellent coining that described its essence very well as the subsequent history shows.

5. Stable curves of Deligne and Mumford

5.1. **The moduli space M_g of stable curves.** In the paper [DM69] Deligne and Mumford compactified the moduli space of nonsingular curves by adding stable curves — a class of curves with mild singularities. Roughly speaking we have

$$\begin{aligned} &\text{the moduli of smooth curves}\\ &\quad = \text{the set of all isomorphism classes of smooth curves}\\ &\quad \subset \text{the set of all isomorphism classes of stable curves}\\ &\quad = \text{the Deligne-Mumford compactification.}\end{aligned}$$

The Deligne-Mumford compactification M_g frequently appears in diverse branches such as the quantum gravity in physics and Gromov-Witten invariants in connection with mirror symmetry. Also widely known are Kontsevich's solution of Witten conjecture, and his cellular decomposition of M_g by ribbon graphs.

Let us recall the definition of stable curves:

Definition 5.2. Let C be a possibly reducible, connected projective curve of genus at least two. A curve C is called *moduli-stable* if the following conditions are satisfied:

(1) it is locally a curve on a nonsingular algebraic surface (or a two-dimensional complex manifold) defined by an equation $x = 0$ or $xy = 0$ in terms of local coordinates x, y,
(2) if a nonsingular rational curve C' is an irreducible component of C, then C' intersects the other irreducible components of C at least at three points.

It is easy to see that the automorphism group of any moduli-stable curve is finite. Since diverse stabilities appear in the context, we call a Deligne-Mumford stable curve a moduli-stable curve in what follows to distinguish the terminology strictly.

5.3. **Another stability.** Any moduli-stable curve is stable in the following sense [DM69]:

(i) Any given one parameter family of curves, after a suitable process of surgeries, namely after pulling back, taking normalizations and by contracting excessive irreducible rational components, can be modified into a one parameter family of moduli-stable curves,
(ii) if any fibre of the above family is moduli-stable, then fibres of the family are unchanged by the above surgeries.

What is the relationship between moduli-stability and GIT-stability?

Theorem 5.4. *For a connected curve C of genus greater than one, the following are equivalent:*

(1) *C is moduli-stable,*
(2) *Any Hilbert point of C of large degree is Kempf-stable,*
(3) *Any Chow point of C of large degree is Kempf-stable.*

We note that a Hilbert point (or a Chow point) of a curve C is Kempf-stable if and only if it is properly-stable because the automorphism group of C of genus greater than one is finite. The equivalence of (1) and (2) is due to [**?**], while the equivalence of (1) and (3) is due to [Mumford77]. A Hilbert point and a Chow point are the points which completely describe the embedding of C into the projective space, each being a point of a projective space of very big dimension. These are a kind of Plücker coordinates of the Grassman variety in the sense that the Plücker coordinates describe completely a subspace embedded in a fixed vector space.

5.5. **What is our problem?** A simple and clear theorem like Theorem 5.4 which intrinsically characterizes moduli-stable curves is our goal in our compactification problem of moduli spaces. In general the phenomenon of degeneration of algebraic varieties is so complicated that it may be far beyond classifying completely. Compactifying a moduli space may be achieved by minimizing the scale of degeneration of the algebraic varieties. Therefore one needs to collect only significant degeneration, and one needs to ignore less important ones. From this standpoint, it would be our central problem to understand the meaning of the most significant degeneration or the fundamental principle behind it.

So far in this section we reviewed the known results about moduli-stable curves. From the next section on we consider the same problems about abelian varieties. One of our goals is to complete the following diagram:

$$\begin{aligned}
\text{the}\,&\text{moduli of smooth AVs (= abelian varieties)}\\
&= \{\text{smooth polarized AVs} + \text{extra structure}\}/\text{ isom.}\\
&\subset \{\text{smooth polarized AVs or}\\
&\qquad \text{singular polarized degenerate AVs} + \text{extra structure}\}/\text{ isom.}\\
&= \ \text{the new compactification of the moduli of AVs}
\end{aligned}$$

It is also another important goal to characterize those degenerate varieties (schemes) which appear as natural limits of abelian varieties (Theorem 7.4).

6. Moduli theory of cubic curves

6.1. **Stability of cubic curves.** Now let us recall the compact moduli theories of cubic curves. Let k be an algebraically closed field of characteristic $\neq 3$. There are two compact moduli theories of cubic curves from the viewpoint of GIT. First we recall that $\mathrm{SL}(3,k)$ acts on $H^0(\mathbf{P}^2, O(1))$, hence on $V := H^0(\mathbf{P}^2, O(3))$=the space of ternary cubic forms.

Definition 6.2. Le $f \in V$. Then

(1) f is *semistable* if there is an $\mathrm{SL}(3,k)$ invariant homogeneous polynomial H on V such that $H(f) \neq 0$.
(2) f is *Kempf-stable* if the $\mathrm{SL}(3,k)$-orbit of f in V is closed.

We recall that Kempf-stable implies semistable, whereas properly-stable is Kempf-stable with finite stabilizer. The stabilizer group of a Kempf-stable point can be infinite. Let us say that a cubic curve is Kempf-stable/semistable if the equation defining the curve is Kempf-stable/semistable. See Table 1.

Table 1. Stability of cubic curves

curves (sing.)	stability	stab. gr.
smooth elliptic	Kempf-stable	finite
3 lines, no triple point	Kempf-stable	2 dim
a line+a conic, not tangent	semistable, not Kempf-stable	1 dim
irreducible, a node	semistable, not Kempf-stable	$\mathbf{Z}/2\mathbf{Z}$
3 lines, a triple point	not semistable	1 dim
a line+a conic, tangent	not semistable	1 dim
irreducible, a cusp	not semistable	1 dim

6.3. **Moduli spaces of cubic curves.** The first compactification of the moduli of nonsingular cubic curves is the quotient

$$SQ_{1,1} := V(\text{semistable})//\,\mathrm{SL}(3) \simeq \mathbf{P}^1.$$

This is the theory of the j-invariant of elliptic curves.

The second compactification is given in Theorem 6.4. A pair of cubic curves with level 3-structure is defined to be isomorphic if there is an linear isomorphism of the cubic curves mapping e_i to e_i. By the theorem of Hesse, the cubic curves $C(\mu)$ and $C(\mu')$ with level 3-structure are isomorphic iff $\mu = \mu'$ and the isomorphism is the identity morphism of $C(\mu)$. Thus we see that

$$\begin{aligned}&\{\text{smooth cubics } + \text{level 3 structure}\}/\text{ isom.}\\ &\qquad = \{\text{smooth Hesse cubics } + \text{ level 3 structure}\}/\text{ isom.}\\ &\qquad = \{\text{smooth Hesse cubics } + \text{ level 3 structure}\}.\end{aligned}$$

The following is a prototype for all the rest.

Theorem 6.4. *Let $G(3)$ be the Heisenberg group of level 3. Then*

$$\begin{aligned} SQ_{1,3} :&= \left\{\begin{array}{l}\textit{Kempf-stable cubic curves}\\ \textit{with level 3-structure}\end{array}\right\}/\textit{isom.}\\ &= \left\{\begin{array}{l}\textit{cubic curves invariant under } G(3)\\ \textit{with level 3-structure}\end{array}\right\}\\ &= \left\{\begin{array}{l}\textit{Hesse cubics}\\ \textit{with level 3-structure}\end{array}\right\},\end{aligned}$$

which compactifies

$$\begin{aligned} A_{1,3} :&= \left\{\begin{array}{l}\textit{smooth cubic curves}\\ \textit{with level 3-structure}\end{array}\right\}/\textit{isom.}\\ &= \left\{\begin{array}{l}\textit{smooth Hesse cubics}\\ \textit{with level 3-structure}\end{array}\right\}.\end{aligned}$$

The compactification $SQ_{1,3}$ ($\simeq \mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$) is projective over $\mathbf{Z}[\zeta_3, 1/3]$, and it is the fine moduli scheme for families of Kempf-stable cubic curves with level 3 structure over reduced base schemes.

We note that $A_{1,3}$ is $\mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$ with four points ∞, ζ_3^k deleted.

7. Compactification of the moduli in higher dimension

7.1. **PSQAS and TSQAS.** Let R be a complete discrete valuation ring and $k(\eta)$ the fraction field of R. Given an abelian variety $(G_\eta, \mathcal{L}_\eta)$ over $k(\eta)$ with an ample line bundle $\mathcal{L}_\eta$, we have Faltings-Chai degeneration data for it by a finite base change if necessary. In [Nakamura99] for the Faltings-Chai degeneration data, we constructed two natural R-flat projective degenerating families $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ of abelian varieties with generic fiber isomorphic to $(G_\eta, \mathcal{L}_\eta)$. The family $(Q, \mathcal{L})$ is the most naive choice with $\mathcal{L}$ an ample line bundle, while the family $(P, \mathcal{L})$ with $\mathcal{L}$ $(= \mathcal{L}_P)$ the pull back of $\mathcal{L}$ $(= \mathcal{L}_Q)$ on Q is the normalization of $(Q, \mathcal{L})$ after a certain finite minimal base change so that the closed fiber P_0 of P may be reduced.

We call the closed fiber $(P_0, \mathcal{L}_0)$ of $(P, \mathcal{L})$ a torically stable quasi-abelian scheme (abbr. TSQAS), while we call the closed fiber $(Q_0, \mathcal{L}_0)$ of $(Q, \mathcal{L})$ a projectively stable quasi-abelian scheme (abbr. PSQAS) [Nakamura99].

7.2. **Symplectic finite abelian groups.** Let K be a finite symplectic abelian group, namely a finite abelian group with e_K a nondegenerate alternating bimultiplicative form, which we call a symplectic form on K. Let $e_{\min}(K)$ (resp. $e_{\max}(K)$) be the minimum (resp. the maximum) of elementary divisors of K. To be more explicit, let $K = H \oplus H^\vee$, $H = (\mathbf{Z}/e_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/e_g\mathbf{Z})$ with $e_1|e_2|\cdots|e_g$ and $H^\vee = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{G}_m)$. Then $e_{\min}(K) = e_1$ and $e_{\max}(K) = e_g$. Moreover we set

$$e_K(z+\alpha, w+\beta) = \beta(z)\alpha(w)^{-1}$$

for $z, w \in H$, $\alpha, \beta \in H^\vee$. Then e_K is a symplectic form on K. Let $N := e_{\min}(K)$ and $M := e_{\max}(K)$.

Let $\mu_M = \{z \in \mathbf{G}_m; z^M = 1\}$ and $G(K)$ the Heisenberg group, that is a central extension of μ_M by K with its commutator form equal to e_K. If k is algebraically closed, $G(K)$ is unique up to isomorphism. If $g = 1$, $e_1 = 3$, then $G(K) = G(3)$ in the notation of §1.

The classical level-K structures on abelian varieties are generalized as level-$G(K)$ structures on PSQASes and TSQASes. The group scheme $G(K)$ has an essentially unique irreducible representation of weight one over $\mathbf{Z}[\zeta_N, 1/N]$. In [Nakamura99] this fact played a substantial role in constructing a canonical compactification $SQ_{g,K}$ of the moduli space $A_{g,K}$ of abelian varieties with (non-classical and non-commutative) level-K structure. We note that, for any closed field k over $\mathbf{Z}[\zeta_N, 1/N]$, $A_{g,K}(k)$ is the same as the set of all isomorphism classes of abelian varieties with level-K structure in the classical sense.

Theorem 7.3. *Suppose $e_{\min}(K) \geq 3$. Let (Z, L) be a g-dimensional K-PSQAS over an algebraically closed field k of characteristic prime to $|K|$. Then (Z, L) is $G(K)$-equivariantly embedded into $\mathbf{P}(V(K) \otimes k)$ by the natural limits of theta functions. In particular, the image of (Z, L) is a $G(K)$-invariant closed subscheme of $\mathbf{P}(V(K) \otimes k)$.*

By Theorem 7.3, any K-PSQAS (Z, L) is a point of the Hilbert scheme

$$\mathrm{Hilb}_K := \mathrm{Hilb}^P_{\mathbf{P}(V(K))}$$

where $P(n) = \chi(Z, nL) = n^g\sqrt{|K|}$, $g = \dim Z$.

In what follows we say that $\operatorname{Spec} k$ is a geometric point of $\operatorname{Spec}\mathbf{Z}[\zeta_M, 1/M]$ if k is an algebraically closed field containing ζ_M and M is invertible in k. We denote by $\mathrm{SL}_{\pm}(V(K)\otimes k)$ the subgroup of $\mathrm{GL}(V(K)\otimes k)$ consisting of matrices with determinant ± 1.

Theorem 7.4. *Suppose $e_{\min}(K) \geq 3$. Let $M = e_{\max}(K)$. Let $\operatorname{Spec} k$ be a geometric point of $\operatorname{Spec}\mathbf{Z}[\zeta_M, 1/M]$ and $(Z, L) \in \mathrm{Hilb}_K(k)$. Suppose that (Z, L) is smoothable into an abelian variety whose Heisenberg group is isomorphic to $G(K)$. Then the following are equivalent:*

1. *the n-th Hilbert points of (Z, L) are Kempf-stable for any large n,*
2. *a subgroup of $\mathrm{SL}_{\pm}(V(K)\otimes k)$ conjugate to $G(K)$ stabilizes (Z, L),*
3. *(Z, L) is a K-PSQAS over k.*

This theorem follows from [Nakamura99, Theorem 0.3] and [NS06, Theorem 2].

We note that any nonsingular PSQAS is an abelian scheme, which is known to be Kempf-stable by [Kempf78]. For comparison with Theorem 5.4, we restate Theorem 7.4 in a much simpler form:

Theorem 7.5. *Any of the following three objects is the same:*

1. *a degenerate abelian variety whose Hilbert points are* Kempf-stable*,*
2. *a degenerate abelian variety which is* stable under the action of *$G(K)$, the Heisenberg group,*
3. *a K-PSQAS, namely a degenerate abelian variety which is* moduli-stable *(in the sense similar to stable curves).*

Theorem 7.6. *Assume $e_{\min}(K) \geq 3$. Let $M = e_{\max}(K)$. Then there is a projective $\mathbf{Z}[\zeta_M, 1/M]$-subscheme $SQ_{g,K}$ of Hilb_K such that for any geometric point $\operatorname{Spec} k$ of $\operatorname{Spec}\mathbf{Z}[\zeta_M, 1/M]$,*

$$\begin{aligned} SQ_{g,K}(k) &= \left\{ (Z,L);\ \begin{array}{l} (Z,L) \text{ is a Kempf-stable} \\ \text{degenerate AV over } k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} / isom. \\ &= \left\{ (Z,L);\ \begin{array}{l} (Z,L) \text{ is a degenerate AV over } k \\ \text{invariant under } G(K) \\ \text{with level } G(K)\text{-structure} \end{array} \right\} \\ &= \left\{ (Z,L);\ \begin{array}{l} (Z,L) \text{ is a } K\text{-PSQAS over } k \\ \text{invariant under } G(K) \\ \text{with level } G(K)\text{-structure} \end{array} \right\}, \end{aligned}$$

which compactifies

$$\begin{aligned} A_{g,K}(k) &= \left\{ (Z,L);\ \begin{array}{l} (Z,L) \text{ is an abelian variety over } k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} / isom. \\ &= \left\{ (Z,L);\ \begin{array}{l} (Z,L) \text{ is an abelian variety over } k \\ \text{invariant under } G(K) \\ \text{with level } G(K)\text{-structure} \end{array} \right\}. \end{aligned}$$

By definition $G(3) = G(K)$ and $SQ_{1,3} = SQ_{1,K}$ if $K = (\mathbf{Z}/3\mathbf{Z})^2$.

Theorem 7.7. *Suppose $e_{\min}(K) \geq 3$. Let $M := e_{\max}(K)$. The functor $\mathcal{SQ}_{g,K}$ of projectively stable quasi-abelian schemes over reduced base schemes with level $G(K)$-structure is representable by the projective $\mathbf{Z}[\zeta_M, 1/M]$-scheme $SQ_{g,K}$.*

Theorem 7.6 or Theorem 7.7 shows that $SQ_{g,K}$ is a nice compactification of the moduli space of abelian varieties. It is shown by Theorem 7.4 to be also natural from the viewpoint of GIT. If one chooses $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$, all of the above theorems are reduced to Theorem 6.4.

The following theorem [Nakamura10] constructs another canonical compactification of the moduli space $A_{g,K}$ by proper reduced degenerate abelian schemes $(P_0, \mathcal{L}_0)$, which we call *torically stable quasi-abelian schemes* (TSQASes).

Theorem 7.8. *If $e_{\min}(K) \geq 3$, the functor of g-dimensional torically stable quasi-abelian schemes with level-$G(K)$ structure over reduced base algebraic spaces has a complete separated reduced-coarse (hence reduced) moduli algebraic space $SQ^{\mathrm{toric}}_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$. Moreover, there is a canonical bijective finite birational morphism $\mathrm{sq} : SQ^{\mathrm{toric}}_{g,K} \to SQ_{g,K}$. Hence in particular, $SQ^{\mathrm{toric}}_{g,K}$ is projective. Moreover the normalization of $SQ^{\mathrm{toric}}_{g,K}$ is isomorphic to that of $SQ_{g,K}$.*

This theorem suggests that the normalization of $SQ_{g,K}$ is isomorphic to the Voronoi compactification.

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