GRÖBNER BASES OF SYZYGIES

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1. Initial modules

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates over a field K, F a free module with basis $e_1, \cdots, e_m$ and $U \subset F$ a submodule. We say that $m \in F$ is a *monomial*, if for some i, the element m is of the form ue_i, where u is a monomial. A submodule $U \subset F$ is called a *monomial module*, if it is generated by monomials. Then U is a monomial module if and only if for each j there exist monomial ideals I_j such that $U = I_1e_1 \oplus I_2e_2 \oplus \cdots \oplus I_re_r$. In particular, U is finitely generated.

A *monomial order* of the monomials of F is a total order $<$ satisfying the following two conditions:

(1) $m < um$ for all monomials $m \in F$ and all monomials $u \neq 1$ in S;
(2) if $m_1 < m_2$, then $um_1 < um_2$ for all monomials $m_1, m_2 \in F$ and all monomials $u \in S$.

Given a monomial order $<$ on S, there are two standard methods to define monomial orders on F. For $u, v \in \mathrm{Mon}(S)$ and $i, j \in \{1, 2, \dots, r\}$, we define

Position over coefficient: $ue_i > ve_j$, if $i < j$ or $i = j$ and $u > v$;

Coefficient over position: $ue_i > ve_j$, if $u > v$ or $u = v$ and $i < j$.

For example, if $<$ is the lexicographic order on S and $F = Se_1 \oplus Se_2$. Then $x_2e_1 > x_1e_2$, if the position is given more importance than the coefficient, and $x_1e_2 > x_2e_1$ in the opposite case.

We call the monomial order on F which is the (reverse) lexicographic order on the coefficients and gives priority to the position, the *(reverse) lexicographic order* on F (with respect to the given order).

Let $U \subset F$ be a submodule of F, and $<$ a monomial order of F. We let $\mathrm{in}_<(U)$ be the submodule of F which is generated by the monomials $\mathrm{in}(f)$ for *all* $f \in U$. The monomial module $\mathrm{in}_<(U)$ is called the *initial module* of U. Since $\mathrm{in}_<(U)$ is finitely generated, there exist elements $f_1, \dots, f_m \in U$ such that $\mathrm{in}_<(U)$ is generated by $\mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_m)$. Any such system of elements of U is a called a *Gröbner basis* of U with respect to $<$

Proposition 1.1. *Any Gröbner basis of U is a system of generators of U.*

For $f, g \in F$ we construct an element which is obtained as a linear combination of f and g such that their leading terms cancel. Say, $\mathrm{in}_<(f) = ue_i$ and $\mathrm{in}_<(g) = ve_j$. Obviously, if $i \neq j$, there is no linear combination of f and g such that the leading terms can cancel. Thus an analogue to S-polynomials can only be defined if $i = j$. In that case we set

$$S(f,g) = \frac{\mathrm{lcm}(u,v)}{cu}f - \frac{\mathrm{lcm}(u,v)}{dv}g, \tag{1}$$

where c is the coefficient of $\mathrm{in}_<(f)$ in f and d is the coefficient of $\mathrm{in}_<(g)$ in g. We call $S(f,g)$ the *S-element* of f and g

Suppose that $f_1,\dots,f_m$ is Gröbner basis of U. Then $f_1,\dots,f_m$ is a system of generators of U. We choose a free S-module G with basis $g_1,\dots,g_m$, and let $\varepsilon\colon G\to U$ be the epimorphism defined by $\varepsilon(g_i)=f_i$ for $i=1,\dots,m$. The kernel of ε will be denoted by $\mathrm{Syz}(f_1,\dots,f_m)$. Our task is to compute $\mathrm{Syz}(f_1,\dots,f_m)$, which amounts to compute a system of generators of $\mathrm{Syz}(f_1,\dots,f_m)$. The elements of $\mathrm{Syz}(f_1,\dots,f_m)$ are called *relations* of U (with respect to the presentation $G\to U$). Notice that $\sum_{i=1}^m s_ig_i$ is a relation, if and only if $\sum_{i=1}^m s_if_i=0$.

For each pair f_i,f_j with $i<j$, whose initial monomials involve the same basis element of F, the element $S(f_i,f_j)$ reduces to zero with respect to $f_1,\dots,f_m$. In other words, for each such pair we have an equation

$$(2)\; S(f_i,f_j)=q_{ij,1}f_1+q_{ij,2}f_2+\cdots+q_{ij,m}f_m \quad\text{with}\quad \mathrm{in}_<(q_{ij,k}f_k)<\mathrm{in}_<(S(f_i,f_j)),$$

which is a standard expression for $S(f_i,f_j)$. Recall that $S(f_i,f_j)=u_{ij}f_i-u_{ji}f_j$, where the terms u_{ij} and u_{ji} are chosen such that the leading terms of $u_{ij}f_i$ and $u_{ji}f_j$ are the same, so that they cancel in $S(f_i,f_j)$.

Equation (2) gives rise to the following relation:

$$(3)\qquad r_{ij}=u_{ij}g_i-u_{ji}g_j-q_{ij,1}g_1-q_{ij,2}g_2-\cdots-q_{ij,m}g_m.$$

2. Hilbert's syzygy theorem via Gröbner bases

Our goal is to show that each finitely generated free S-module has a free resolution of length at most n, where n is the number of variables of the polynomial ring S. This is the celebrated *syzygy theorem* of Hilbert. We prove this theorem by using Gröbner bases following the arguments given by Schreyer in his dissertation, who found this constructive proof of Hilbert's syzygy theorem. The essential idea is to choose suitable monomial orders in the computation of the syzygies.

Let F be a free S-module with basis $e_1,\dots,e_r$ and $<$ a monomial order on F. Let $U\subset F$ be generated by $f_1,\dots,f_m$, G a free S-module with basis $g_1,\dots,g_m$, and $\varepsilon\colon G\to U$ the epimorphism with $\varepsilon(g_j)=f_j$ for $j=1,\dots,m$. We define a monomial order on G, again denoted $<$, as follows. Let ug_i and vg_j be monomials in G. Then we set

$$ug_i<vg_j \iff \mathrm{in}_<(uf_i)<\mathrm{in}_<(vf_j),\ \text{or}\quad \mathrm{in}_<(uf_i)=\mathrm{in}_<(vf_j)\quad\text{and}\quad j<i.$$

Let us verify that $<$ is a monomial order on G. In order to see that $<$ is a total order on the monomials of G, we have to show that either $ug_i<vg_j$ or $ug_i\geq vg_j$.

Assume that $ug_i\not< vg_j$. Then $\mathrm{in}_<(uf_i)\not<\mathrm{in}_<(vf_j)$, and either $\mathrm{in}_<(uf_i)\neq\mathrm{in}_<(vf_j)$ or $j\geq i$. In the first case $\mathrm{in}_<(uf_i)>\mathrm{in}_<(vf_j)$, since $<$ is a total order on F. It follows in this case that $ug_i>vg_j$. In the second case $\mathrm{in}_<(uf_i)=\mathrm{in}_<(vf_j)$ and $j\geq i$. In this case $ug_i\geq vg_j$, by the definition of $<$ on G.

Next we check condition (1) and (2) for monomial orders as defined before:

(1) Let $w\in\mathrm{Mon}(S)$, $w\neq 1$. Then $\mathrm{in}_<(uf_i)<w\,\mathrm{in}_<(uf_i)=\mathrm{in}_<(wuf_i)$, therefore $ug_i<wug_i$.

(2) Let $ug_i < vg_j$ and $w \in \text{Mon}(S)$. If $\text{in}_<(uf_i) < \text{in}_<(vf_j)$, then $\text{in}_<(wuf_i) = w\,\text{in}_<(uf_i) < w\,\text{in}_<(vf_j) = \text{in}_<(wvf_j)$, and so $wug_i < wvg_j$. On the other hand, if $\text{in}_<(uf_i) = \text{in}_<(vf_j)$, then $j < i$ and $\text{in}_<(wuf_i) = \text{in}_<(wvf_j)$. So again, $wug_i < wvg_j$.

We call this monomial order defined on G the monomial order induced by $f_1, \dots, f_m$ (and the monomial order $<$ on F).

The crucial result whose proof can be found [2, Theorem 15.10] is now the following:

Theorem 2.1 (Schreyer)**.** *Let F be a free S-module with basis $e_1, \dots, e_r$, and $<$ a monomial order on F. Let $U \subset F$ be a submodule of F with Gröbner basis $\mathcal{G} = \{f_1, \dots, f_m\}$. Then the relations r_{ij} arising from the S-elements of the f_i as described in (3) form a Gröbner basis of $\text{Syz}(f_1, \dots, f_m) \subset G$ with respect to the monomial order induced by $f_1, \dots, f_m$. Moreover, one has*

$$\text{in}_<(r_{ij}) = u_{ij} g_i,$$

where u_{ij} is defined as in (3).

The monomial order induced by $f_1, \dots, f_m$ allows some flexibility, since we are free to relabel the elements of the Gröbner basis as we want. Doing this in a clever way we obtain

Corollary 2.2. *With the notation introduced in Theorem 2.1, let the f_i be indexed in such way such that whenever $\text{in}_<(f_i)$ and $\text{in}_<(f_j)$ for some $i < j$ involve the same basis element, say $\text{in}_<(f_i) = ue_k$ and $\text{in}_<(f_j) = ve_k$, then $u > v$ with respect to the lexicographic order induced by $x_1 > x_2 > \dots > x_n$. Then it follows, that if for some $t < n$ the variables $x_1, \dots, x_t$ do not appear in the initial forms of the f_j, then the variables $x_1, \dots, x_{t+1}$ do not appear in the initial forms of the r_{ij}.*

Proof. By Theorem 2.1 we have $\text{in}_<(r_{ij}) = (\text{lcm}(u, v)/v)e_k$. Since $u > v$, and since u and v are monomials in the variables $x_{t+1}, \dots, x_n$, it follows that the exponent of x_{t+1} in u is bigger that of v. Thus $\text{lcm}(u, v)/v$ is a monomial in the variables $x_{t+2}, \dots, x_n$, as desired.

As a consequence of Corollary 2.2 we finally obtain

Theorem 2.3 (Hilbert's syzygy theorem)**.** *Let M be a finitely generated S-module over the polynomial ring $S = K[x_1, \dots, x_n]$. Then M admits a free S-resolution*

$$0 \to F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \to 0$$

of length $p \leq n$.

Proof. Let $U \subset F$ be a submodule of the free S-module F with basis $e_1, \dots, e_r$. Let $<$ be a monomial order on F, and $f_1, \dots, f_m$ be a Gröbner basis of U. Finally, let $t \leq n$ be the largest integer such that the variables $x_1, \dots, x_t$ do not appear in any of the initial forms of the f_i. We prove by induction on $n - t$, that U has a free S-resolution of length $\leq \max\{0, n - t - 1\}$

If $t \geq n - 1$, then $\text{in}_<(U) = \bigoplus_{j=1}^r I_j e_j$, where for each j, there exists a monomial ideal $J_j \subset K[x_n]$ such that $I_j = J_j S$. Since all monomial ideals in $K[x_n]$ are principal, it follows that U is free.

If $t < n$, we may assume that the Gröbner basis $f_1, \dots, f_m$ is labeled as described in Corollary 2.2. Then Theorem 2.1 together with Corollary 2.2 imply that $\text{Syz}(f_1, \dots, f_m)$

has a Gröbner basis with the property that the variables $x_1, \dots, x_{t+1}$ do not appear in any of the leading monomials of the elements of the Gröbner basis. Thus, by induction, $\mathrm{Syz}(f_1, \dots, f_m)$ has a free S-resolution of length $\leq n-t-2$. Composing this resolution with the exact sequence $0 \to \mathrm{Syz}(f_1, \dots, f_m) \to G \to U \to 0$, we obtain for U a free S-resolution of length $\leq n-t-1$, as desired.

Now let M be an arbitrary finitely generated S-module. Then $M \cong F/U$, where F is a finitely generated free S-module. We may assume that $n > 0$. Then by the preceding arguments U has a free S-resolution of length $\leq n-1$. This implies that M has a free S-resolution of length $\leq n$. □

3. $\mathbb{Z}^n$-graded modules

The objective of this section is to present a result for the syzygies of a $\mathbb{Z}^n$-graded modules, due to Fløystad and the author [3], which is of similar nature as that of Schreyer discussed in the previous section.

Let F be a $\mathbb{Z}^n$-graded free S-module with homogeneous basis $e_1, \dots, e_m$ and $\deg e_i = \mathbf{a}_i$ for $i = 0, \dots, m$. Then $F_{\mathbf{a}}$ is the K-vector space spanned by all monomials $\mathbf{x}^{\mathbf{a}-\mathbf{a}_i} e_i$ for which $\mathbf{a} - \mathbf{a}_i \in \mathbb{Z}^n_{\geq 0}$.

We fix a monomial order on S and let $<$ be the monomial order on F induced by the monomial order on S which gives priority to the position over the coefficients.

Let $M \subset F$ be a $\mathbb{Z}^n$-graded submodule. Then $\mathrm{in}_<(M)$ is generated by all elements $\mathrm{in}_<(u)$ where $u \in M$ is homogeneous. Let u be homogeneous of degree $\mathbf{a}$, say, $u = \sum_i c_i u_i e_i$ with $c_i \in K$, $u_i \in \mathrm{Mon}(S)$ and $\deg u_i + \deg e_i = \mathbf{a}$ for all i with $c_i \neq 0$. Then $\mathrm{in}_<(u) = u_j e_j$, where $j = \min\{i\colon\ c_i \neq 0\}$. Thus we see that $\mathrm{in}_<(M)$ depends only on the basis $\mathcal{F} = e_1, \dots, e_m$ of F and *not* on the given monomial order on S. Hence we denote the initial module of M by $\mathrm{in}_{\mathcal{F}}(M)$.

Our considerations so far can be summed up as follows:

Lemma 3.1. *With the assumptions and notation introduced we have*

$$\mathrm{in}_{\mathcal{F}}(M) = \bigoplus_{i=1}^{m} I_j e_j,$$

where $I_j \cong (M \cap \bigoplus_{k=j}^m Se_k)/(M \cap \bigoplus_{k=j+1}^m Se_k)$ *for* $j = 1, \dots, m$.

We call the basis $\mathcal{F} = e_1, \dots, e_m$ of F *lex-refined*, if $\deg(e_1) \geq \deg(e_2) \geq \ldots \geq \deg(e_m)$ in the lexicographical order.

In the following we present a result which is a sort of analogue to the theorem of Schreyer. Let M be a $\mathbb{Z}^n$-graded S-module, and

$$\mathbb{F}\colon \quad \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

a $\mathbb{Z}^n$-graded free resolution of M. We set $Z_p(\mathbb{F}) = \mathrm{Im}(\varphi_p)$ for all p. Then $Z_p = Z_p(\mathbb{F})$ is the pth syzygy module of M with respect to the resolution $\mathbb{F}$.

Theorem 3.2. *Let $1 \leq p \leq n$ be an integer, and $\mathcal{F}$ a lex-refined basis of F_{p-1}. Then $\mathrm{in}_{\mathcal{F}}(Z_p) = \bigoplus_{j=1}^m I_j e_j$, where the minimal set of monomial generators of each I_j belongs to $K[x_p, \dots, x_n]$.*

Proof. The statement is trivial for $p = 1$. We may therefore assume that $p \geq 2$. Let $n \in Z_p$ be a homogeneous element of Z_p with $\text{in}(n) = u_i e_i$ and such that u_i is a minimal generator of I_i. Let k be the smallest number such that x_k divides $\text{in}(n) = u_i e_i$, and suppose that $k < p$. Then $x_1, \ldots, x_{p-2}$ is a regular sequence on Z_{p-2}, where we set $Z_{p-2} = M$ if $p = 2$. We denote by 'overline' reduction modulo $(x_1, \ldots, x_{k-1})$. It follows that the sequence

$$0 \longrightarrow \bar{Z}_p \longrightarrow \bar{F}_{p-1} \xrightarrow{\bar{\varphi}_{p-1}} \bar{F}_{p-2}$$

is exact. Here $\bar{\varphi}_{p-1} = \bar{\varepsilon}$, if $p = 2$. Hence $\bar{Z}_p$ may be identified with its image in $\bar{F}_{p-1}$.

Thus $\bar{n}$ can be written as

$$\bar{n} = c_i u_i \bar{e}_i + c_{i+1} u_{i+1} \bar{e}_{i+1} + \cdots \quad \text{with} \quad c_j \in K \quad \text{and} \quad u_j \in \text{Mon}(S) \quad \text{and} \quad c_i \neq 0.$$

Since $u_j \in K[x_k, \ldots, x_n]$ for all j with $c_j \neq 0$ and since $\bar{n}$ is homogeneous, it follows $\deg_t \bar{e}_j = \deg_t \bar{e}_i$ for all $t \leq k-1$ and all j with $c_j \neq 0$. (Here, for any homogeneous element r, we denote by $\deg_t r$ the tth component of $\deg r$.) Therefore, since x_k divides u_i, it follows that x_k divides $u_j \neq 0$ for $j > i$ with $c_j \neq 0$, because $\deg \bar{e}_i = \deg e_i \geq \deg e_j = \deg \bar{e}_j$ for $j > i$. This implies that x_k divides $\bar{n}$. Thus there exist $w \in \bar{F}_{p-1}$ such that $\bar{n} = x_k w$. It follows that $x_k \bar{\varphi}_{p-1}(w) = \bar{\varphi}_{p-1}(\bar{n}) = 0$. Since x_k is a nonzero divisor on $\bar{F}_{p-2}$, we see that $\bar{\varphi}_{p-1}(w) = 0$. This implies that $w \in \bar{Z}_p$. Let $m = d_r v_r e_r + \cdots + d_i v_i e_i + \cdots$ be a homogeneous element in F_{p-1} such that $\bar{m} = w$ with $v_j \in \text{Mon}(S)$ and $d_j \in K$ for all j, and $d_r \neq 0$. Then $r \leq i$ and $u_i = x_k v_i$.

Suppose that $r < i$. Since $x_j \nmid u_i$ for all $j < k$, and since m is homogeneous it follows that

$$\text{(4)} \qquad \deg_t v_r e_r = \deg_t v_i e_i = \deg_t e_i \quad \text{for all} \quad t < k.$$

On the other hand, since $\bar{n} = x_k \bar{m} = d_r x_k \bar{v_r} \bar{e_r} + \cdots$, we see that $x_k \bar{v_r} = 0$, and this implies that v_r is divisible by some x_j with $j < k$. Let s be the smallest such integer. Then form (4) we deduce that $\deg_j e_r = \deg_j e_i$ for $j < s$ and $\deg_s e_r < \deg_s e_i$. Hence $\deg e_r < \deg e_i$ (with respect to the lexicographic order), contradicting the choice of our basis. Thus $r = i$, and consequently $v_i \in I_i$. But this is again a contradiction, since $u_i = x_k v_i$ and since u_i is a minimal generator of I_i. □

Theorem 3.2 has a remarkable consequence for the Stanley depth of syzygies. A *Stanley decomposition* of a finitely generated $\mathbb{Z}^n$-graded S-module M is a direct sum decomposition $M = \bigoplus_{i=1}^m u_i K[Z_i]$ of M as a $\mathbb{Z}^n$-graded K-vector space, where each u_i is a homogeneous element of M, $K[Z_i]$ is a polynomial ring is a set of variables $Z_i \subset \{x_1, \ldots, x_n\}$, and each $u_i K[Z_i]$ is a free $K[Z_i]$-submodule of M. The minimum of the numbers $|Z_i|$ is called the Stanley depth of this decomposition. The *Stanley depth* of M, denoted $\text{sdepth}\, M$, is the maximal Stanley depth of a Stanley decomposition of M. In his paper [5] Stanley conjectured that $\text{sdepth}\, M \geq \text{depth}\, M$. This conjecture is widely open.

Here we show (see [3])

Theorem 3.3. *Let M be a finitely generated $\mathbb{Z}^n$-graded module, and let $F_{\bullet}$ be a free resolution. Then for $p \geq 1$ the p'th syzygy module Z_p has Stanley depth greater than or equal to p, or it is a free module.*

Proof. Let $\mathcal{F}$ be a lex-refined basis for F_p. If $p \geq n$ then Z_p is free, so suppose $p < n$. By Theorem 3.2, $\text{in}_{\mathcal{F}}(Z_p) = \bigoplus_{j=1}^{m} I_j e_j$, where the minimal set of monomial generators of each of the monomial ideals I_j belongs to $K[x_p, \ldots, x_n]$. But then $\operatorname{sdepth} I_j \geq p$. In fact, Cimpoeaş [1, Corollary 1.5] showed that the Stanley depth of any finitely generated $\mathbb{Z}^n$-graded torsionfree S-module is at least 1. Hence the asserted inequality for the Stanley depth of I_j follows from [4, Lemma 3.6]. Now the desired inequalities for the Stanley depths of the syzygy modules follow from the simple fact that $\operatorname{sdepth}(\text{in}_{\mathcal{F}}(Z_p)) \geq \max\{\operatorname{sdepth} I_1, \ldots, \operatorname{sdepth} I_m\}$. □

References

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