Limit stable objects on Calabi-Yau 3-folds

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Abstract

The notion of limit stability is introduced in [31] to give a limiting degeneration of Bridgeland stability. In this article, we give the survey of the results in [31], [30]. We introduce the limit stability, counting invariants of limit stable objects, and see that the wall-crossing phenomena yields the rationality conjecture of Pandharipande-Thomas generating functions.

1 Motivations

1.1 triangulated categories, stability conditions

Let $\mathcal{D}$ be a triangulated category, e.g. bounded derived category of coherent sheaves $D^b(X)$ on an algebraic variety X. Its objects consist of bounded complexes of coherent sheaves,

$$\cdots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots \to \mathcal{F}^j \to 0 \to \cdots ,$$

where $\mathcal{F}^i \in \mathrm{Coh}(X)$. Historically such a class of categories was introduced to formulate the generalization of several duality theories, such as Poincaré duality, Serre duality. (cf. [3], [10].) On the other hand, the notion of triangulated categories draw much attention recently from the viewpoint of string theory. In terms of string theory, an object in the derived category of coherent sheaves is considered to represent a D-brane of type B, and a conjectural symmetry (Homological mirror symmetry) between the category of A-branes (Fukaya category) and B-branes (derived category) is proposed by Kontsevich [19].

In 2002, an important notion of stability conditions on triangulated categories was introduced by Bridgeland [8]. For a triangulated category $\mathcal{D}$, he associates a space $\mathrm{Stab}(\mathcal{D})$, which has a structure of complex manifold. So we have the following correspondence,

$$\text{triangulated category} \longrightarrow \text{complex manifold}$$

There are several motivations to introduce the complex manifold $\mathrm{Stab}(\mathcal{D})$.

- Classically there is a notion of stability condition on vector bundles on curves. (cf. [24].) We want to generalize this notion to objects in derived categories. For each $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the associated notion of σ-semistable objects in $\mathcal{D}$. So each point $\sigma \in \mathrm{Stab}(\mathcal{D})$ gives a generalization of the classical notion of stability condition. In terms of string theory, σ-semistable objects are considered to be the D-branes of BPS-state.

- The space $\mathrm{Stab}(\mathcal{D})$ is considered to describe the (extended) stringy Kähler moduli space, which should be isomorphic to the moduli space of complex structures on the mirror side. Thus it is an interesting problem to compare the space $\mathrm{Stab}(\mathcal{D})$ with the moduli space of the complex structures under mirror symmetry.

Since the theory of stability conditions on triangulated categories has been proposed recently, the theory is not so developed yet. One of the big issues is the existence problem of stability conditions, especially on the derived category of coherent sheaves on projective Calabi-Yau 3-folds. We will address this problem later.

1.2 Counting invariants of semistable objects

Let X be a smooth projective Calabi-Yau 3-fold. Let $I_n(X,\beta)$ be the moduli space of ideal sheaves $I_C \subset \mathcal{O}_X$ such that $C \subset X$ is one dimensional with

$$\mathrm{ch}_2(\mathcal{O}_C) = \beta, \quad \mathrm{ch}_3(\mathcal{O}_C) = n,$$

where $\beta \in H^4(X,\mathbb{Z})$ and $n \in H^6(X,\mathbb{Z}) \cong \mathbb{Z}$. In other words, $I_n(X,\beta)$ is a component of the moduli space of rank one Gieseker stable sheaves on X with trivial determinants. By the work of [28], there is the virtual fundamental cycle on $I_n(X,\beta)$, whose virtual dimension is zero. The Donaldson-Thomas invariant is defined by the integration of the virtual cycle,

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The invariant $I_n(X,\beta)$ is interesting in connection with Gromov-Witten theory. (cf. [23].) Note that the structure sheaf $\mathcal{O}_C$ may contain zero dimensional subsheaves, so $I_{n,\beta}$ does not directly count curves. In order to avoid that issue, a variant of Donaldson-Thomas type invariants is introduced by Pandharipande-Thomas [26], via counting stable pairs. By definition a stable pair consist of data (F,s), where $F \in \mathrm{Coh}(X)$ is pure one dimensional sheaf, and s is a morphism

$$s \colon \mathcal{O}_X \longrightarrow F,$$

such that $\mathrm{Coker}(s)$ is zero dimensional. The moduli space of stable pairs (F,s) with $(\mathrm{ch}_2(F), \mathrm{ch}_3(F)) = (\beta, n)$ is constructed in [26] as a projective variety, and shown to have a perfect symmetric obstruction theory. In particular there is the virtual fundamental cycle on $P_n(X,\beta)$, and one can define the invariant,

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The invariant $P_{n,\beta}$ is considered to count objects in the derived category, by viewing stable pairs (F,s) as two term complexes,

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X \stackrel{s}{\longrightarrow} F \longrightarrow 0 \longrightarrow \cdots \in D^b(X). \tag{1}$$

Here the degree of $\mathcal{O}_X$ is -1 and the degree of F is zero. Conjecturally the objects (1) are stable with respect to a certain stability condition on $D^b(X)$. Note that an object E given in (1) satisfies the following condition,

$$\mathrm{ch}(E) = (-1, 0, \beta, n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6, \quad \det E = \mathcal{O}_X, \tag{2}$$

Under the above background, we suggest the following story.

- For a projective Calabi-Yau 3-fold X, let $\mathcal{D} = D^b(X)$. We expect that there are stability conditions $\sigma, \tau \in \mathrm{Stab}(\mathcal{D})$ such that ideal sheaves $I_C[1]$ and objects (1) become stable with respect to σ, τ respectively.

- We expect that for any $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the algebraic moduli stack of σ-semistable objects $E \in \mathcal{D}$ with fixed phase and satisfy 2. We denote that moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$. For a particular choice of σ, the stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ should be the gerb over $I_n(X,\beta)$ or $P_n(X,\beta)$.

- We expect that there is the generalized Donaldson-Thomas invariant,

$$DT_{n,\beta}(\sigma) \in \mathbb{Q},$$

  counting σ-semistable objects $E \in \mathcal{D}$ satisfying (2). $DT_{n,\beta}(\sigma)$ should be defined as the integration of the "logarithm" of the moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ in the Hall algebra of $\mathcal{D}$, after multiplying Behrend's weight function [2]. This procedure (expect multiplication of weight function) follows from Joyce's sequent works [14], [15], [16], [16], [17]. It should be possible to use the motivic milnor fiber idea of Kontsevich-Soibelman [20] to involve weight function into Joyce's invariants. A particular choice of σ yields $DT_{n,\beta}(\sigma) = I_{n,\beta}$ or $P_{n,\beta}$. However $DT_{n,\beta}(\sigma)$ give new invariants by deforming σ.

- We want to know how $DT_{n,\beta}(\sigma)$ varies under change of σ. In principle, there is a wall and chamber structure on $\mathrm{Stab}(\mathcal{D})$ so that $DT_{n,\beta}(\sigma)$ does not change if σ deforms in a chamber. However if σ crosses a wall, then the invariant $DT_{n,\beta}(\sigma)$ jumps and its difference should be expressed in terms of the structure of the Ringel-Hall Lie algebra associated to $\mathcal{D}$. Thus we should have the wall-crossing formula of the invariants $DT_{n,\beta}(\sigma)$.

- Applying the wall-crossing formula of $DT_{n,\beta}(\sigma)$, we expect that several properties or equalities of the generating functions of sheaf countings are realized. For instance, DT-PT correspondence [26], DT-NCDT correspondence [27], flop formula of DT-invariants [11], and the rationality conjecture of the generating functions of DT or PT-invariants should be explained by wall-crossing formula. (cf. [30].)

At this moment, there are several technical difficulties to realize the above story. One of them is to find stability conditions, which will be discussed in the next section.

2 Stability conditions

2.1 Bridgeland's stability conditions

First let us give the definition of stability conditions introduced in [8].

Definition 2.1. A stability condition on a triangulated category $\mathcal{D}$ consists of data $\sigma = (Z, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$, and Z is a group homomorphism,

$$Z \colon K(\mathcal{D}) \longrightarrow \mathbb{C},$$

which satisfies the following axiom.

- For a non-zero object $0 \neq E \in \mathcal{A}$, we have

$$Z(E) \in \mathbb{H} := \{r \exp(i\pi\phi) \mid 0 < \phi \leq 1, r > 0\}.$$

  Especially one can choose the argument $\arg Z(E) \in (0, \pi]$ uniquely. An object $E \in \mathcal{A}$ is said to be Z-(semi)stable if for any non-zero object $F \subset E$, one has

$$\arg Z(F) \leq (<) \arg Z(E).$$

- There is a Harder-Narasimhan property, i.e. any $E \in \mathcal{A}$ admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

  such that each $F_i = E_i/E_{i-1}$ is Z-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here we give some examples.

Example 2.2. (i) Let $\mathcal{D} = D^b(C)$ for a smooth projective curve C, and $Z \colon K(C) \to \mathbb{C}$ be

$$Z(E) = -\deg(E) + \mathrm{rk}(E) \cdot i.$$

Then the pair $(Z, \mathrm{Coh}(C))$ determines a stability condition on $\mathcal{D}$. In this case, an object $E \in \mathrm{Coh}(C)$ is Z-semistable if and only if it is a semistable sheaf in the sense of [24].

(ii) Let A be a finite dimensional k-algebra with k a field, and $\mathcal{D} = D^b(\mathcal{A})$ where $\mathcal{A} = \mathrm{mod}\, A$ is the abelian category of finitely generated right A-modules. Then there is a finite number of simple objects $S_1, \cdots, S_N \in \mathcal{A}$ which generates $\mathcal{A}$. One can choose $Z \colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z(S_i) \in \mathcal{H}$ for all $1 \leq i \leq N$. Then the pair $(Z, \mathcal{A})$ determines a stability condition on $\mathcal{D}$.

So far, the spaces $\mathrm{Stab}(\mathcal{D})$ for several $\mathcal{D}$ have been studied in detail. For instance, see [9], [6], [7], [29], [25], [22], [13], [4], [33], [34]. On the other hand, the following problem has been a big issue in studying stability conditions.

Problem 2.3. *Given a triangulated category $\mathcal{D}$, do we have an example of a stability condition on $\mathcal{D}$, i.e.* $\mathrm{Stab}(\mathcal{D}) \neq \emptyset$*?*

The above problem is non-trivial especially for the case $\mathcal{D} = D^b(X)$, where X is a smooth projective variety with $\dim X \geq 2$. In this case, one can show that there is no stability condition $(Z, \mathcal{A})$ with $\mathcal{A} = \mathrm{Coh}(X)$. As an analogue of Example 2.2 (i), one might try to construct Z to be the group homomorphism

$$Z(E) = -c_1(E) \cdot \omega + \mathrm{rk}(E) \cdot i,$$

for a fixed ample divisor ω. However the pair $(Z, \mathrm{Coh}(X))$ does not give a stability condition since $Z([\mathcal{O}_x]) = 0$ for a closed point $x \in X$. When $\dim X = 2$, the examples of stability conditions are constructed by tilting the abelian category $\mathrm{Coh}(X)$, (cf. [9].) However we do not know any example of stability conditions when $\dim X \geq 3$, except the case that there is a derived equivalence $D^b(X) \cong D^b(A)$ for a finite dimensional algebra A. (e.g. $X = \mathbb{P}^3$.)

From the viewpoint of mirror symmetry, the most important case is when X is a projective Calabi-Yau 3-fold. In this case, there are some ideas coming from string theory. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is ample } \}.$$

Let $Z_{(B,\omega)} \colon K(X) \to \mathbb{C}$ be

$$Z_{(B,\omega)}(E) = -\int e^{-(B+i\omega)} \operatorname{ch}(E)\sqrt{\operatorname{td}_X}.$$

We can state the following conjecture.

Conjecture 2.4. *For $\omega \gg 0$, there should exist the heart of a bounded t-structure $\mathcal{A}_{(B,\omega)} \subset D^b(X)$ such that the pair $\sigma_{(B,\omega)} = (Z_{(B,\omega)}, \mathcal{A}_{(B,\omega)})$ is a stability condition on $D^b(X)$.*

The above conjecture holds true if $\dim X \leq 2$.

2.2 Limit stability

The idea of limit stability is to construct the stability conditions corresponding to $\omega \to \infty$ in the notation of Conjecture 2.4. In order to see this, let us take non-zero $E \in \mathrm{Coh}(X)$ and observe the value $Z_{(B,\omega)}(E)$ for $\omega \to \infty$. The result is as follows, ($d = \dim \mathrm{Supp}(E)$,)

$$\arg Z_{(B,\omega)}(E) \longrightarrow \begin{cases} \pi, & d = 0, \\ \pi/2, & d = 1, \\ 0, & d = 2, \\ -\pi/2, & d = 3. \end{cases}$$

By the above observation, it is reasonable to guess that $\mathcal{A}_{(B,\omega)}$ should be an approximation of the category spanned by $E \in \mathrm{Coh}(X)$ with $d \leq 1$ and $E[1] \in \mathrm{Coh}(X)[1]$ with $d \geq 2$. The heart of the perverse t-structure, introduced by Bezrukavnikov [5] and Kashiwara [18] provide the appropriate heart of t-structure. Let $\mathcal{T}$, $\mathcal{F}$ be the pair of subcategories of $\mathrm{Coh}(X)$ defined as

$$\begin{aligned} \mathcal{T} &= \{E \in \mathrm{Coh}(X) \mid \dim \mathrm{Supp}(E) \leq 1\}, \\ \mathcal{F} &= \{E \in \mathrm{Coh}(X) \mid \mathrm{Hom}(\mathcal{T}, E) = 0\}. \end{aligned}$$

It is easy to see that $(\mathcal{T}, \mathcal{F})$ determines a torsion theory on $\mathrm{Coh}(X)$.

Definition 2.5. We define the category $\mathcal{A}^p$ to be

$$\mathcal{A}^p := \{E \in D^b(X) \mid \mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^i(E) = 0 \text{ for } i \neq 0, -1\}.$$

$\mathcal{A}^p$ is the tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$, and hence $\mathcal{A}^p$ is the heart of a bounded t-structure on $D^b(X)$. By the above asymptotic behavior of $Z_{(B,\omega)}$ for $\omega \to \infty$, it is easy to see the following lemma. (See [31, Lemma 2.20].)

Lemma 2.6. *For a non-zero $E \in \mathcal{A}^p$, we have*

$$Z_{(B,m\omega)}(E) \in e^{\frac{\pi i}{4}}\mathbb{H},$$

for $m \gg 0$, depending on the numerical classes of E.

By the above lemma, one can uniquely determine the argument,

$$\arg Z_{(B,m\omega)}(E) \in (\pi/4, 5\pi/4),$$

for $m \gg 0$.

Definition 2.7. An object $E \in \mathcal{A}^p$ is called $B + i\omega \in A(X)_{\mathbb{C}}$-limit (semi)stable if for any non-zero subobject $F \subset E$ in $\mathcal{A}^p$, we have

$$\arg Z_{(B,m\omega)}(F) < (\leq) \arg Z_{(B,m\omega)}(E),$$

for $m \gg 0$.

Remark 2.8. (i) Since the choice of $m \gg 0$ depends on $E \in \mathcal{A}^p$, the above definition of stability does not give a stability condition on $D^b(X)$ in the sense of Bridgeland. The limit stability is interpreted as a kind of limiting degeneration of Bridgeland stability.

(ii) A similar kind of stability is introduced by Bayer [1] independently. He introduces the notion of polynomial stability on the category of more general perverse coherent sheaves.

In the following, we give some examples of limit (semi)stable objects.

Example 2.9. (i) Let F be a μ-stable vector bundle on X. Then $F[1] \in \mathcal{A}^p$ and it is $B + i\omega$-limit stable for any $B + i\omega \in A(X)_{\mathbb{C}}$.

(ii) For $T \in \mathcal{T} \subset \mathrm{Coh}(X)$, we can easily see that F is $B + i\omega$-limit semistable if and only if F is a (B, ω)-twisted semistable sheaf, i.e. for any non-zero subsheaf $F' \subset F$, one has $\nu_{(B,\omega)}(F') \leq \nu_{(B,\omega)}(F)$, where

$$\nu_{(B,\omega)}(F) = \frac{\mathrm{ch}_3(F) - B\,\mathrm{ch}_2(F)}{\omega\,\mathrm{ch}_2(F)} \in \mathbb{R}.$$

(iii) Let $x \in X$ be a closed point and $I_x \subset \mathcal{O}_X$ be the ideal sheaf. Then I_x is a Gieseker stable sheaf, but $I_x[1] \in \mathcal{A}^p$ is not $B + i\omega$-limit semistable. In fact we have the exact sequence in $\mathcal{A}^p$,

$$0 \longrightarrow \mathcal{O}_x \longrightarrow I_x[1] \longrightarrow \mathcal{O}_X[1] \longrightarrow 0,$$

with $\arg Z_{(B,m\omega)}(\mathcal{O}_x) > \arg Z_{(B,m\omega)}(I_x[1])$ for $m \gg 0$, which destabilizes $I_x[1]$.

We have the following property, whose proof is seen in [31, Theorem 2.29].

Proposition 2.10. *For $B+i\omega \in A(X)_{\mathbb{C}}$, we have the following.*

(i) For a non-zero $E \in \mathcal{A}^p$, there exists a filtration in $\mathcal{A}^p$,

$$E_0 \subset E_1 \subset \cdots \subset E_n = E, \tag{3}$$

such that each $F_i = E_i/E_{i+1}$ is $B+i\omega$-limit semistable with

$$\arg Z_{(B,m\omega)}(F_i) > \arg Z_{(B,m\omega)}(F_{i+1})$$

for $m \gg 0$, i.e. (3) is a Harder-Narasimhan filtration.

(ii) For a $B+i\omega$-limit semistable object $E \in \mathcal{A}^p$, there exists a filtration in $\mathcal{A}^p$,

$$E_0 \subset E_1 \subset \cdots \subset E_n = E, \tag{4}$$

such that each $F_i = E_i/E_{i+1}$ is $B+i\omega$-limit stable with

$$\arg Z_{(B,m\omega)}(F_i) = \arg Z_{(B,m\omega)}(F_{i+1})$$

for $m \gg 0$, i.e. (4) is a Jordan-Hölder filtration.

3 Moduli theory of stable objects and counting invariants

Let us take elements,

$$\beta \in H^4(X,\mathbb{Z}), \quad n \in H^6(X,\mathbb{Z}) \cong \mathbb{Z}.$$

We consider the moduli problem of limit stable objects $E \in \mathcal{A}^p$, satisfying the following condition,

$$(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)) = (-1, 0, \beta, n), \quad \det E = \mathcal{O}_X. \tag{5}$$

The moduli problem of objects in the derived category has been studied in some articles, see [12], [21], [32]. Here we use the algebraic space constructed by Inaba [12], which provides a "mother space" of our moduli problem. Let $\mathcal{M}$ be the functor,

$$\mathcal{M}\colon (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Set}),$$

which sends a $\mathbb{C}$-scheme S to a family of simple complexes $\mathcal{E} \in D^b(X \times S)$, (up to isomorphism,) where an object $E \in D^b(X)$ is called a *simple complex* if

$$\mathrm{Hom}(E,E) = \mathbb{C}, \quad \mathrm{Ext}^{-1}(E,E) = 0. \tag{6}$$

Then Inaba [12] shows that the étale sheafication of $\mathcal{M}$, denoted by $\mathcal{M}^{\mathrm{et}}$, is an algebraic space of locally finite type. Let $\mathcal{M}_0^{\mathrm{et}}$ be the closed fiber at $[\mathcal{O}_X] \in \mathrm{Pic}(X)$ with respect to the following morphism,

$$\det\colon \mathcal{M}^{\mathrm{et}} \ni E \longmapsto \det E \in \mathrm{Pic}(X).$$

Since any limit stable object $E \in \mathcal{A}^p$ satisfies (6), there is a subspace (in the sense of functor)

$$\mathcal{M}_n^{(B,\omega)}(X,\beta) \subset \mathcal{M}_0^{\mathrm{et}}, \tag{7}$$

paramerizing $B+i\omega$-limit stable objects $E \in \mathcal{A}$ satisfying (5). In [31, Theorem 3.20, Theorem 4.7], we proved the following.

Theorem 3.1. *(i) The space* $\mathcal{M}_n^{(B,\omega)}(X,\beta)$ *is a separated algebraic space of finite type over* $\mathbb{C}$.

(ii) If $B = k\omega$ *for* $k \in \mathbb{R}$*, we have*

$$\mathcal{M}_n^{(B,\omega)}(X,\beta) \cong P_n(X,\beta), \quad (k \ll 0,)$$
$$\mathcal{M}_n^{(B,\omega)}(X,\beta) \cong P_{-n}(X,\beta), \quad (k \gg 0.)$$

4 Counting invariants of limit stable objects

For $B + i\omega \in A(X)_{\mathbb{C}}$, $\beta \in H^4(X,\mathbb{Z})$ and $n \in \mathbb{Z}$, let

$$\mathcal{M}_n^{(B,\omega)}(X,\beta) \subset \mathcal{M}_0^{\mathrm{et}},$$

be the algebraic space constructed in Theorem 3.1. In [2], Behrend constructs a canonical constructible function on any scheme M, (more generally on any Deligne Mumford stack M,)

$$\nu_M \colon M \longrightarrow \mathbb{Z},$$

such that if M is proper and carries a symmetric obstruction theory, one has

$$\sharp^{\mathrm{vir}}(M) = \sum_{a \in \mathbb{Z}} a e(\nu_M^{-1}(a)),$$

where $\sharp^{\mathrm{vir}}(M)$ is the integration over the virtual cycle, and $e(*)$ is the topological euler number. In our situation, let

$$\nu_{\mathcal{M}} \colon \mathcal{M}_n^{(B,\omega)}(X,\beta) \longrightarrow \mathbb{Z},$$

be Behrend's constructible function.

Definition 4.1. We define the invariant $L_{n,\beta}(B,\omega) \in \mathbb{Z}$ by

$$L_{n,\beta}(B,\omega) = \sum_{a \in \mathbb{Z}} a e(\nu_{\mathcal{M}}^{-1}(a)).$$

Remark 4.2. Suppose that $\mathcal{M}_n^{(B,\omega)}(X,\beta)$ is a proper algebraic space. Then by the same argument as in [26, Lemma 2.10], there is a virtual fundamental class,

$$[\mathcal{M}_n^{(B,\omega)}(X,\beta)]^{\mathrm{vir}} \in A_0(\mathcal{M}_n^{(B,\omega)}(X,\beta)),$$

and our invariant $L_{n,\beta}(B,\omega))$ coincides with the integration over the virtual class,

$$L_{n,\beta}(B,\omega) = \int_{[\mathcal{M}_n^{(B,\omega)}(X,\beta)]^{\mathrm{vir}}} 1.$$

By Theorem 3.1 (ii), the invariants $L_{n,\beta}(B,\omega)$ relate to $P_{n,\beta}$ as follows.

Theorem 4.3. *Suppose* $B = k\omega$ *for* $k \in \mathbb{R}$*. Then*

$$L_{n,\beta}(B,\omega) = P_{n,\beta}, \quad (k \ll 0), \quad L_{n,\beta}(B,\omega) = P_{-n,\beta}, \quad (k \gg 0).$$

5 Relationship to the rationality conjecture

In [26], Pandharipande and Thomas proposed the following conjecture, called the rationality conjecture.

Conjecture 5.1. *The generating series*

$$P_{\beta}(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n$$

is the Laurent expansion of a rational function of q, invariant under $q \leftrightarrow 1/q$.

In the paper [31], the author stated that the wall-crossing phenomena of the invariant $L_{n,\beta}(B,\omega)$ should be relevant to the above conjecture. In [30], the rationality of the closely related series,

$$P_{\beta}^{eu}(q) = \sum_{n \in \mathbb{Z}} e(P_n(X,\beta))$$

was proved, by using the wall-crossing formula of the motivic invariants, something like the topological euler number $e(\mathcal{M}_{n,\beta}(X,\beta))$. More precisely, we considered a coarse version of limit stability, called μ-limit stability, and showed the existence of the algebraic moduli stack of μ-limit semistable $E \in \mathcal{A}^p$ satisfying (2),

$$\mathfrak{M}_n^{(B,\omega)}(X,\beta),$$

as an algebraic stack of finite type. Then we constructed the "euler number" of the moduli stack

$$J_{n,\beta}(B,\omega) = \text{“}e(\mathfrak{M}_n^{(B,\omega)}(X,\beta))\text{”} \in \mathbb{Q}.$$

One might guess that the euler number of the algebraic stack is obtained as taking the quotient of the euler number of the stabilizer groups. However this does not make sense, since the euler number of the stabilizer groups might be zero. Instead, Joyce [17] proposes another construction, using Hall algebras associated to abelian categories. The invariant $J_{n,\beta}(B,\omega)$ is constructed along with the same argument of Joyce's work [17]. As in Theorem 4.3, the invariant $J_{n,\beta}(B,\omega)$ satisfies,

$$J_{n,\beta}(k\omega,\omega) = e(P_n(X,\beta)), \quad (k \ll 0),$$
$$J_{n,\beta}(k\omega,\omega) = e(P_{-n}(X,\beta)), \quad (k \gg 0).$$

We set

$$L_{n,\beta} = J_{n,\beta}(0,\omega) \in \mathbb{Q}.$$

Let $\tilde{\mathfrak{M}}_n(X,\beta)$ be the moduli stack of one dimensional ω-Gieseker semistable sheaves $E \in \mathrm{Coh}(X)$, which satisfies $(\mathrm{ch}_2(E), \mathrm{ch}_3(E)) = (\beta, n)$. We denote by $N_{n,\beta}$ the Joyce-type invariant,

$$N_{n,\beta} = \text{“}e(\tilde{\mathfrak{M}}_n(X,\beta))\text{”} \in \mathbb{Q}.$$

Let us introduce the following generating series,

$$L_{\beta}(q) = \sum_{n \in \mathbb{Z}} L_{n,\beta} q^n, \quad N_{\beta}(q) = \sum_{n \in \mathbb{Z}} n N_{n,\beta} q^n.$$

Using the arguments of wall-crossing formula developed by Joyce [17], we showed the following in [30, Theorem 4.7].

Theorem 5.2. *We have the following equality of the generating series,*

$$\sum_{\beta} P_{\beta}(q)v^{\beta} = \left(\sum_{\beta} L_{\beta}(q)v^{\beta}\right)\left(\sum_{\beta} N_{\beta}(q)v^{\beta}\right).$$

It can be shown that $L_{\beta}(q)$ is a polynomial of $q^{\pm 1}$, and $N_{\beta}(q)$ is a rational function of q. Furthermore they are both invariant under $q \leftrightarrow 1/q$. As a corollary we have the following. (See [30, Corollary 5.3].)

Corollary 5.3. *The generating series $P_{\beta}(q)$ is the Laurent expansion of a rational function of q, invariant under $q \leftrightarrow 1/q$.*

It should be possible to show Conjecture 2.4 as soon as we can involve Behrend's constructible function into the proof of Theorem 5.2. Such technical issue is now under progress by the work of Kontsevich-Soibelman [20].

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