多重標準形式の拡張とその応用

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The purpose is to review the recent development of the extension results of twisted pluricanonical forms and its applications. This text is for non-experts in this topic and hence proofs will not be provided.

1. Invariance of Plurigenera

Let us start with recalling basic notions. We let X be a smooth complex projective variety or a compact complex manifold, and denote by K_X a canonical divisor, or the canonical line bundle.

$$P_m(X) := \dim H^0(X, \mathcal{O}_X(mK_X))$$

is the m-genus, or m th-plurigenus ($m > 0$), which is a birational discrete invariant. The growth order of $P_m(X)$ is called the "Kodaira dimension" $\kappa(X)$ of X, i.e.,

$$P_m(X) \sim m^{\kappa(X)}$$

for every large and enough divisible m. In case $P_m(X) = 0$ for all $m > 0$, we understand as $\kappa(X) = -\infty$. It is known $\kappa(X) \in \{-\infty, 0, 1, \ldots, n = \dim X\}$.
The X is called "of general type"
$\Longleftrightarrow$ $\kappa(X) = n = \dim X$. Then, "the volume of K_X" is defined to be

$$\mathrm{vol}\,(K_X) := \limsup_{m\to\infty} \frac{P_m(X)}{m^n/n!} > 0, \qquad P_m(X) = \frac{\mathrm{vol}\,(K_X)}{n!} m^n + o(m^n).$$

$\Longleftrightarrow$ There exists $m > 0$ such that the rational map associated to the pluricanonical system $|mK_X|$ gives a birational map $\Phi_{|mK_X|} : X \dashrightarrow \mathbb{P} = \mathbb{P}^{P_m(X)-1}$ onto its image.

In this section, we will discuss in the following situation.

Basic set up in §1.

X : a normal variety, $(C, 0)$: a germ of a smooth curve,
$\pi : X \longrightarrow C$: a projective, surjective morphism with connected fibers,
$X_t = \pi^{-1}(t)$ the fiber of $t \in C$ (as a divisor). □

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If $\pi : X \longrightarrow C$ is smooth, for every fixed m, $P_m(X_t) = h^0(X_t, mK_X|_{X_t})$ is upper semi-continuous as a fucntion of $t \in C$ (in Zariski topology), i.e., $P_m(X_t)$ may only jump up at some special points. The Hodge theory in fact shows that $P_1(X_t) = \dim \mathcal{H}^{n,0}(X_t, \mathbb{C})$ is independent of $t \in C$. The problem of the invariance of plurigenera is to ask the parameter independence of $P_m(X_t)$ for all $m > 0$. This was observed by Nakayama [N1] assuming the minimal model program (MMP).

The breakthrough was brought by Siu in case fibers are of general type.

Theorem 1.1. *Siu* [S1]. *Assume $\pi : X \longrightarrow C$ is smooth, and every X_t is of general type. Then $P_m(X_t)$ is independent of $t \in C$ for any m.*

More precisely, every $\sigma \in H^0(X_t, mK_{X_t})$ extends $\widetilde{\sigma} \in H^0(X, mK_X)$. This is the so-called an "analytic proof". After Siu, an "algebraic proof" is given, and applied to the deformation theory of certain type of singularities which appear in MMP. Refer [KMM] for the definitions of terminal and canonical singularities.

Theorem 1.2. *Kawamata* [K1].

(1) If X_0 has canonical singularities at most, then so does X_t general.

(2) All X_t are of general type and have canonical singularities at most, then $P_m(X_t)$ is independent of $t \in C$.

Theorem 1.3. *Nakayama* [N2].

(1) If X_0 has terminal singularities at most, then so does X_t general.

(2) If $\pi : X \longrightarrow C$ is smooth and the "abundance conjecture" holds true for general X_t, then $P_m(X_t)$ is independent of $t \in C$.

Here the plurigenera for possibly singular varieties is defined as $P_m(X_t) := P_m(\widetilde{X}_t)$ for a smooth model $\widetilde{X}_t$ of X_t.

The general case is also solved by Siu. That means, without assuming X_t is of general type nor the "abundance conjecture", we have

Theorem 1.4. *Siu* [S2]. *Assume $\pi : X \longrightarrow C$ is smooth. Then $P_m(X_t)$ is independent of $t \in C$.*

In case $\pi : X \longrightarrow C$ may not be smooth, we also have

Theorem 1.5. *T-* [T1].

(1) All $X_t = \pi^{-1}(t)$ have canonical singularities at most, then $P_m(X_t)$ is independent of $t \in C$.

(2) Let X_0 be a special fiber with $\mathrm{Supp}\,(X_0) = \sum_{i \in I} X_i$, *and let X_t be a general fiber. Then $\sum_{i \in I} P_m(X_i) \leq P_m(X_t)$.*

The second statement (2) is the "lower semi-continuity" of plurigenera, which is conjectured by Nakayama [N1]. If the singularities of X_t are mild, this "lower semi-continuity" plus the usual "upper semi-continuity" imply (1).

In Siu's proof, there is a pair of inductions, which he calls "twin tower proof". Păun simplified and generalized Siu's proof with only one induction; "one tower proof". The following form is the most general extension statement for the moment.

Theorem 1.6. *Păun* [P]. *Assume $\pi : X \longrightarrow C$ is smooth. Let L be a holomorphic line bundle on X with a singular Hermitian metric h such that the curvature is semi-positive, and $h|_{X_0}$ is well-defined. Then $\sigma \in H^0(X_0, K_{X_0}^{\otimes m} \otimes L \otimes \mathcal{I}(h|_{X_0}))$ extends $\widetilde{\sigma} \in H^0(X, K_X^{\otimes m} \otimes L \otimes \mathcal{I}(h))$. Here $\mathcal{I}(h) \subset \mathcal{O}_X$ is the "multiplier ideal sheaf" of h.*

Here are some well-known open problems.

Problem 1.7. (1) Give an "algebraic proof" of Theorem 1.4, i.e., the invariance of $P_m(X_t)$ in case X_t is not necessarily of general type.

(2) Give the "Kähler version" of the invariance of $P_m(X_t)$. In the proof of theorems quoted above, the projectivity or algebraicity of the map $\pi : X \longrightarrow C$ is crucial, especially in the first step of the induction. It is well-known that there is a counter-example without Kählerian assumption, by Iku Nakamura.

(3) "Log-version". Assume $\pi : X \longrightarrow C$ is smooth, and let Δ be an effective ($\mathbb{Q}$-)divisor with simple normal crossing support on X. Find a meaningful statement on the behavior of, like, $h^0(X_t, mK_{X_t} + \Delta_t)$, $h^0(X_t, m(K_{X_t} + \Delta_t))$, $h^0(X_t, mK_{X_t} + (m-1)\Delta_t)$. These kind of extensions are technically important, but it is sometime difficult to see what the right statement will be. For example the invariance of $h^0(X_t, m(K_{X_t} + \Delta_t))$ is quite unlikely in case $K_X = 0$.

2. Material for Proof

We recall basic notions and theorems, which are used in the proof of the results in §1.

Definition 2.1. Let X be a smooth variety, and let D be an effective $\mathbb{Q}$-divisor on X.

(1) Let $\mu : X' \longrightarrow X$ be a log-resolution of D so that $\mathrm{Supp}\,(\mu^*D + \mathrm{Exc}(\mu))$ becomes simple normal crossing. Then the "multiplier ideal sheaf" of D is

$$\mathcal{J}(D) = \mathcal{J}(X, D) := \mu_*\mathcal{O}_{X'}(K_{X'/X} - \llcorner \mu^*D \lrcorner) \subset \mathcal{O}_X.$$

(2) The pair (X, D) is called klt, "Kawamata log-terminal" $\Longleftrightarrow \mathcal{J}(X, D) = \mathcal{O}_X$.

(3) The pair (X, D) is called lc, "log-canonical" $\Longleftrightarrow \mathcal{J}(X, (1-\varepsilon)D) = \mathcal{O}_X$ for any $0 < \varepsilon < 1$.

For example, for $D=\sum d_jD_j$ an effective $\mathbb{Q}$-divisor with simple normal crossing, (X,D) is klt $\Longleftrightarrow 0\le d_j<1$ for any j, (X,D) is lc $\Longleftrightarrow 0\le d_j\leqq 1$ for any j.

(4) The "non-klt locus" of the pair (X,D) is $\mathrm{Nklt}\,(X,D)=\mathrm{Supp}\,\mathcal{O}_X/\mathcal{J}(D)$.

Theorem 2.2. *Nadel, Kawamata-Viehweg vanishing (*[L, 9.4.8]*). Let X be a smooth projective variety, D an effective $\mathbb{Q}$-divisor, and L a divisor on X. Assume $L-D$ is ample. Then $H^q(X,\mathcal{O}_X(K_X+L)\otimes\mathcal{I}(D))=0$ for $q>0$.*

In case $D=0$, this is Kodaira's vanishing, which was used to construct global sections of some power of positive line bundles.

Theorem 2.3. *Effective global generation of multiplier ideal sheaves (*[L, 9.4.26]*). Let X,D,L be as above with $L-D$ ample. Let H be a very ample divisor. Then $\mathcal{O}_X(K_X+L+mH)\otimes\mathcal{J}(D)$ is generated by global sections, for any $m\ge\dim X$.*

The point is that m is independent from L and D.

In the analytic proof, the following extension with L^2-estimate is very important, and which is the difference between the analytic proof and the algebraic one.

Theorem 2.4. *Ohsawa-Takegoshi L^2-extension* [OT]. *Let $X\subset\mathbb{C}^n$ be a bounded pseudoconvex domain, $\varphi\in PSH(X)$ a plurisubharmonic function on X, and $S=X\cap\{z_n=0\}$. Then for any $f\in H^0(S,\mathcal{O})$ with $\int_S|f|^2e^{-\varphi}dV_{n-1}<\infty$, there exists $F\in H^0(X,\mathcal{O})$ such that $F|_S=f$ and*

$$\int_X|F|^2e^{-\varphi}dV_n\le C_{diamX}\int_S|f|^2e^{-\varphi}dV_{n-1}.$$

Here C_{diamX} is a positive constant depending only on the diameter of X, independent from φ and f.

There is also a manifold version of this theorem, for example, for $\pi:X\longrightarrow(C,0)$ as in §1 and π is smooth, and L is a holomorphic line bundle on X with a singular Hermitian metric h such that the curvature is semi-positive and that $h|_{X_0}$ is well-defined. Then $\sigma\in H^0(X_0,K_{X_0}\otimes L\otimes\mathcal{I}(h|_{X_0}))$ extends $\widetilde{\sigma}\in H^0(X,K_X\otimes L\otimes\mathcal{I}(h))$ with L^2-estimate. Here the smoothness assumption of $\pi:X\longrightarrow(C,0)$ is important.

Naive explanation of the invariance of $P_m(X_t)$. Let $\pi:X\longrightarrow(C,0)$ be as in §1. We take $m\in\mathbb{N}$ and H a sufficiently π-ample divisor on X. By induction on p of pK_X+H, we find a good decomposition

$$L:=(\ell m-1)K_X+H=A+D$$

so that $H^0(X,\ell mK_X+H)\fallingdotseq H^0(X,K_X\otimes L\otimes\mathcal{I}(D))$, $H^0(X_0,K_{X_0}\otimes L\otimes\mathcal{I}(D|_{X_0}))\fallingdotseq H^0(X_0,\ell mK_{X_0}+H)$, and that $H^0(X,K_X\otimes L\otimes\mathcal{I}(D))\longrightarrow H^0(X_0,K_{X_0}\otimes L\otimes\mathcal{I}(D|_{X_0}))$ is surjective, for all $\ell>0$.

Because H is fixed and ℓ is arbitrary, the extension statement for $\ell m K_X + H$ may leads to the similar extension statement for $mK_X + \frac{1}{\ell}H$. If K_X is π-big (in case Theorem 1.1), mK_X absorbs $\frac{1}{\ell}H$ part. If K_X is only π-pseudo-effective (in case Theorem 1.4), we prove the surjection or extension with uniform L^2-estimates, then pass to the limit $\ell \to \infty$: $mK_X + \frac{1}{\ell}H \rightsquigarrow mK_X$. Then we obtain a surjectioin $H^0(X, mK_X) \longrightarrow H^0(X_0, mK_{X_0})$. $\square$

3. Extension of Twisted Pluricanonical Forms

So far, we discussed on a family of varieties. Here we discuss extension properties on a compact variety. We will mention their applications in the next section.

Theorem 3.1. *T-* [T2]. *Let X be a smooth projective variety, and $S \subset X$ a smooth hypersurface. Let L be an integral divisor on X such that $L \sim_{\mathbb{Q}} A + D$ with A an ample $\mathbb{Q}$-divisor, and D an effective $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp} D$ and $(S, D|_S)$ is klt. Then the natural map*

$$H^0(X, m(K_X + S + L)) \longrightarrow H^0(S, m(K_S + L|_S))$$

is surjective for any $m > 0$.

Recall that a divisor L is called "big"

$\iff$ $\Phi_{|mL|} : X \dashrightarrow \mathbb{P}$ is birational onto its image for a large m.

$\iff$ $L \sim_{\mathbb{Q}} A + D$ into A an ample $\mathbb{Q}$-divisor and D an effective $\mathbb{Q}$-divisor.

Theorem 3.2. *Hacon-McKernan* [HM1]. *Let X be a smooth projective variety, $S \subset X$ be a smooth hypersurface. Let L be an effective $\mathbb{Q}$-divisor (not necessarily integral) such that $L \sim_{\mathbb{Q}} A + D$ with A an ample $\mathbb{Q}$-divisor, and D an effective $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp} D$ and $(S, D|_S)$ is klt. Assume there exists $m \in \mathbb{N}$ and an integral divisor M such that $M \sim_{\mathbb{Q}} m(K_X + S + L)$, and that for some $p \in \mathbb{N}$, the base locus of $|pM|$ does not contain any stratum of $S + \ulcorner L \urcorner$ plus a mild condition (that is $(X, S + L)$ is log-smooth and $\lfloor L \rfloor = 0$ so that $(X, S + L)$ is plt). Then the natural map*

$$H^0(X, m(K_X + S + L)) \longrightarrow H^0(S, m(K_S + L|_S))$$

is surjective for such m.

These two extension theorems can be proved by a similar mannar as in §1 and §2. Extensions from higher codimensional subvarieties are also possible.

Theorem 3.3. *T-* [T2]. *Let L be an integral divisor, and $V \subset X$ a subvareity. Assume there is a decomposition $L \sim_{\mathbb{Q}} A + D$ into A an ample $\mathbb{Q}$-divisor and D an effective $\mathbb{Q}$-divisor such that V is a maximal lc center for (X, D). Then*

$$\operatorname{vol}_{X|V}(K_X + L) \geq \operatorname{vol}(K_{\widetilde{V}}).$$

Here $\widetilde{V}$ is any smooth model of V.

Definition 3.4. Let $V \subset X$ be a subvariety of $\dim V = d$.

(1) $V \subset X$ is a "maximal lc center" for (X, D), if V is an irreducible component of $\mathrm{Nklt}\,(X, D)$, and $V \not\subset \mathrm{Nklt}\,(X, (1-\varepsilon)D)$.

(2) We denote by $H^0(X|V, mL) := \mathrm{Image}[H^0(X, mL) \longrightarrow H^0(V, mL)]$. The "restricted volume" of L along V is

$$\mathrm{vol}_{X|V}(L) := \limsup_{m\to\infty} \frac{h^0(X|V, mL)}{m^d/d!}.$$

In case L is ample, we have $\mathrm{vol}_{X|V}(L) = L^d \cdot V$. □

The (usual) "volume" of L is defined by

$$\mathrm{vol}_X(L) := \limsup_{m\to\infty} \frac{h^0(X, mL)}{m^n/n!},$$

where $n = \dim X$. Then L is big if and only if $\mathrm{vol}_X(L) > 0$.

4. Applications

§4.1. Boundedness of pluricanonical maps.

Recall that X is of general type,

$\Longleftrightarrow \Phi_{|mK_X|} : X \dashrightarrow \mathbb{P}$ is birational onto its image for large m,

$\Longleftrightarrow \mathrm{vol}\,(K_X) := \limsup_{m\to\infty} \dfrac{P_m(X)}{m^n/n!} > 0$, where $n = \dim X$.

The following theorems guarantee that there exist lower bounds, depending only on the dimension.

Theorem 4.1. *Hacon-McKernan* [HM1]*, Tsuji* [Ts]*, T-* [T2]*. Fix $n \in \mathbb{N}$. Then there exists $m_n \in \mathbb{N}$ depending only on n such that $\Phi_{|mK_X|} : X \dashrightarrow \mathbb{P}$ is birational onto its image, for any n-dimensional smooth projective variety X of general type, and for any $m \geq m_n$.*

Theorem 4.2. *Hacon-McKernan* [HM1]*, Tsuji* [Ts]*, T-* [T2]*. Fix $n \in \mathbb{N}$. Then there exists $v_n > 0$ depending only on n such that*

$$\mathrm{vol}\,(K_X) \geq v_n$$

for any n-dimensional smooth projective variety X of general type.

The basic strategy for the proof is induction on $n = \dim X$ for both Theorem 4.1 and 4.2, to apply the extension statements in §3. For a big divisor $L := (m-1)K_X$, we try to find a decomposition $L \sim_{\mathbb{Q}} A + D$ into A an ample $\mathbb{Q}$-divisor and D an effective $\mathbb{Q}$-divisor, such that $V \subset X$ is a maximal lc center for (X, D). Then by Theorem 3.3,

$$\mathrm{vol}_{X|V}(K_X + L) \geq \mathrm{vol}\,(K_{\widetilde{V}}).$$

Induction hypothesis implies that $m^d \mathrm{vol}_{X|V}(K_X) \geq v_d$ with $d = \dim V < n$. We need to mind the following:

(i) How to find m (smaller is better),

(ii) What is V, and how to construct $L \sim_{\mathbb{Q}} A + D$,

(iii) Estimate on $\mathrm{vol}_{X|V}(K_X) \Longrightarrow$ Estimate on $\mathrm{vol}\,(K_X)$.

§4.2. Shokurov's rationally chain connectedness conjecture.

In §4.2–§4.4, we will discuss uniruledness of subvarieties in special position. A variety V is called "uniruled", if there exists a dominant map : $\mathbb{P}^1 \times W^{d-1} \dashrightarrow V^d$. The following is a useful uniruledness criterion.

Theorem 4.3. *Miyaoka-Mori* [MM]*, Boucksom-Demailly-Păun-Peternell* [BDPP]*. A smooth projective variety X is uniruled if and only if K_X is not pseudo-effective.*

The pseudo-effectivity of the canonical bundle of a subvariety plus the extension statements in §3 imply that the existence of certain twisted pluricanonical forms on the total space. Then we see, the subvariety may not be in the base locus of the twisted pluricanonical system. Thus the subvariety may be in general position if the canonical bundle is pseudo-effective.

Theorem 4.4. *Hacon-McKernan* [HM2]*. Let (X, Δ) be a dlt pair, and $f : Y \longrightarrow X$ be a birational morphism from a normal variety Y. Then any fiber $f^{-1}(x)$ is rationally chain connected.*

We do not define "dlt", but we only mention the relations: klt $\Longrightarrow$ dlt $\Longrightarrow$ lc.

§4.3. Uniruledness of stable base locus.

Conjecture 4.5. *(Ueno, ..., around '75) Let X be a smooth projective variety of general type. Then every irreducible component V of the stable base locus of K_X has negative Kodaira dimension; $\kappa(V) = -\infty$. In terms of MMP, V should be uniruled.*

For a divisor L, the "stable base locus" is $\mathrm{SBs}\,(L) = \bigcap_{m>0} \mathrm{Bs}\,|mL|$.

Theorem 4.6. *T-* [T3]*. Let X be a smooth projective variety.*

(1) Assume K_X is big. Then every irreducible compoment of $\mathrm{SBs}\,(K_X)$ *is uniruled.*

(2) Assume $K_X = 0$. Let L be a big divisor on X. Then every irreducible component of $\mathrm{SBs}\,(L)$ *is uniruled.*

Outline of proof. Let L be a big divisor. Assume (i) $V \subset \mathrm{SBs}\,(L)$ is an irreducible component, and (ii) $V \subset \mathrm{SBs}\,(K_X + aL)$ for some or any large $a \in \mathbb{N}$. We see (i) $\Longrightarrow$ (ii), if $K_X = 0$, or if $L = K_X$ is big.

By (i), we can find an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} bL$ for some $b \in \mathbb{Q}_{>0}$ and V is a maximal lc center for (X, D). By Kawamata's subadjunction, we have $(K_X + D)|_V \succeq K_V$. Then we apply the extension statements in §3 to obtain "almost surjection":

$$H^0(X, m(K_X + D)) \ \cdots\twoheadrightarrow\ H^0(V, mK_V)$$

Then, this "surjection" plus (ii) imply $H^0(V, mK_V) = 0$ for all sufficiently large m, i.e., $\kappa(V) = -\infty$. A refinement of this argument plus Theorem 4.3 imply that V is uniruled. □

Question 4.7. (Tomari) Configuration of $\mathrm{NAmp}\,(K_X)$ in case K_X is (nef-) big??

For a divisor L, the "non-ample locus" is $\mathrm{NAmp}\,(L) = \bigcap_{m>0} \mathrm{SBs}\,(mL - A)$ for any given ample divisor A.

§4.4. Degeneration of algebraic varieties with trivial canonical divisor.

Let X be a normal variety with only canonical singularities, $(C, 0)$ a germ of a smooth curve, and $\pi : X \longrightarrow C$ a projective surjective morphism with connected fibers. Let $X_0 = \sum_{i \in I} m_i F_i$ be the irreducible decomposition. Assume $K_{X_t} \sim_{\mathbb{Q}} 0$ for general X_t. Then we can see that $K_X \sim_{\mathbb{Q}} \sum_{j \in J} r_j F_j$ uniquely, for a subset $J \subset I$ with $0 \le |J| < |I|$ and $r_j \in \mathbb{Q}_{>0}$ for $j \in J$. Note that we do not require in general that X is smooth, nor $\pi : X \to C$ is relatively minimal, nor a semi-stable degeneration.

Theorem 4.8. *T-* [T4]. *(0) Every F_i is either uniruled, or $\kappa(F_i) = 0$. In addition, there exists at most one F_i with $\kappa(F_i) = 0$.*

(1) Case $|I| = 1$; $X_0 = m_1 F_1$.

(1.1) F_1 is non-normal, or F_1 contains a codim 2 singular locus of X, then F_1 is uniruled.

(1.2) Assume F_1 is normal, and F_1 does not contain any codim 2 singular loci of X (then $K_{F_1} \sim_{\mathbb{Q}} 0$). Then, F_1 is uniruled if and only if F_1 has a singularity worse than canonical.

(2) Case $|I| \ge 2$. If $J \cup \{i\} \ne I$, then F_i is uniruled. In particular

(2.1) K_X is supported by uniruled divisors.

(2.2) If a component of X_0, say F_1 is not uniruled, then $J = I \setminus \{1\}$.

(2.3) If $\pi : X \longrightarrow C$ is relatively minimal, i.e., K_X is π-nef (then $K_X \sim_{\mathbb{Q}} 0$), then all F_i are uniruled.

References

[BDPP] Boucksom S., Demailly J.-P., Păun M. and Peternell Th., *The pseudo-effective cone of a compact Kähler manifold and varieties of non-negative Kodaira dimension*, arXiv:math.AG/0405285.

[HM1] Hacon C. and M^{c}Kernan J., *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006) 1–25.

[HM2] Hacon C. and M^{c}Kernan J., *Shokurov's rational connectedness conjecture*, Duke. Math. J. **138** (2007) 119–136.

[K1] Kawamata Y., *Deformations of canonical singularities*, J. Amer. Math. Soc. **12** (1999) 85–92.

[K2] Kawamata Y., *On the extension problem of pluricanonical forms*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 193–207, Contemp. Math. **241**, Amer. Math. Soc., Providence, RI, 1999.

[KMM] Kawamata Y., Matsuda K. and Matsuki K., *Introduction to the minimal model problem*, Algebraic Geometry (Sendai, 1985), Advanced Studies in Pure Math. **10** (1987), North-Holland, 283–360.

[L] Lazarsfeld R., *Positivity in algebraic geometry I, II*, Ergebnisse der Math. und ihrer Grenzgebiete (3), **48-49**, Springer-Verlag, Berlin, 2004.

[MM] Miyaoka Y. and Mori S., *A numerical criterion of uniruledness*, Ann. Math. **124** (1986) 65–69.

[N1] Nakayama N., *Invariance of the plurigenera of algebraic varieties under minimal model conjectures*, Topology **25** (1986) 237–251.

[N2] Nakayama N., *Zariski-decomposition and abundance*, MSJ Memoirs **14**, Math. Soc. Japan, 2004.

[OT] Ohsawa T. and Takegoshi K., *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987) 197–204.

[P] Păun M., *Siu's invariance of plurigenera: a one tower proof*, J. Differential Geom. **76** (2007) 485–493.

[S1] Siu Y.-T., *Invariance of plurigenera*, Invent. Math. **134** (1998) 661–673.

[S2] Siu Y.-T., *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semi-positively twisted plurigenera for manifolds not necessarily of general type*, Complex Geometry (Göttingen, 2000), I. Bauer eds., (2002), Springer, Berline, 223–277.

[T1] Takayama S., *On the invariance and the lower semi-continuity of plurigenera of algebraic varieties*, J. Algebraic Geom. **16** (2006) 1-18.

[T2] Takayama S., *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. **165** (2006) 551–587.

[T3] Takayama S., *On the uniruledness of stable base loci*, J. Differential Geom. **78** (2008) 521–541.

[T4] Takayama S., *On uniruled degenerations of algebraic varieties with trivial canonical divisor*, Math. Z. **259** (2008) 487–501.

[Ts] Tsuji H., *Pluricanonical systems on projective varieties of general type II*, Osaka J. Math. **44** (2007) 723–764.

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