Generic Newton polygons of Ekedahl-Oort strata: Oort's conjecture

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Abstract

We study the moduli space of principally polarized abelian varieties in positive characteristic. We determine the Newton polygon of any generic point of each Ekedahl-Oort stratum, by proving Oort's conjecture on intersections of Newton polygon strata and Ekedahl-Oort strata. This result tells us a combinatorial algorithm determining the optimal upper bound of the Newton polygons of principally polarized abelian varieties with given isomorphism type of p-kernel. We expect that "the generic parts" of Ekedahl-Oort strata can be beautifully described. In the last section we confirm the expectation for the supersingular Ekedahl-Oort strata.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. For an abelian variety A over k, we have two objects:

the p-divisible group: $A[p^\infty] = \varinjlim_i \mathrm{Ker}(p^i : A \to A)$ and

the p-kernel: $A[p] = \mathrm{Ker}(p : A \to A)$.

By the Dieudonné-Manin classification, the isogeny classes of p-divisible groups are classified by Newton polygons (cf. §2.2). On the other hand, the set of the isomorphism classes of $A[p]$ are naturally identified with a subset ${}^I W_g$ of the Weyl group W_g of Sp_{2g} (cf. §2.3). An element of ${}^I W_g$ is called a *final element* of W_g.

The following question is still open in general:

For each $w \in {}^I W_g$, which Newton polygons can occur as the Newton polygons $\mathcal{N}(A)$ of principally polarized abelian varieties (A, η) with $A[p]_{/\simeq} = w$?

A purpose of this note is to give a combinatorial algorithm determining the optimal upper bound $b(w)$ of such Newton polygons. The precise definition of $b(w)$ is as follows: any (A, η) with $A[p]_{/\simeq} = w$ satisfies $\mathcal{N}(A) \prec b(w)$ and there exists (A', η') satisfying $A'[p]_{/\simeq} = w$ and $\mathcal{N}(A') = b(w)$. We shall explain below the non-trivial fact that $b(w)$ exists.

For the purpose above, we investigate some stratifications and foliations on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g in characteristic p. Let ξ be a symmetric Newton polygon. We write $\mathcal{W}^0_\xi$ for the open Newton polygon stratum (cf. §2.2)

$$\mathcal{W}^0_\xi = \{(A, \eta) \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\}.$$

For $w \in {}^I W_g$, let $\mathcal{S}_w$ be the Ekedahl-Oort stratum:

$$\mathcal{S}_w = \{(A, \eta) \in \mathcal{A}_g \mid A[p]_{/\simeq} = w\}$$

(see §2.3 for details). Oort proved $\mathcal{S}_w \neq \emptyset$ for every $w \in {}^I W$, and Ekedahl and van der Geer proved that $\mathcal{S}_w$ is irreducible if $\mathcal{S}_w$ is not contained in the supersingular locus $\mathcal{W}^0_\sigma$ (see §2.3). Let $\xi(w)$ denote the Newton polygon of the generic point of $\mathcal{S}_w$ if $\mathcal{S}_w$ is not contained in the supersingular locus and otherwise the supersingular Newton polygon. We call $\xi(w)$ the *generic Newton polygon* of $\mathcal{S}_w$. Since the Newton polygon goes down (or stays) w.r.t. $\prec$ under any specialization (Grothendieck-Katz [16], Th. 2.3.1 on p. 143), we see that $\xi(w)$ fulfills the conditions defining $b(w)$; thus $b(w)$ exists and $\xi(w) = b(w)$.

Let $H(\xi)$ denote the minimal p-divisible group of Newton polygon ξ, and let $\mathcal{Z}_\xi$ be the the central stream (cf. §2.4):

$$\mathcal{Z}_\xi = \{(A, \eta) \in \mathcal{A}_g \mid A[p^\infty]_\Omega \simeq H(\xi)_\Omega \text{ for some alg. closed field } \Omega\}.$$

Let $\overline{\mathcal{S}_w}$ denote the Zariski closure of $\mathcal{S}_w$ in $\mathcal{A}_g$. Our main result is

Main theorem. *For any final element w of W_g, we have $\mathcal{Z}_{\xi(w)} \subset \overline{\mathcal{S}_w}$.*

The main theorem is closely related to [23], (6.9):

Oort's conjecture. *If $\mathcal{W}^0_\xi \cap \mathcal{S}_w \neq \emptyset$, then $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$.*

Indeed in [10, Cor. 3.7] it was proved that the main theorem and the conjecture are equivalent (see Cor. 2.4.5 below). Thus we obtain

Corollary I. *Oort's conjecture is true.*

Here is another corollary (see §3 for the proof).

Corollary II. $\xi(w)$ *is the biggest element of the set* $\{\ \xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\ \}$ *with respect to* $\prec$.

This is a generalization of [8]. From this we have a purely combinatorial algorithm determining the generic Newton polygon $\xi(w)$, see Rem. 3.0.7 for the detail. Please see a beautiful similarity between Cor. II and the result [6], Th. 5.4.11 of Goren and Oort in the case of Hilbert modular varieties over inert primes.

Now we expect that the generic part $\mathcal{S}_w^0 := \mathcal{S}_w \cap \mathcal{W}^0_{\xi(w)}$ of $\mathcal{S}_w$ for every $w \in {}^I W$ can be beautifully described. In the last section we confirm this expectation in the case that $\mathcal{S}_w$ is contained in the supersingular locus $\mathcal{W}_\sigma$. More precisely, we describe some union of $\mathcal{S}_w$'s contained in the supersingular locus in terms of Deligne-Lusztig varieties.

2 Stratifications

In this section we collect some fundamental facts on the Newton polygon stratification, the Ekedahl-Oort stratification and the central streams.

2.1 The Dieudonné theory

Let K be a perfect field of characteristic p and $W(K)$ the ring of Witt vectors with coordinates in K. Let A_K be the p-adic completion of the associative ring

$$W(K)[\mathcal{F},\mathcal{V}]/(\mathcal{F}x - x^\rho\mathcal{F}, \mathcal{V}x^\rho - x\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \forall x \in W(K))$$

with the Frobenius automorphism ρ of $W(K)$. A *Dieudonné module over* $W(K)$ is a left A_K-module which is finitely generated as a $W(K)$-module. There is a canonical categorical equivalence $\mathbb{D}$ (the covariant Dieudonné functor) from the category of p-torsion finite commutative group schemes (resp. p-divisible groups) over K to the category of Dieudonné modules over $W(K)$ which are of finite length (resp. free as $W(K)$-modules). We have $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$ for the Frobenius F and the Verschiebung V on finite commutative group schemes (resp. p-divisible groups).

2.2 The Newton polygon stratification

A pair (m, n) of non-negative integers with $\gcd(m, n) = 1$ is called a *segment*. For a series of segments (m_i, n_i) $(i = 1, \dots, \mathfrak{t})$ satisfying $\lambda_1 \leq \cdots \leq \lambda_{\mathfrak{t}}$ with $\lambda_i = m_i/(m_i + n_i)$, putting $P_j := (\sum_{i=1}^{j}(m_i + n_i), \sum_{i=1}^{j} m_i) \in \mathbb{R}^2$ for $0 \leq j \leq \mathfrak{t}$, we denote by $\sum_{i=1}^{\mathfrak{t}}(m_i, n_i)$ the line graph in $\mathbb{R}^2$ passing through $P_0, \dots, P_{\mathfrak{t}}$ in this order. We call such a line graph a *Newton polygon.* We say, for two Newton polygons ξ, ξ' with the same end point, that $\xi' \prec \xi$ if no point of ξ is below ξ'. A Newton polygon $\sum_{i=1}^{\mathfrak{t}}(m_i, n_i)$ is said to be *symmetric* if $\lambda_i + \lambda_{\mathfrak{t}+1-i} = 1$ for all $i = 1, \dots, \mathfrak{t}$. The symmetric Newton polygon $\sum_{i=1}^{\mathfrak{t}}(1, 1)$ is called *supersingular.*

For a segment (m, n), we define a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(G_{m,n}) = E_{\mathbb{F}_p}/E_{\mathbb{F}_p}(\mathcal{F}^m - \mathcal{V}^n). \tag{1}$$

By the Dieudonné-Manin classification [19], for any p-divisible group X over a field K of characteristic p, there is an isogeny over an algebraically closed field Ω containing K from X to $\bigoplus_{i=1}^{\mathfrak{t}} G_{m_i,n_i}$ for some finite set $\{(m_i, n_i)\}$ of segments. Thus we get a Newton polygon $\sum_{i=1}^{\mathfrak{t}}(m_i, n_i)$, which is denoted by $\mathcal{N}(X)$. For an abelian variety A, we have its Newton polygon $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$. Note $\mathcal{N}(A)$ is symmetric.

For a symmetric Newton polygon ξ of height $2g$, we define its *NP-stratum* by

$$\mathcal{W}_\xi = \{(A, \eta) \in \mathcal{A}_g \,|\, \mathcal{N}(A) \prec \xi\}.$$

Grothendieck and Katz ([16], Th. 2.3.1 on p. 143) proved that $\mathcal{W}_\xi$ is closed in $\mathcal{A}_g$; we consider this is a closed subscheme by giving it the induced reduced scheme structure. We also define the *open NP-stratum* by

$$\mathcal{W}_\xi^0 = \{(A, \eta) \in \mathcal{A}_g \,|\, \mathcal{N}(A) = \xi\};$$

similarly we regard $\mathcal{W}_\xi^0$ as a locally closed subscheme of $\mathcal{A}_g$.

2.3 The Ekedahl-Oort stratification

The main reference for the EO-stratification is [22]. For a formulation in terms of Weyl groups, see [5], [20] and [21].

Definition 2.3.1. (1) A finite locally free commutative group scheme G over $\mathbb{F}_p$-scheme S is said to be a BT_1 over S if it is annihilated by p and $\mathrm{Im}(V : G^{(p)} \to G) = \mathrm{Ker}(F : G \to G^{(p)})$.

(2) Assume k is perfect. Let G be a BT_1 over k. A *polarization* on G a non-degenerate alternating pairing $\langle\ ,\ \rangle$ on $\mathbb{D}(G)$ satisfying $\langle \mathcal{F}x, y\rangle = \langle x, \mathcal{V}y\rangle^{\rho}$ for all $x, y \in \mathbb{D}(G)$. Such a pair $(G, \langle\ ,\ \rangle)$ is called a *polarized* BT_1.

The following classification of polarized BT_1's is due to Oort [22], (9.4) and Moonen-Wedhorn [21], (5.4), also see Moonen [20] for $p > 2$.

Theorem 2.3.2. *Let k be an algebraically closed field. There is a canonical bijection*

$$\mathcal{E}: \quad \{\textit{polarized } \mathrm{BT}_1 \ \textit{ over } k\} / \simeq \ \xrightarrow{\ \sim\ } {}^I W_g.$$

Remark 2.3.3. Instead of ${}^I W_g$, Oort used the set of elementary sequences. See below for the definition of elementary sequences. The above formulation in terms of Weyl groups is due to Moonen-Wedhorn.

One usually identifies W_g with

$$\{ w \in \mathrm{Aut}\{1, \ldots, 2g\} \mid w(i) + w(2g + 1 - i) = 2g + 1 \}. \qquad (2)$$

Let $\{s_1, \ldots, s_g\}$ be the set of simple reflections, where $s_i = (i, i+1) \cdot (2g - i, 2g + 1 - i)$ for $i < g$ and $s_g = (g, g + 1)$. Let $I = \{s_1, \ldots, s_{g-1}\}$ and let $W_{g,I}$ be the subgroup of W_g generated by elements of I. We denote by ${}^I W_g$ the set of $(I, \emptyset)$-reduced elements of W_g (cf. [2], Chap. IV, Ex. §1, 3); this is a set of representatives of $W_{g,I} \backslash W_g$. Recall ${}^I W_g$ can be written as

$${}^I W_g = \left\{ \ w \in W_g \ \middle| \ w^{-1}(1) < \cdots < w^{-1}(g) \right\}. \qquad (3)$$

For non-negative integer $c \leq g$, we put

$${}^I W_g^{[c]} = \left\{ w \in {}^I W_g \ \middle| \ w(i) = i, \ \forall \ i \leq g - c \right\}, \qquad (4)$$

and set ${}^I W_g^{(c)} = {}^I W_g^{[c]} - {}^I W_g^{[c-1]}$ with ${}^I W_g^{[-1]} = \emptyset$.

Definition 2.3.4. A *symmetric final sequence* of length $2g$ is a map

$$\psi: \quad \{0, \ldots, 2g\} \longrightarrow \{0, \ldots, g\}$$

such that $\psi(i-1) \leq \psi(i) \leq \psi(i-1) + 1$ for $1 \leq i \leq 2g$ with $\psi(0) = 0$ and $\psi(2g - i) = g - i + \psi(i)$.

For each element w of ${}^I W_g$, we define

$$\psi_w(i) = \sharp\{a \in \{1, \ldots, i\} \mid w(a) > g\}; \qquad (5)$$

then ψ_w becomes a symmetric final sequence. The map $w \mapsto \psi_w$ gives a bijection from ${}^I W_g$ to the set of symmetric final sequences of length $2g$. An *elementary sequence* of length g is the restriction of a symmetric final sequence of length $2g$ to $\{1, \dots, g\}$. Obviously we can recover the symmetric final sequence from its elementary sequence.

Lemma 2.3.5. *For $w \in {}^I W_g$, we have $w \in {}^I W_g^{[c]}$ if and only if $\psi_w(g-c) = 0$.*

Proof. If $w \in {}^I W_g^{[c]}$, then $w(i) = i$ for $i \leq g - c$ by definition; hence we obtain $\psi_w(g-c) = 0$ by (5). Conversely assume $\psi_w(g-c) = 0$. Then we have $w(i) \leq g$ for all $i \leq g - c$. Since $w \in {}^I W_g$, i.e., $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g)$, we have $w(i) = i$ for all $i \leq g - c$. $\square$

The map $\mathcal{E}$ in Theorem 2.3.2 is defined as follows. Let G be a polarized BT_1 and let N be its Dieudonné module. We define an operation $\mathcal{V}^{-1}$ on the set of Dieudonné submodules N' of N by $\mathcal{V}^{-1}N' := \mathcal{V}^{-1}(N' \cap \mathcal{V}(N))$, and inductively we define a Dieudonné submodule sN' of N for any word s of $\mathcal{F}$ and $\mathcal{V}^{-1}$. As shown in [22], (2.4) we have that there exists a unique $w \in {}^I W_g$ satisfying $\mathrm{rk}(\mathcal{F}sN) = \psi_w(\mathrm{rk}\, sN)$ and $\mathrm{rk}(\mathcal{V}^{-1}sN) = g + \mathrm{rk}\, sN - \psi_w(\mathrm{rk}\, sN)$ for any word s. Now $\mathcal{E}(G)$ is defined to be this w.

Let G be a polarized BT_1. Let $w = \mathcal{E}(G)$ and put $\psi := \psi_w$. Recall [22], (9.4) that the Dieudonné module $N = \mathbb{D}(G)$ of G can be described as follows:

$$N = \bigoplus_{i=1}^{2g} k b_i \tag{6}$$

with the $\mathcal{F}$ and $\mathcal{V}$-operations defined by

$$\mathcal{F}(b_i) := \begin{cases} b_{\psi(i)} & \text{if} \quad w(i) > g, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$\mathcal{V}(b_j) := \begin{cases} b_i & \text{if} \quad j = g + i - \psi(i) \text{ with } w(i) \leq g \text{ and } w(j) \leq g, \\ -b_i & \text{if} \quad j = g + i - \psi(i) \text{ with } w(i) \leq g \text{ and } w(j) > g, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and the polarization $\langle\ ,\ \rangle$ defined by

$$\langle b_i, b_{2g+1-j} \rangle = \begin{cases} 1 & \text{if} \quad i = j \quad \text{and} \quad w(i) > g, \\ -1 & \text{if} \quad i = j \quad \text{and} \quad w(i) \leq g, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{9}$$

For each $w \in {}^IW_g$ the Ekedahl-Oort stratum $\mathcal{S}_w$ is the subset of $\mathcal{A}_g$ defined by

$$\mathcal{S}_w = \{(A, \eta) \in \mathcal{A}_g \mid \mathcal{E}(A[p]) = w\}.$$

Oort [22], (3.2) proved that $\mathcal{S}_w$ is locally closed in $\mathcal{A}_g$; we consider $\mathcal{S}_w$ as a locally closed subscheme by giving it the reduced induced scheme structure.

Recall the result of Ekedahl and van der Geer:

Theorem 2.3.6 ([5], Theorem 11.5)**.** *Assume $w \in {}^IW^{(c)}$ with $c > \lfloor g/2 \rfloor$. Then $\mathcal{S}_w$ is irreducible.*

Remark 2.3.7. Let $w \in {}^IW^{(c)}$. From Lemma 2.3.5 we have $c \le \lfloor g/2 \rfloor \iff \psi_w(\lfloor (g+1)/2 \rfloor) = 0$. Recall [3], (3.7), Step 2 that this is also equivalent to that $\mathcal{S}_w$ is contained in the supersingular locus, also see [8] for a generalization.

Recently Wedhorn proved

Theorem 2.3.8 ([26])**.** *For any two $w, w' \in {}^IW$, we have $\mathcal{S}_{w'} \subset \overline{\mathcal{S}_w}$ if and only if there exists an element u of W_I such that $u^{-1} \cdot w' \cdot (w_{0,I} \cdot u \cdot w_{0,I}) \le w$ with respect to the Bruhat-Chevalley order $\le$. Here $w_{0,I}$ is the element of W_I sending i to $g+1-i$ for any $i = 1, \dots, g$.*

Remark 2.3.9. For $w \in W$ and $1 \le i, j \le 2g$, we define $r_w(i,j) := \sharp\{a \le i \mid w(a) \le j\}$. It is known (cf. [5], §2.1) that for two $w, w' \in W$ the Bruhat-Chevalley order is described as $w' \le w \Leftrightarrow r_{w'}(i,j) \ge r_w(i,j)$ for all $1 \le i, j \le 2g$.

2.4 Central streams

We first recall Oort's theory [24] on minimal p-divisible groups.

Definition 2.4.1. For non-negative integers m, n with $\gcd(m,n) = 1$, we define a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$P_{m,n} := \mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p e_i \tag{10}$$

with $\mathcal{F}, \mathcal{V}$ operations: $\mathcal{F}e_i = e_{i+n}$ and $\mathcal{V}e_i = e_{i+m}$ for all $i \in \mathbb{Z}_{\ge 0}$, where e_i ($i \in \mathbb{Z}_{\ge m+n}$) are defined as satisfying $e_{i+m+n} = pe_i$ for $i \in \mathbb{Z}_{\ge 0}$.

For a Newton polygon $\xi = \sum_{l=1}^{\mathfrak{t}} (m_l, n_l)$, we write

$$H(\xi) = \bigoplus_{l=1}^{\mathfrak{t}} H_{m_l,n_l} \qquad \text{and} \qquad P(\xi) = \bigoplus_{l=1}^{\mathfrak{t}} P_{m_l,n_l}. \tag{11}$$

Note the Newton polygon of $H(\xi)$ is equal to ξ.

Definition 2.4.2. A p-divisible group X is called *minimal* if there exists an isomorphism over an algebraically closed field from X to $H(\xi)$ for a certain Newton polygon ξ. If a BT_1 G is isomorphic to $H(\xi)[p]$ over an algebraically closed field, we call G (and its final element) *minimal*.

Let ξ be a symmetric Newton polygon. The *central stream* of ξ is defined to be

$$\mathcal{Z}_\xi = \{(A, \eta) \in \mathcal{A}_g \mid A[p^\infty]_\Omega \simeq H(\xi)_\Omega \text{ for some alg. closed field } \Omega\}.$$

Theorem 2.4.3 (Oort, [24])**.** *Let X be a p-divisible group over an algebraically closed field Ω. If $X[p] \simeq H(\xi)[p] \otimes \Omega$, then $X \simeq H(\xi) \otimes \Omega$.*

Let w_ξ be the element of ${}^I W$ corresponding to $H(\xi)[p]$. Then Th. 2.4.3 implies

$$\mathcal{Z}_\xi = \mathcal{S}_{w_\xi}. \tag{12}$$

By Th. 2.3.6, $\mathcal{Z}_\xi$ is irreducible if ξ is not supersingular.

We recall the results on the configuration of the central streams, see [10], §3. Let $\overline{\mathcal{Z}_\xi}$ denote the Zariski closure of $\mathcal{Z}_\xi$ in $\mathcal{A}_g$.

Theorem 2.4.4. *$\mathcal{Z}_\zeta \subset \overline{\mathcal{Z}_\xi}$ if and only if $\zeta \prec \xi$.*

Corollary 2.4.5. *If $\mathcal{Z}_{\xi(w)} \subset \overline{\mathcal{S}_w}$ holds, then Oort's conjecture is true.*

Proof. If $\mathcal{W}_\xi^0 \cap \mathcal{S}_w \neq \emptyset$, we have $\xi \prec \xi(w)$ by Grothendieck-Katz; then Th. 2.4.4 implies $\mathcal{Z}_\xi \subset \overline{\mathcal{Z}_{\xi(w)}}$. By the assumption $\mathcal{Z}_{\xi(w)} \subset \overline{\mathcal{S}_w}$, we have $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$. $\square$

3 Main results

Let k be an algebraically closed field of characteristic p. Let $x \in \mathcal{W}_\xi^0(k)$. Oort defined the isogeny leaf $\mathcal{I}_x$ in $\mathcal{W}_\xi^0$, see [23], (4.2), and showed that $\mathcal{I}_x$ is closed in $\mathcal{W}_\xi^0$ and proper over k, see [23], (4.11).

The following theorem is the essential part of the paper [11].

Theorem 3.0.6. *Assume $w \in {}^I W$ is not minimal. Then for a geometric point x of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$, a component of $\mathcal{I}_x \cap \mathcal{S}_w$ has dimension > 0.*

Proof. See [11]. $\square$

Let us show every results in §1 from this theorem:

Proof of (Th. 3.0.6 $\Rightarrow$ Main theorem). If w is minimal, then $\mathcal{Z}_{\xi(w)} = \mathcal{S}_w$; hence the main theorem is obviously true. Assume w is not minimal. Assume the main theorem is true for all w' with $\mathcal{S}_{w'} \subsetneq \overline{\mathcal{S}_w}$. (The smallest case w.r.t. $\subset$ is the superspecial case $w = \mathrm{id}$ and in this case w is minimal.) According to Th. 3.0.6 there exists a geometric point x of $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$ such that a component of $\mathcal{I}_x \cap \mathcal{S}_w$ has dimension > 0. Since $\mathcal{I}_x$ is proper and $\mathcal{S}_w$ is quasi-affine, there exists w' with $S_{w'} \subsetneq \overline{S_w}$ such that we have $\mathcal{S}_{w'} \cap \mathcal{I}_x \neq \emptyset$. Note $S_{w'} \subset \overline{S_w}$ implies $\xi(w') \prec \xi(w)$, and $\mathcal{I}_x \subset \mathcal{W}^0_{\xi(w)}$ and $\mathcal{S}_{w'} \cap \mathcal{I}_x \neq \emptyset$ imply $\xi(w) \prec \xi(w')$; hence we obtain $\xi(w) = \xi(w')$. By the hypothesis of induction, we have $\mathcal{Z}_{\xi(w')} \subset \overline{\mathcal{S}_{w'}}$. Then $\mathcal{Z}_{\xi(w)} = \mathcal{Z}_{\xi(w')} \subset \overline{\mathcal{S}_{w'}} \subset \overline{\mathcal{S}_w}$. $\square$

Proof of (Main theorem $\Rightarrow$ Cor. II). The main theorem says that $\xi(w) \in \{\xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\}$. Let ξ be a symmetric Newton polygon with $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$. Then by Grothendieck-Katz, we have $\xi \prec \xi(w)$. $\square$

Remark 3.0.7. Cor. II gives a purely combinatorial algorithm determining $\xi(w)$. Indeed first note $\mathcal{Z}_\xi = \mathcal{S}_{w_\xi}$ and there is an algorithm determining w_ξ for concretely given ξ ([10]); then by using Wedhorn's result Th. 2.3.8 and Rem. 2.3.9 we can check whether $\mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$ for concretely given ξ and w; thus we can explicitly describe the set $\{\xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\}$; finally find the biggest element in the set, which exists and is equal to $\xi(w)$.

4 Ekedahl-Oort strata contained in the supersingular locus

In this section, we describe some union of Ekedahl-Oort strata contained in the supersingular locus in terms of Deligne-Lusztig varieties, and show the reducibility of such Ekedahl-Oort strata.

4.1 Notations

Let c be a non-negative integer $\leq g$. Let W_c be the Weyl group of the smaller symplectic group Sp_{2c}. Let $W_{c,J}$ be the subgroup of W_c generated by the elements of $J = \{s_1, \ldots, s_{c-1}\}$. Put $W_c := W_{c,J}\backslash W_c/W_{c,J}$. We define a map

$$\mathfrak{r}: \quad {}^I W_g^{(c)} \longrightarrow \overline{W}_c \tag{13}$$

by sending w to the class of $v \in W_c$ determined by $v(i) = w(g-c+i)-(g-c)$ for all $1 \leq i \leq c$. We denote by $\overline{W}'_c$ the image of $\mathfrak{r}$.

Assume $c \leq \lfloor g/2 \rfloor$. For $w' \in \overline{W}'_c$, we consider the union

$$\mathcal{J}_{w'} = \bigcup_{\mathfrak{r}(w)=w'} \mathcal{S}_w. \tag{14}$$

For each c, we fix once and for all a symplectic vector space $(L_0, \langle\ ,\ \rangle)$ over $\mathbb{F}_{p^2}$ of dimension $2c$ and a maximal totally isotropic subspace C_0 over $\mathbb{F}_{p^2}$ of L_0. Let $\mathrm{Sp}(L_0)$ denote the symplectic group over $\mathbb{F}_{p^2}$ associated to $(L_0, \langle\ ,\ \rangle)$. Let X be the flag variety $\mathrm{Sp}(L_0)/\mathrm{P}_0$ over $\mathbb{F}_{p^2}$, where P_0 denotes the Siegel parabolic subgroup, i.e., the parabolic subgroup of $\mathrm{Sp}(L_0)$ stabilizing C_0. For $w' \in \overline{W}_c$, let $\mathrm{X}(w')$ denote the (generalized) Deligne-Lusztig variety in X related to w':

$$\{\mathrm{P} \in \mathrm{Sp}_{2c}/\mathrm{P}_0 \mid {}^h\mathrm{P} = \mathrm{P}_0, {}^h\mathrm{Fr}(\mathrm{P}) = {}^{w'}\mathrm{P}_0 \text{ for } \exists h \in \mathrm{Sp}_{2c}\}.$$

4.2 A description of $\mathcal{J}_{w'}$

Theorem 4.2.1. *Assume $c \leq \lfloor g/2 \rfloor$. For each $w' \in \overline{W}'_c$, there exists a finite surjective morphism*

$$\mathrm{G}(\mathbb{Q})\backslash \mathrm{X}(w') \times \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K} \longrightarrow \mathcal{J}_{w'}$$

over $\mathbb{F}_{p^2}$, which is bijective on geometric points, see (15) *below for the definition of the quaternion unitary group* G *over* $\mathbb{Z}$ *with* $\mathrm{K} = \prod_l \mathrm{G}(\mathbb{Z}_l)$.

Let E be a supersingular elliptic curve over $\mathbb{F}_p$ (see [18], 1.2). Put $\mathcal{O} = \mathrm{End}(E)$ (the endomorphism ring over $\overline{\mathbb{F}_p}$). We denote $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ by B. Note B is the quaternion algebra over $\mathbb{Q}$ ramified only at p and ∞ over $\mathbb{Q}$. The quaternion unitary group G over $\mathbb{Z}$ in Th. 4.2.1 is defined by

$$\mathrm{G}(R) = \{h \in \mathrm{GL}_g(\mathcal{O} \otimes_{\mathbb{Z}} R) \mid {}^t\overline{h}\varphi h = \varphi\} \tag{15}$$

for any commutative ring R.

We remark that in [9] we proved Th. 4.2.1 for $g \geq 2$. If $g = 1$, then we have ${}^I W_1^{(0)} = \{\mathrm{id}\}$ and the Ekedahl-Oort stratum $\mathcal{S}_{\mathrm{id}}$ is the (finite) set consisting of the supersingular elliptic curves; hence see Deuring [4] and

Igusa [14] for this case. Also Ibukiyama-Katsura-Oort [13] and Katsura-Oort [15] have investigated the case of $g = 2$ with more detailed results.

Hoeve [12] seems to have refined Th. 4.2.1, where he found a description of individual Ekedahl-Oort strata contained in the supersingular locus in terms of "fine" Deligne-Lusztig varieties. Recently Vollaard and Wedhorn [25] obtained an analogous result for unitary Shimura varieties. Also Görtz and Yu [7] studied supersingular Kottwitz-Rapoport strata, where Deligne-Lusztig varieties also appear.

4.3 Reducibility of Ekedahl-Oort strata contained in the supersingular locus

Oort conjectured that (i) $\mathcal{S}_w$ is irreducible if $\mathcal{S}_w$ is not contained in the supersingular locus and (ii) $\mathcal{S}_w$ is reducible for sufficiently large p otherwise. Ekedahl and van der Geer proved (i), see Th. 2.3.6 and Rem. 2.3.7. In this subsection we shall confirm (ii) by showing $\lim_{p\to\infty} H_{g,c} = \infty$.

Assume $c \leq \lfloor g/2 \rfloor$. Let w' be an element of $\overline{W}'_c$. Note a (any) representative of w' is not in $W_{c,J}$. Note $W_{c,J}$ is a maximal parabolic subgroup of W_c. Hence Bonnafé and Rouquier [1], Theorem 2 implies that the Deligne-Lusztig variety $\mathrm{X}(w')$ is irreducible. Then by Theorem 4.2.1 we have

Corollary 4.3.1. *The set of irreducible (connected) components of $\mathcal{J}_{w'}$ is identified with* $\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}$.

We can estimate $H_{g,c} = \sharp\, \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}$ by the mass formula, i.e., we have

$$H_{g,c} \geq 2\mathfrak{m}_{g,c},$$

where the mass $\mathfrak{m}_{g,c}$ is defined to be

$$\mathfrak{m}_{g,c} = \sum_{\gamma\in \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}^\infty)/\mathrm{K}} \frac{1}{\sharp\, \mathrm{G}(\mathbb{Q}) \cap \gamma\,\mathrm{K}\,\gamma^{-1}}.$$

By using Prasad's mass formula, we can show

Proposition 4.3.2. *We have*

$$\mathfrak{m}_{g,c} = \prod_{i=1}^{g} \frac{(2i-1)!\zeta(2i)}{(2\pi)^{2i}} \cdot \binom{g}{2c}_{p^2} \cdot \prod_{i=1}^{g-2c} (p^i + (-1)^i) \prod_{i=1}^{c} (p^{4i-2} - 1),$$

where $\zeta(s)$ is the Riemann zeta function and

$$\binom{g}{r}_q := \frac{\prod_{i=1}^{g}(q^i-1)}{\prod_{i=1}^{r}(q^i-1)\prod_{i=1}^{g-r}(q^i-1)} \quad \in \mathbb{Z}[q].$$

Corollary 4.3.3. *If $w \in {}^I W_g^{(c)}$ with $c \leq \lfloor g/2 \rfloor$ (i.e., $\mathcal{S}_w$ is contained in the supersingular locus), then $\mathcal{S}_w$ is reducible for sufficiently large p.*

Proof. By Corollary 4.3.1 the Hecke action on the set of connected components of $\mathcal{J}_{w'}$ is transitive, where $w' = \mathfrak{r}(w)$. By the definition of $\mathcal{S}_w$, the Hecke action stabilizes $\mathcal{S}_w$. Hence the number of connected components of $\mathcal{S}_w$ is greater than or equal to that of $\mathcal{J}_{w'}$. By Prop. 4.3.2, we have $\lim_{p\to\infty} H_{g,c} \geq \lim_{p\to\infty} \mathfrak{m}_{g,c} = \infty$. This means that $\mathcal{S}_w$ is reducible for sufficiently large p. $\square$

References

[1] C. Bonnafé and R. Rouquier: *On the irreducibility of Deligne-Lusztig varieties.* C. R. Math. Acad. Sci. Paris **343** (2006), no. 1, 37–39.

[2] N. Bourbaki: *Lie groups and Lie algebras. Chapters 4–6.* Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[3] C.-L. Chai and F. Oort: Monodromy and irreducibility of leaves. 30 pages, preprint. See `http://www.math.uu.nl/people/oort/`.

[4] M. Deuring: *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper.* Abh. Math. Sem. Hansischen Univ. **14**, (1941), 197–272.

[5] T. Ekedahl and G. van der Geer: Cycle classes of the E-O stratification on the moduli of abelian varieties. To appear in: Algebra, Arithmetic and Geometry Volume I: In Honor of Y. I. Manin Series: Progress in Mathematics, Vol. 269 Tschinkel, Yuri; Zarhin, Yuri G. (Eds.) 2008.

[6] E. Z. Goren and F. Oort: Stratifications of Hilbert Modular Varieties. J. Algebraic Geom. **9** (2000), no. 1, 111–154.

[7] U. Görtz and C.-F. Yu: *Supersingular Kottwitz-Rapoport strata and Deligne-Lusztig varieties.* Preprint. arXiv: 0802.3260v1

[8] S. Harashita: Ekedahl-Oort strata and the first Newton slope strata. J. Algebraic Geom. **16** (2007) 171–199.

[9] S. Harashita: Ekedahl-Oort strata contained in the supersingular locus and Deligne-Lusztig varieties. To appear in J. Algebraic Geom.

[10] S. Harashita: Configuration of the central streams in the moduli of abelian varieties. Preprint. See `http://www.ms.u-tokyo.ac.jp/~harasita`.

[11] S. Harashita: Generic Newton polygons of Ekedahl-Oort strata: Oort's conjecture. Preprint. See `http://www.ms.u-tokyo.ac.jp/~harasita`.

[12] M. Hoeve: *Ekedahl-Oort strata in the supersingular locus.* Preprint. arXiv: 0802.4012

[13] T. Ibukiyama, T. Katsura and F. Oort: *Supersingular curves of genus two and class numbers.* Compositio Math. **57** (1986), pp. 127-152.

[14] J. Igusa: *Class number of a definite quaternion with prime discriminant.* Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 312–314.

[15] T. Katsura and F. Oort: Families of supersingular abelian surfaces. Compositio Math. **62** (1987), no. 2, 107–167.

[16] N. M. Katz: Slope filtration of F-crystals. Journ. Géom. Alg. Rennes, Vol. I, Astérisque **63** (1979), Soc. Math. France, 113-164.

[17] H. Kraft: Kommutative algebraische p-Gruppen (mit Anwendungen auf p-divisible Gruppen und abelsche Varietäten). Sonderforsche. Bereich Bonn, September 1975. Ms. 86 pp.

[18] K.-Z. Li and F. Oort: *Moduli of Supersingular Abelian Varieties.* Lecture Notes in Math. **1680** (1998).

[19] Yu. I. Manin: Theory of commutative formal groups over fields of finite characteristic. Uspehi Mat. Nauk **18** (1963) no. 6 (114), 3–90; Russ. Math. Surveys **18** (1963), 1-80.

[20] B. Moonen: Group schemes with additional structures and Weyl group cosets. In: Moduli of abelian varieties (Ed. C. Faber, G. van der Geer, F. Oort), Progr. Math., **195**, Birkhäuser, Basel, 2001; pp. 255–298.

[21] B. Moonen and T. Wedhorn: Discrete invariants of varieties in positive characteristic. Int. Math. Res. Not. 2004, no. **72**, 3855–3903.

[22] F. Oort: A stratification of a moduli space of abelian varieties. In: Moduli of abelian varieties (Ed. C. Faber, G. van der Geer, F. Oort), Progr. Math., **195**, Birkhäuser, Basel, 2001; pp. 345–416.

[23] F. Oort: Foliations in moduli spaces of abelian varieties. J. Amer. Math. Soc. **17** (2004), no. 2, 267–296.

[24] F. Oort: Minimal p-divisible groups. Ann. of Math. (2) **161** (2005), no. 2, 1021–1036.

[25] I. Vollaard and T. Wedhorn: *The supersingular locus of the Shimura variety of $GU(1, n-1)$ II.* Preprint. arXiv: 0804.1522v1

[26] T. Wedhorn: Specialization of F-zips. Preprint. arXiv: 0507175v1

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