On j-multiplicity

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This is a joint work with B. Ulrich.

The notion of j-multiplicity was introduced by Achilles and Manaresi in [1] and the theory was developed in [4], [2] and [3]. The j-multiplicity $\mathrm{j}(I)$ is an invariant of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$. If I is $\mathfrak{m}$-primary, then $\mathrm{j}(I)$ coincides with the usual multiplicity $\mathrm{e}(I)$. In this note we give a length formula of j-multiplicity which enables us to compute $\mathrm{j}(I)$ of a given ideal I.

Let us begin with the definition of j-multiplicity. It can be defined for a finitely generated module L over a positively graded Noetherian ring $T = \oplus_{n\geq 0} T_n$ such that $(T_0, \mathfrak{m})$ is local and $T = T_0[T_1]$. We assume that $T_0/\mathfrak{n}$ is an infinite field. Let d be a positive integer with $\dim_T L \leq d$. We denote the Krull dimension of $L/\mathfrak{n}L$ as an T-module by $\ell(T, L)$ and call it the analytic spread of L. Let $W = \mathrm{H}^0_{\mathfrak{n}T}(L)$, which is the 0-th local cohomology module of L with respect to $\mathfrak{n}T$. By the Artin-Rees lemma, we see that $W \cap \mathfrak{n}^k L = 0$ for $k \gg 0$. Then $W = \oplus_{n\geq 0} \mathrm{H}^0_{\mathfrak{n}}(L_n)$ can be embedded in $L/\mathfrak{n}^k L$ as a graded $T/\mathfrak{n}^k T$-module. Because $T/\mathfrak{n}^k T$ is a standard graded ring over an Artinian local ring and $\dim_T L/\mathfrak{n}^k L = \ell(T, L) \leq d$, there exists an integer $\alpha \geq 0$ such that

$$\mathrm{length}_{T_0}\ \mathrm{H}^0_{\mathfrak{n}}(L_n) = \frac{\alpha}{(d-1)!}\, n^{d-1} + (\text{terms of lower degree})$$

for $n \gg 0$. This number α is called the j-multiplicity of the T-module L and is denoted by $\mathrm{j}_d(T, L)$.

Lemma 1 (cf. [4]) $\mathrm{j}_d(T, L) \neq 0$ *if and only if* $\ell(T, L) = d$.

Lemma 2 (cf. [4]) *Let* $d \geq 2$ *and* $\dim_{T_0} L_n < \dim_T L$ *for any* $n \geq 0$. *We choose* $f \in T_1$ *generally so that the following two conditions are satisfied;*

(1) *f is T_+-filter regular for L,*

(2) $\ell(T, L/fL + W) \le d - 2$.

Then we have $\dim_T L/fL \le d-1$ *and* $\mathrm{j}_d(T,L) = \mathrm{j}_{d-1}(T, L/fL)$.

Now we consider a Noetherian local ring $(R, \mathfrak{m})$ with $|R/\mathfrak{m}| = \infty$ and a finitely generated R-module M. We take an ideal I of R and a positive integer d with $\dim_R M \le d$. We set $\mathrm{j}_d(I, M) = \mathrm{j}_d(\mathrm{gr}_I R, \mathrm{gr}_I M)$ and call it the j-multiplicity of I with respect to M. Let us simply denote $\mathrm{j}_{\dim R}(I, R)$ by $\mathrm{j}(I)$. By Lemma 1 and Lemma 2, we have the following assertion.

Lemma 3 $\mathrm{j}(I) \neq 0$ *if and only if* $\ell(I) = \dim R > 0$, *where $\ell(I)$ denotes the usual analytic spread of I.*

Lemma 4 *Let $d \ge 2$ and* $\dim_R M/IM < \dim_R M$. *Then, for a general element $a \in I$, we have* $\dim_R M/aM \le d-1$ *and* $\mathrm{j}_d(I, M) = \mathrm{j}_{d-1}(I, M/aM)$.

In the case where $\dim_R M/IM = \dim_R M$, we need the following result.

Lemma 5 *Let* $\overline{M} = M/\mathrm{H}^0_I(M)$. *Then* $I^n M/I^{n+1}M \cong I^n\overline{M}/I^{n+1}\overline{M}$ *for* $n \gg 0$, *and so* $\mathrm{j}_d(I, M) = \mathrm{j}_d(I, \overline{M})$. *Furthermore, if* $\overline{M} \neq 0$, *we have* $\dim_R \overline{M}/I\overline{M} < \dim_R \overline{M}$.

By applying Lemma 4 and Lemma 5 successively, we get the next result.

Theorem 6 *Let $1 \le i < d$. Then, choosing sufficiently generic elements $a_1, \dots, a_i$ of I, we have*

$$\dim_R M/((a_1, \dots, a_i)M :_M I^\infty) \le d - i \quad \text{and}$$
$$\mathrm{j}_d(I, M) = \mathrm{j}_{d-i}(I, M/((a_1, \dots, a_i)M :_M I^\infty)),$$

where $(a_1, \dots, a_i)M :_M I^\infty = \cup_{n>0}((a_1, \dots, a_i)M :_M I^n)$.

Corollary 7 *Choosing sufficiently generic elements* $a_1, \ldots, a_{d-1}$ *and* a_d *of* I*, we have*

$$\mathrm{j}_d(I, M) = \mathrm{length}_R\ M/((a_1, \ldots, a_{d-1})M :_M I^\infty) + a_d M\,.$$

Lemma 8 *Suppose* $\dim_R M \leq 1$*. We put*

$$\mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim\ R/\mathfrak{p} = 1 \text{ and } I \not\subseteq \mathfrak{p}\}\,.$$

Then we have

$$\mathrm{j}_1(I, M) = \sum_{\mathfrak{p} \in \mathcal{P}} \mathrm{length}_{R_\mathfrak{p}}\ M_\mathfrak{p} \cdot \mathrm{e}_I(R/\mathfrak{p})\,.$$

Applying Lemma 4, Lemma 5 and Lemma 8, we can give another proof for the additivity of j-multiplicity, which was first proved in [4].

Theorem 9 *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of* R*-modules. Then we have*

$$\mathrm{j}_d(I, M) = \mathrm{j}_d(I, L) + \mathrm{j}_d(I, N)\,.$$

Then we get the additive formula of j-multiplicity similarly as the usual multiplicity.

Theorem 10 $\displaystyle \mathrm{j}_d(I, M) = \sum_{\mathfrak{p} \in \mathrm{Assh}_R\ M} \mathrm{length}_{R_\mathfrak{p}}\ M_\mathfrak{p} \cdot \mathrm{j}_d(I, R/\mathfrak{p}).$

Moreover we get the following.

Theorem 11 *Let* $1 \leq i < d$ *and* $a_1, \ldots, a_i$ *be sufficiently generic elements of* I*. We set*

$$\mathcal{P}_i = \{\mathfrak{p} \in \operatorname{Spec} R \mid (a_1, \ldots, a_i) \subseteq \mathfrak{p}\,,\ I \not\subseteq \mathfrak{p} \text{ and } \dim\ R/\mathfrak{p} = d - i\}\,.$$

Then $\mathcal{P}_i \cap \operatorname{Supp}_R M$ *is finite and*

$$\mathrm{j}_d(I, M) = \sum_{\mathfrak{p} \in \mathcal{P}_i} \mathrm{length}_{R_\mathfrak{p}}\ M_\mathfrak{p}/(a_1, \ldots, a_i)M_\mathfrak{p} \cdot \mathrm{j}_{d-i}(I, R/\mathfrak{p})\,.$$

As an application of the theory stated above, we get the following assertion.

Example 12 *Let* $R = K[[X, Y, Z]]$ *be the formal power series ring over an infinite field* K*. Let* $\mathfrak{p}$ *be the defining ideal of a space monomial curve:* $X = t^k, Y = t^\ell, Z = t^m$*, where* k, ℓ *and* m *are positive integers with*

$$\mathrm{G}CD\{k, \ell, m\} = 1\,.$$

Then $\mathfrak{p}$ *is generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{pmatrix},$$

where α*,* β*,* γ*,* α'*,* β' *and* γ' *are positive integers. Replacing the variables* X, Y *and* Z*, we may assume*

$$k\alpha = \min\{k\alpha, \ell\beta', m\gamma', \ell\beta, m\gamma, k\alpha'\}\,.$$

Then we have $\mathrm{j}(\mathfrak{p}) = \alpha\beta(\gamma + \gamma')$*.*

We give a sketch of proof for this example. We put $f = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}, g = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'}$ and $h = Y^{\beta+\beta'} - X^\alpha Z^\gamma$. Then $\mathfrak{p} = (f, g, h)$ and the ideal generated by general two elements in $\mathfrak{p}$ can be written in the form $(af - g, bf - h)$ with $0 \neq a, b \in K$. We put $\xi = af - g$ and $\eta = bf - h$. It is easy to see that

$$\begin{aligned} (\xi, \psi) :_R \mathfrak{p}^\infty &= (\xi, \eta) :_R f \\ &= (X^\alpha + aY^{\beta'} + bZ^{\gamma'}, Y^\beta + aZ^\gamma + bX^{\alpha'})\,. \end{aligned}$$

Therefore, by Theorem 6, we get

$$\mathrm{j}(\mathfrak{p}) = \mathrm{length}_R\ R/\mathfrak{p} + (X^\alpha + aY^{\beta'} + bZ^{\gamma'}, Y^\beta + aZ^\gamma + bX^{\alpha'})\,.$$

Let $A = K[[t^k, t^\ell, t^m]]$. Then ϕ induces an isomorphism $R/\mathfrak{p} \xrightarrow{\sim} A$, which implies

$$\mathrm{j}(\mathfrak{p}) = \mathrm{length}_A\ A/(t^{k\alpha}u, t^{\ell\beta}u)A\,,$$

where $u = 1 + at^{\ell\beta' - k\alpha} + bt^{m\gamma' - k\alpha} = 1 + at^{m\gamma - \ell\beta} + bt^{k\alpha' - \ell\beta} \in K[[\, t\,]]$. Therefore we get $\mathrm{j}(\mathfrak{p}) = \alpha\beta(\gamma + \gamma')$ since

$$\begin{aligned} & \mathrm{length}_A\ A/(t^{k\alpha}u, t^{\ell\beta}u)A \\ = \ & \mathrm{length}_A\ A/(t^{k\alpha}, t^{\ell\beta})A \\ = \ & \mathrm{length}_R\ R/(X^\alpha, Y^\beta)R + \mathfrak{p} \\ = \ & \mathrm{length}_R\ R/(X^\alpha, Y^\beta, Z^{\gamma+\gamma'})R\,. \end{aligned}$$

References

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