

An example of stable Higgs bundles for which the Bogomolov inequality fails

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1 Introduction

The Bogomolov inequality for semistable vector bundles on smooth complex projective n -folds X reads

$$c_2(\mathcal{E})A^{n-2} \geq \frac{r-1}{2r}c_1(\mathcal{E})^2A^{n-2},$$

where A is an ample divisor and \mathcal{E} is an A -semistable vector bundle of rank r on X . In case \mathcal{E} is A -stable with vanishing $c_1(\mathcal{E})$, the lower bound of this inequality $c_2(\mathcal{E}) \geq 0$ is attained if and only if \mathcal{E} admits the structure of a flat hermitian bundle associated with an irreducible unitary representation of the fundamental group $\pi_1(X)$, thereby establishing the one-to-one *Kobayashi-Hitchin correspondence* between the stable bundles with vanishing Chern classes and the irreducible unitary representation of $\pi_1(X)$ [2]. The Bogomolov inequality is natural enough to have several proofs by completely different approaches (geometric invariant theory [1]; characteristic p method [3]; the theory of effective cones on ruled surfaces [8]; Yang-Mills theory of connections [2]).

Because of this naturality, the Bogomolov inequality extends to certain classes of generalized vector bundles, including parabolic bundles and orbibundles. Another important class of generalized vector bundles is that of Higgs bundles (see [9]), and Simpson [?] succeeded in generalizing the inequality also to stable Higgs bundles through a generalized version of Yang-Mills theory. In contrast to the aforementioned cases, an algebro-geometric proof of the Bogomolov inequality is so far not available for Higgs bundles except for very special ones: some standard examples listed in Section 1 as Examples 0, 1 and 2, and the bundles of small ranks 2, 3 [7]. One of the implications of Simpson's result is that, if a stable Higgs bundle has trivial

Chern classes, then it comes from an irreducible representation of $\pi_1(X)$ to the special linear group $SL(r, \mathbf{C})$.

In this note, we give several examples of stable Higgs bundles with trivial Chern classes.

2 Higgs bundles: definition and examples

Let \mathcal{E} be a vector bundle on a complex manifold X and $\theta : \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E}$ an \mathcal{O}_X -linear mapping. The pair (\mathcal{E}, θ) is said to be a *Higgs bundle* if the natural composite map $\theta \wedge \theta : \mathcal{E} \rightarrow \Omega_X^2 \otimes \mathcal{E}$ identically vanishes. Alternatively, \mathcal{E} is a Higgs bundle if an \mathcal{O}_X -linear action of the sheaf of the local vector fields Θ_X on \mathcal{E} is given in such a way such that $\xi_1(\xi_2(e)) = \xi_2(\xi_1(e))$ for arbitrary $\xi_i \in \Theta_X$ and $e \in \mathcal{E}$. In other words, a Higgs bundle is a vector bundle with a $\text{Sym } \Theta_X$ -module structure, where

$$\text{Sym } \Theta_X = \bigoplus_{i=0}^{\infty} \text{Sym}^i \Theta_X$$

is the symmetric tensor algebra generated by Θ_X . Higgs subsheaves are, by definition, $\text{Sym } \Theta_X$ -submodules.

Given an ample divisor A on X , the notion of A -(semi)stable Higgs bundles is naturally defined. Namely, a Higgs bundle \mathcal{E} is A -stable if

$$\frac{c_1(\mathcal{S})A^{n-1}}{\text{rank } \mathcal{S}} < \frac{c_1(\mathcal{E})A^{n-1}}{\text{rank } \mathcal{E}}$$

for any nontrivial Higgs subsheaf $\mathcal{S} \subset \mathcal{E}$, $\mathcal{S} \neq 0, \mathcal{E}$, where $n = \dim X$.

Historically Higgs structures were introduced in the study of moduli of integrable connections [5]. Let \mathcal{E} be a vector bundle with an integrable connection $\nabla_0 : \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E}$. Given another integrable connection ∇ , the difference $\theta = \nabla - \nabla_0$ is a Higgs bundle structure and this correspondence translates the moduli of the integrable connections on a flat vector bundle \mathcal{E} into the moduli of the Higgs bundle structures.

With the above definition in mind, we give below several examples of Higgs bundles.

Example 0. An ordinary vector bundle is viewed as a Higgs bundle with trivial zero action of Θ_X . In this case, the Higgs stability is nothing but the usual stability.

Starting from given Higgs bundles, we can construct new Higgs bundles by taking tensor products, duals and pull-backs.

Given two Higgs bundles $\mathcal{E}_1, \mathcal{E}_2$, the tensor bundle $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Higgs bundle by defining $\xi(e_1 \otimes e_2) = \xi(e_1) \otimes e_2 + e_1 \otimes \xi(e_2)$ for $\xi \in \Theta_X$. The dual bundle \mathcal{E}^\vee of a Higgs bundle is again a Higgs bundle by $\langle e | \xi(e^\vee) \rangle = -\langle \xi(e) | e^\vee \rangle$, where $e \in \mathcal{E}, e^\vee \in \mathcal{E}^\vee, \xi \in \Theta_X$.

If $g: X \rightarrow Y$ is a morphism between complex manifolds and \mathcal{E} is a Higgs bundle on Y , then the pull-back $g^*\mathcal{E}$ is naturally a $\text{Sym } g^*\Theta_Y$ -module. Then the natural \mathcal{O}_Y -algebra homomorphism $\text{Sym } \Theta_X \rightarrow \text{Sym } g^*\Theta_Y$ defines a canonical Higgs bundle structure of $g^*\mathcal{E}$.

Example 1. Let X be a complex manifold. The symmetric tensor algebra $\mathbf{E}_0^\infty(X) = \text{Sym } \Theta_X$ is naturally a Higgs bundle of infinite rank and so is its

ideal. In particular, the graded ideal $\mathbf{E}_{l+1}^\infty(X) = \bigoplus_{i=l+1}^\infty \text{Sym}^i \Theta_X$ is a Higgs subbundle of infinite rank. Given $l \geq k \geq 0$, the subquotient $\mathbf{E}_k^l(X) = \mathbf{E}_k^\infty(X)/\mathbf{E}_{l+1}^\infty(X)$ is a coherent Higgs bundle isomorphic to $\bigoplus_{i=k}^l \text{Sym}^i \Theta_X$.

The action of Θ_X on $\mathbf{E}_k^l(X)$ is given by zero on $\text{Sym}^l \Theta_X$ and by the standard multiplication $\Theta \otimes \text{Sym}^i \Theta_X \rightarrow \text{Sym}^{i+1} \Theta_X$ on the other components.

If $K_X A > 0$ and Θ_X is A -semistable as an ordinary vector bundle [resp. If $K_X A \geq 0$ and Θ_X is semistable], then $\mathbf{E}_0^l(X)$ is an A -stable [resp. A -semistable] Higgs bundle. If K_X is ample and $A = K_X$, then the Yau inequality [10]

$$c_2(X)K_X^{n-2} \geq \frac{\dim X - 1}{2 \dim X} K_X^n$$

yields the Bogomolov inequality for $\mathbf{E}_k^l(X)$.

Example 2. Given integers $l \geq k \geq 0$, we define the Higgs bundle $\mathbf{F}_l^k(X)$

as the vector bundle $\bigoplus_{i=k}^l \text{Sym}^i \Omega_X^1$ with the Θ_X -action defined by 0 on

$\text{Sym}^k \Omega_X^1$ and by (-1) times the standard contraction map $\Theta_X \otimes \text{Sym}^i \Omega_X^1 \rightarrow \text{Sym}^{i-1} \Omega_X^1$. $\mathbf{F}_l^k(X)$ is the dual $\mathbf{E}_0^\infty(X)$ -module $\mathcal{H}om_{\mathbf{E}_0^\infty(X)}(\mathbf{E}_k^l(X), \mathcal{O}_X)$ of $\mathbf{E}_k^l(X)$, where \mathcal{O}_X is viewed as a Higgs bundle with trivial Θ_X -action. For $l \geq m \geq k$, $\mathbf{F}_m^k(X)$ is naturally a Higgs subbundle of $\mathbf{F}_l^k(X)$ with the quotient $\mathbf{F}_l^k(X)/\mathbf{F}_m^k(X)$ isomorphic to $\mathbf{F}_l^{m+1}(X)$. The stability condition and the Bogomolov inequality for \mathbf{F}_l^k are similar as for \mathbf{E}_k^l .

Example 3. Let $m \geq l \geq 0$ be integers. We define a Higgs bundle structure on

$$\text{Sym}_{i=0}^l \text{Sym}^i \Omega_X^1 \oplus \bigoplus_{i=m-l}^0 \text{Sym}^i \Theta_X \otimes \text{Sym}^{m+1} \Omega_X^1$$

by defining the action of $\xi \in \Theta_X$ as follows:

- For $\alpha \in \text{Sym}^i \Omega_X^1$, $l \geq i \geq 0$, $\xi(\alpha)$ is the $(-1) \times$ the natural contraction $\in \text{Sym}^{i-1} \Omega_X^1$.
- For $\alpha \in \text{Sym}^{l-m} \Theta_X \otimes \text{Sym}^{m+1} \Omega_X^1$, $\xi(\alpha)$ is defined by the composition of the natural product $\xi\alpha \in \text{Sym}^{m-l+1} \Theta_X \otimes \text{Sym}^{m+1} \Omega_X$ and the contraction map $\text{Sym}^{m-l+1} \Theta_X \otimes \text{Sym}^{m+1} \Omega_X \rightarrow \text{Sym}^l \Omega_X^1$.
- For $\alpha \in \text{Sym}^i \Theta_X \otimes \text{Sym}^{m+1} \Omega_X^1$, $i < m-l$, $\xi(\alpha)$ is the natural product $\xi\alpha \in \text{Sym}^{i+1} \Theta_X \otimes \text{Sym}^{m+1}$.

This Higgs bundle is an extension of $\mathbf{E}_0^{m-l}(X) \otimes \text{Sym}^{m+1} \Omega_X^1$ by $\mathbf{F}_0^l(X)$.

Example 4. Let $f_i : X \rightarrow C_i$ be a surjective morphism onto a curve. Let $\mathcal{F}_i = \mathcal{O}_{C_i} \oplus \Omega_{C_i}^1$ be the standard Higgs bundle on C_i . Then $\mathcal{E}_i = f_i^* \mathcal{F}_i$ is a Higgs subsheaf of $\mathcal{O}_X \oplus \Omega_X^1$. It is straight forward to check that $\mathcal{E}_i^\circ = \mathcal{E}_i(-f_i^* K_{C_i}/2)$ has trivial c_1 and c_2 . If H is ample on X and $g(C_i) \geq 2$, then \mathcal{E}_i° is H -stable, and hence flat. $\mathcal{E}^\circ = \bigotimes_i \mathcal{E}_i^\circ$ is also an H -semistable, flat Higgs bundle of rank 2^r .

When $X = C_1 \times C_2$ and the f_i are the two canonical projections, we get an H -stable, flat Higgs bundle

$$\mathcal{O}_X(-\frac{K_X}{2}) \oplus \Omega_X^1(-\frac{K_X}{2}) \oplus \mathcal{O}(\frac{K_X}{2}) \subset (\mathcal{O}_X \oplus \Omega_X^1 \oplus \text{Sym}^2 \Omega_X^1) \otimes \mathcal{O}_X(-\frac{K_X}{2}).$$

3 Hirzebruch's Kummer covers $X^{(n)}$ attached to the complete quadrilateral on \mathbb{P}^2

We briefly review Hirzebruch's construction of Kummer covers of projective plane branching along a complete quadrilateral [4].

Take general four points P_1, \dots, P_4 on \mathbb{P}^2 , and let $L_{ij} = L_{ji}$ denote the line connecting P_i and P_j ($i \neq j$). The reduced divisor $D = \bigcup L_{ij}$ is the so called complete quadrilateral consisting of six lines, the P_i being the triple points of D . D has extra three double points of the form $L_{i_1, i_2} \cap L_{j_1, j_2}$, where $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Exactly three singular points of D lies on each L_{ij} , two of which are the triple points P_i, P_j and one a double point. Thus the Euler number of the non-singular locus of D is $6 \times (2 - 3) = -6$, while that of D is $-6 + 4 + 3 = 1$. Therefore the Euler number of of the complement of D is given by $e(\mathbb{P}^2 \setminus D) = 3 - 1 = 2$.

Let $\mu : X \rightarrow \mathbb{P}^2$ be the blowing up at the four triple points P_1, \dots, P_4 and let $E_i \subset X$ denote the exceptional divisor over P_i . X is a Del Pezzo surface

of degree five with very ample anticanonical divisor $-K_X \sim 3H - \sum E_i$, where H stands for the pullback of the hyperplane of \mathbb{P}^2 . The effective divisor μ^*D is supported by a reduced effective divisor

$$\tilde{D} \sim \mu^* \sum L_{ij} - 2 \sum E_i \sim 6\mu^*H - 2 \sum E_i \sim -2K_X.$$

\tilde{D} has only simple normal crossings as singularities and consists of ten irreducible components: four exceptional curve E_i and six strict transforms \tilde{L}_{ij} . The \tilde{L}_{ij} meet each other at the three points lying over the double points of D , while each E_i contains three singular points of \tilde{D} . Hence \tilde{D} has exactly $3 + 4 \times 3 = 15$ double points, so that $e(\tilde{D}) = 4 \times (2 - 3) + 6 \times (2 - 3) + 15 = 5$.

Given a positive integer n , there exists a finite Kummer covering $\pi^{(n)} : X^{(n)} \rightarrow X$ of degree n^5 branching along \tilde{D} [4]. The function field of $X^{(n)}$ is simply obtained by adjoining the n -th roots $\sqrt[n]{l_{ij}/l_{12}}$ ($\{i, j\} \neq \{1, 2\} \in \{1, 2, 3, 4\}$) to $\mathbb{C}(\mathbb{P}^2)$, where l_{ij} is a linear differential equation of the line L_{ij} .

$X^{(n)}$ is a smooth projective surface and the local description of $X^{(n)}$ is quite simple: if \tilde{D} is locally defined by the equation $x = 0$ or $xy = 0$, then $\pi^{(n)*} : \mathcal{O}_X \rightarrow \mathcal{O}_{X^{(n)}}$ is given by $(x, y) \mapsto (t^n, u)$ or $(x, y) \mapsto (t^n, u^n)$, where (x, y) and (t, u) are local coordinates of X and $X^{(n)}$. In particular, the inverse image $(\pi^{(n)})^{-1}(p) \subset X^{(n)}$ of a closed point $p \in X$ consists of n^5 [resp. n^4, n^3] points when $p \in X \setminus \tilde{D}$ [resp. $p \in \tilde{D} \setminus \text{Sing}(\tilde{D}), p \in \text{Sing}(\tilde{D})$]. The topological Euler number of $\tilde{X}^{(n)}$ of $X^{(n)}$ is thus given by

$$\begin{aligned} \frac{c_2(X^{(n)})}{n^5} &= \frac{e(X^{(n)})}{n^5} = e(X \setminus \tilde{D}) + \frac{e(\tilde{D} \setminus \text{Sing}(\tilde{D}))}{n} + \frac{e(\text{Sing}(\tilde{D}))}{n^2} \\ &= 2 - \frac{10}{n} + \frac{15}{n^2}. \end{aligned}$$

On the other hand we calculate $K_{X^{(n)}}$ by

$$K_{X^{(n)}} \sim \pi^{(n)*} \left(K_X + \left(1 - \frac{1}{n}\right) \tilde{D} \right) \sim \left(1 - \frac{2}{n}\right) \pi^{(n)*}(-K_X),$$

and hence

$$\frac{c_1(X^{(n)})^2}{n^5} = 5 \left(1 - \frac{2}{n}\right)^2.$$

$X^{(n)}$ has ample canonical divisor if $n \geq 3$ ($X^{(2)}$ is a K3 surface). When $n = 5$, we have $c_1(X^{(5)})^2 = 5^4 \times 9$, $c_2(X^{(5)}) = 5^4 \times 3$, meaning that $X^{(5)}$ is a surface of general type which attains the upper bound of the Miyaoka-Yau inequality $K \leq 3c_2$.

The Del Pezzo surface X carries five linear pencils $|2H - \sum E_i|$, $|H - E_1|$, \dots , $|H - E_4|$, defining five surjective morphisms f_0, f_1, \dots, f_4 from X onto \mathbb{P}^1 . Each f_i of these morphisms has exactly three fibres contained in \tilde{D} , which are the singular fibres of f_i . For f_1 associated with $|H - E_1, \tilde{L}_{1j} + E_j|$, $j = 2, 3, 4$ are such fibres, and so are the three curves $\tilde{L}_{12} + \tilde{L}_{34}$, $\tilde{L}_{13} + \tilde{L}_{24}$, $\tilde{L}_{14} + \tilde{L}_{23}$ for f_0 associated with $|2H - \sum E_i|$.

Upstairs on $X^{(n)}$, there are thus five morphisms $f_0^{(n)}, f_1^{(n)}, \dots, f_4^{(n)}$ onto the curve $C^{(n)}$, an n^2 -sheeted Kummer cover of \mathbb{P}^1 branching at three points, $0, 1, \infty$, say. The pullback line bundle $\mathcal{L}_i^{(n)} = f_i^{(n)*} \omega_{C^{(n)}}$ is an invertible subsheaf of $\Omega_{X^{(n)}}^1$. We easily check that $\mathcal{L}_i^{(n)}$ is saturated in $\Omega_{X^{(n)}}^1$ and that

$$\begin{aligned} \mathcal{L}_0^{(n)} &\sim \left(1 - \frac{3}{n}\right) \pi^{(n)*}(2H - \sigma E_i) \\ \mathcal{L}_i^{(n)} &\sim \left(1 - \frac{3}{n}\right) \pi^{(n)*}(H - E_i), \quad i = 1, 2, 3, 4. \end{aligned}$$

Ishida [6] showed that the natural map

$$\bigoplus_{j=0}^4 f_j^{(n)*} H^0(C^{(n)}, \Omega_{C^{(n)}}^1) \rightarrow H^0(X^{(n)}, \Omega_{X^{(n)}}^1)$$

is an isomorphism. In particular, the irregularity of $X^{(n)}$ is given by

$$q(X^{(n)}) = \frac{5(n-2)(n-1)}{2}.$$

4 Construction of further examples of stable Higgs bundles with vanishing Chern classes

The tensor product $\mathcal{L}^{(n)} = \bigoplus_{i=0}^4 f_i^* \Omega_{C^{(n)}}^1$ is an invertible subsheaf of $\text{Sym}^5 \Omega_{X^{(n)}}^1$.

Easy calculation shows that

$$c_1(\mathcal{L}^{(n)}) = \frac{n-3}{n} \pi^{(n)*} \tilde{D},$$

while

$$K_{X^{(n)}} = \frac{n-1}{2n} \pi^{(n)*} \tilde{D}.$$

This subsheaf induces a Higgs subsheaf

$$\mathcal{E}^{(n)} = \bigoplus_{i=0}^2 \text{Sym}^i \Omega_{X^{(n)}} \oplus \bigoplus_{i=2}^0 \text{Sym}^i \Theta_X \otimes \mathcal{L}^{(n)}$$

of $\bigoplus_{i=0}^2 \text{Sym}^i \Omega_{X^{(n)}} \oplus \bigoplus_{i=2}^0 \text{Sym}^i \Theta_X \otimes \text{Sym}^5 \Omega_{X^{(n)}}^1$ (see example 3). This bundle has rank six. We claim:

Proposition 4.1. $c_2(\mathcal{E}^{(n)})^2 = \frac{5}{12}c_1(\mathcal{E}^{(n)})^2$.

Proposition 4.2. If $n \geq 5$, then $\mathcal{E}^{(n)}$ is $K_{X^{(n)}}$ -semistable. If $n \geq 6$, then $\mathcal{E}^{(n)}$ is $K_{X^{(n)}}$ -stable.

Corollary 4.3. If $n \geq 6$, then $\mathcal{E}^{(n)}$ is projectively flat.

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