The singular Riemann-Roch theorem and Hilbert-Kunz functions

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1 Introduction

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic p, where p is a prime integer. For an $\mathfrak{m}$-primary ideal I and a positive integer e, we set

$$I^{[p^e]} = (a^{p^e} \mid a \in I).$$

It is easy to see that $I^{[p^e]}$ is an $\mathfrak{m}$-primary ideal of A. For a finitely generated A-module M, the function $\ell_A(M/I^{[p^e]}M)$ on e is called the *Hilbert-Kunz funtion* of M with respect to I. It is known that

$$\lim_{e \to \infty} \frac{\ell_A(M/I^{[p^e]}M)}{p^{de}}$$

exists [8], and the real number is called the *Hilbert-Kunz multiplicity*, that is denoted by $e_{HK}(I, M)$. Properties of $e_{HK}(I, M)$ are studied by many authors (Monsky, Watanabe, Yoshida, Huneke, Enescue, e.t.c.).

Recently Huneke, McDermott and Monsky proved the following theorem:

Theorem 1.1 (Huneke, McDermott and Monsky [5]) *Let $(A, \mathfrak{m}, k)$ be a d-dimensional normal local ring of characteristic p, where p is a prime integer. Assume that A is F-finite*[1] *and the residue class field k is perfect.*

1. *For an $\mathfrak{m}$-primary ideal I of A and a finitely generated A-module M, there exists a real number $\beta(I, M)$ that satisfies the following equation*[2]*:*

$$\ell_A(M/I^{[p^e]}M) = e_{HK}(I, M) \cdot p^{de} + \beta(I, M) \cdot p^{(d-1)e} + O(p^{(d-2)e})$$

[1] We say that A is F-finite if the Frobenius map $F : A \to A = {}^1A$ is module-finite. We sometimes denote the e-th iteration of F by $F^e : A \to A = {}^eA$.

[2] Let $f(e)$ and $g(e)$ be functions on e. We denote $f(e) = O(g(e))$ if there exists a real number K that satisfies $|f(e)| < Kg(e)$ for any e.

2. *Let I be an $\mathfrak{m}$-primary ideal of A. Then, there exists a $\mathbb{Q}$-homomorphism $\tau_I : \mathrm{Cl}(A)_{\mathbb{Q}} \longrightarrow \mathbb{R}$ that satisfies*

$$\beta(I, M) = \tau_I \left(\mathrm{cl}(M) - \frac{\mathrm{rank}_A M}{p^d - p^{d-1}} \mathrm{cl}({}^1A) \right),$$

for any finitely generated A-module M. In particular,

$$\beta(I, A) = -\frac{1}{p^d - p^{d-1}} \tau_I \left(\mathrm{cl}({}^1A) \right)$$

is satisfied.

For an abelian group N, $N_{\mathbb{Q}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{Q}$.
It is natural to ask the following question:

Question 1.2 1. *When does* $\mathrm{cl}({}^1A)$ *vanish?*

2. *How does* $\mathrm{cl}({}^eA)$ *behave?*

In the next section, we give a partial answer to this question.

Remark 1.3 The map cl in the theorem as above is called the *determinant map* [1]. Here we recall basic properties on cl.

Let R be a Noetherian normal domain. The group of isomorphism classes of reflexive R-modules of rank 1 is called the *divisor class group* of R, and denoted by $\mathrm{Cl}(R)$. Let $\mathrm{G}_0(R)$ be the Grothendieck group of finitely generated R-modules. Then, there exists the map

$$\mathrm{cl} : \mathrm{G}_0(R) \longrightarrow \mathrm{Cl}(R)$$

that satisfies the following two conditions:

(i) If M is a reflexive module of rank 1, then $\mathrm{cl}(M)$ is just the isomorphism class that contains M.

(ii) Let M be a finitely generated R-module. If the height of the annihilator of M is greater than or equal to 2, then $\mathrm{cl}(M) = 0$.

Example 1.4 1. This example is due to Han-Monsky [3]. Set $A = \mathbb{F}_5[[x_1, \dots, x_4]]/(x_1^4 + \cdots + x_4^4)$ and $m = (x_1, \dots, x_4)A$. Then,

$$\ell_A(A/m^{[p^e]}) = \frac{168}{61} 5^{3e} - \frac{107}{61} 3^e$$

is satisfied. Therefore, in this case, we have $e_{HK}(m, A) = \frac{168}{61}$ and $\beta(m, A) = 0$. We know that there is no hope to extend Theorem 1.1 under the same assumption.

2. Set

$$A = k[[x_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n]]/I_2(x_{ij}),$$

where k is a perfect field of characteristic $p > 0$.

Suppose $m = 2$ and $n = 3$. Then, K.-i. Watanabe proved

$$\ell_A(A/m^{[p^e]}) = (13p^{4e} - 2p^{3e} - p^{2e} - 2p^e)/8.$$

Therefore, we have $e_{HK}(m, A) = \frac{13}{8}$ and $\beta(m, A) = -\frac{1}{4} \neq 0$.

One can prove that, if $m \neq n$, then there exists an maximal primary ideal I (of finite projective dimension) such that $\beta(I, A) \neq 0$.

In Corollary 2.2, we will see that $\beta(I, A) = 0$ if A is a Gorenstein ring.

2 Main Theorem

Here, we state the main theorem. We refer the reader to [7] for a precise proof of the main theorem.

Let $F^e : A \to A = {}^eA$ be the e-th iteration of the Frobenius map F.

Theorem 2.1 *Let (A, m, k) be a d-dimensional Noetherian normal local ring of characteristic p, where p is a prime integer, and assume that A is a homomorphic image of a regular local ring. Assume that k is a perfect field and A is F-finite.*

Then, for each integer $e > 0$, we have

$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2} \mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

The following is an immediate consequence of the above theorem:

Corollary 2.2 *Under the same assumption as in the above theorem, if* $\mathrm{cl}(\omega_A)$ *is a torsion in* $\mathrm{Cl}(A)$*, then* $\beta(I, A) = 0$ *for any maximal primary ideal* I*.*

The following is an analogue of Theorem 2.1 for normal algebraic varieties.

Theorem 2.3 *Let* k *be a perfect field of characteristic* p*, where* p *is a prime integer. Let* X *be a normal algebraic variety over* k *of dimension* d*. Let* $F : X \to X$ *be the Frobenius map*[3]*.*

Then, we have

$$c_1(F_*^e \mathcal{O}_X) = \frac{p^{de} - p^{(d-1)e}}{2} K_X$$

in $\mathrm{Cl}(X)_{\mathbb{Q}} = A_{d-1}(X)_{\mathbb{Q}}$*, where* $c_1(\ )$ *is the first Chern class*[4] *and* K_X *is the canonical divisor of* X*.*

We give an outline of a proof of Theorem 2.1 in the next section.

Example 2.4 1. Set

$$A = k[[x_1, x_2, x_3, y_1, y_2, y_3]] \Big/ I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

$\mathfrak{p} = (x_1, x_2, x_3)A$ and $\mathfrak{q} = (x_1, y_1)A$, where $I_2(\ )$ is the ideal generated by all the 2 by 2 minors of the given matrix. Here, assume that k is a perfect field of characteristic 2. Then, using Hirano's formula [4], we know that

$${}^1A \simeq A^{\oplus 10} \oplus \mathfrak{p} \oplus \mathfrak{q}^{\oplus 5}.$$

Here, recall that $\mathrm{rank}_A {}^1A = p^{\dim A} = 2^4 = 16$.

Then, we have

$$\mathrm{cl}({}^1A) = 10\mathrm{cl}(A) + \mathrm{cl}(\mathfrak{p}) + 5\mathrm{cl}(\mathfrak{q}) = 4\mathrm{cl}(\mathfrak{q})$$

since $\mathrm{cl}(A) = 0$ and $\mathrm{cl}(\mathfrak{p}) + \mathrm{cl}(\mathfrak{q}) = 0$.

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[3] Remark that, under the assumption, F is a finite morphism.

[4] Set $U = X \setminus \mathrm{Sing}(A)$. Since $\mathrm{codim}_X \mathrm{Sing}(A) \geq 2$, the restriction $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism. Here, remark that $F|_U : U \to U$ is flat. Therefore, $(F_*^e \mathcal{O}_X)|_U = (F|_U)_*^e \mathcal{O}_U$ is a vector bundle on U. Here, $c_1(F_*^e \mathcal{O}_X)$ is defined to be the first Chern class $c_1((F_*^e \mathcal{O}_X)|_U) \in \mathrm{Cl}(U) = \mathrm{Cl}(X)$.

On the other hand, it is well known that $\omega_A \simeq \mathfrak{q}$. By Theorem 2.1, we have

$$\mathrm{cl}({}^1 A) = \frac{2^4 - 2^3}{2}\mathrm{cl}(\omega_A) = 4\mathrm{cl}(\mathfrak{q}).$$

2. Let k be a perfect field of characteristic p, where p is a prime integer. Put $X = \mathbb{P}^1_k$. Let $F : X \to X$ be the Frobenius map. Then, we have $F_*\mathcal{O}_X \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{\oplus(p-1)}$, and

$$c_1(F_*\mathcal{O}_X) = c_1(\wedge^p F_*\mathcal{O}_X) = c_1(\mathcal{O}_X(1-p)) = 1-p.$$

Remark that the natural map $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is an isomorphism in this case.

On the other hand, it is well known that $\omega_X \simeq \mathcal{O}_X(-2)$. Therefore, we have $K_X = -2$. By Theorem 2.3, we have

$$c_1(F_*\mathcal{O}_X) = \frac{p-1}{2}K_X = 1-p.$$

3 Proof of Theorem 2.1

Now we start to give an outline of a proof of Theorem 2.1.

Let $(A, \mathfrak{m})$ be a Noetherian local ring that satisfies the assumption in Theorem 2.1.

Since $(A, \mathfrak{m})$ is a homomorphic image of a regular local ring, we have an isomorphism

$$\tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

of $\mathbb{Q}$-vector spaces by the singular Riemann-Roch theorem (Chapter 18 in [2]), where $\mathrm{A}_*(A) = \oplus_{i=0}^d \mathrm{A}_i(A)$ is the Chow group of the affine scheme $\mathrm{Spec}(A)$. Let

$$p : \mathrm{A}_*(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$$

be the projection. We set

$$\tau_{d-1} = p\tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{Cl}(A)_{\mathbb{Q}}.$$

Here, we summarize basic facts on the map τ_{d-1}.

(i) Let $\mathfrak{p}$ be a prime ideal of height 1. There exists a natural identification $\mathrm{A}_{d-1}(A) = \mathrm{Cl}(A)$ by $[\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p})$. By the exact sequence

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0,$$

we have

$$\mathrm{cl}(\mathfrak{p}) = \mathrm{cl}(A) - \mathrm{cl}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

On the other hand, by the top-term property (Theorem 18.3 (5) in [2]), we have $\tau_{d-1}(A/\mathfrak{p}) = [\mathrm{Spec}(A/\mathfrak{p})]$. Therefore we have

$$\tau_{d-1}(A/\mathfrak{p}) = [\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

Let $\mathfrak{q}$ be a prime ideal of height at least 2. By the top-term property, we have $\tau_{d-1}(A/\mathfrak{q}) = 0$.

(ii) By the covariance with proper maps (Theorem 18.3 (1) in [2]), we have

$$\tau_{d-1}({}^{e}A) = p^{(d-1)e}\tau_{d-1}(A)$$

for each $e > 0$.

(iii) We have

$$\tau_{d-1}(A) = \frac{1}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$ by Lemma 3.5 of [6].

Next we prove the following lemma:

Lemma 3.1 *Let $(A, \mathfrak{m})$ be a local ring that satisfies the assumption in Theorem 2.1. Then, for a finitely generated A-module M, we have*

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

Proof. Set $r = \mathrm{rank}_A M$. Then we have an exact sequence

$$0 \to A^r \to M \to T \to 0,$$

where T is a torsion module. By this exact sequence, we obtain

$$\mathrm{cl}(M) = r \cdot \mathrm{cl}(A) + \mathrm{cl}(T) = \mathrm{cl}(T).$$

On the other hand, by the basic fact (iii) as above, we obtain

$$\tau_{d-1}(M) = r \cdot \tau_{d-1}(A) + \tau_{d-1}(T) = \frac{r}{2}\mathrm{cl}(\omega_A) + \tau_{d-1}(T).$$

We have only to prove $\tau_{d-1}(T) = -\mathrm{cl}(T)$.

We may assume that $T = A/\mathfrak{p}$, where $\mathfrak{p} \neq 0$ is a prime ideal of A. If $\mathrm{ht}\,\mathfrak{p} \geq 2$, then we have

$$\tau_{d-1}(A/\mathfrak{p}) = 0 = -\mathrm{cl}(A/\mathfrak{p})$$

by Remark 1.3 and the basic fact (i) as above. If $\mathrm{ht}\,\mathfrak{p} = 1$, then we have

$$\tau_{d-1}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p})$$

by (i) as above. **q.e.d.**

Now we start to prove Theorem 2.1.

By the basic facts (ii) and (iii), we obtain

$$\tau_{d-1}({}^eA) = p^{(d-1)e}\tau_{d-1}(A) = \frac{p^{(d-1)e}}{2}\mathrm{cl}(\omega_A).$$

By Lemma 3.1, we have

$$\tau_{d-1}({}^eA) = -\mathrm{cl}({}^eA) + \frac{\mathrm{rank}_A\,{}^eA}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. It is easy to see that $\mathrm{rank}_A\,{}^eA = p^{de}$. We have obtained

$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. **q.e.d.**

Remark 3.2 By Theorem 2.1 and Lemma 3.1, we have

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{p^d - p^{d-1}}\mathrm{cl}({}^1A).$$

Therefore, we have

$$\beta(I, M) = -\tau_I(\tau_{d-1}(M)).$$

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